Exercise Sheet 5
Quotient Spaces

In what follows $\square := [-1,1] \times [-1,1]$ stands for a closed square. Exercises 5.2.1 and 5.2.2 involve somewhat tedious calculations that you need to do at least once in your lifetime. In Exercises 5.2.5 and 5.2.6 *no formal justification* is required.

1. Easy: check your understanding

Exercise 5.1.1. In the Torus Tic-Tac-Toe the rules are the same as in traditional tic-tac-toe, except here the opposite sides of the board are glued to form a torus. *H*int: the easiest way to analyse the game is to extend the square to tesselation of the entire plane.
(a) Show that the four positions in Figure 1 are equivalent in the Torus Tic-Tac-Toe.

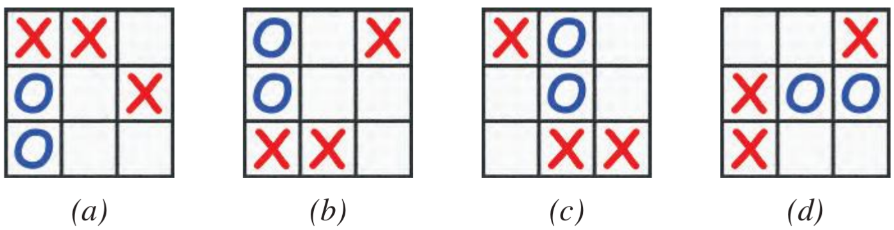


Figure 1: Equivalent positions in Torus Tic-Tac-Toe.

(b) Which of the positions in Figure 2 are equivalent in Torus Tic-Tac-Toe?

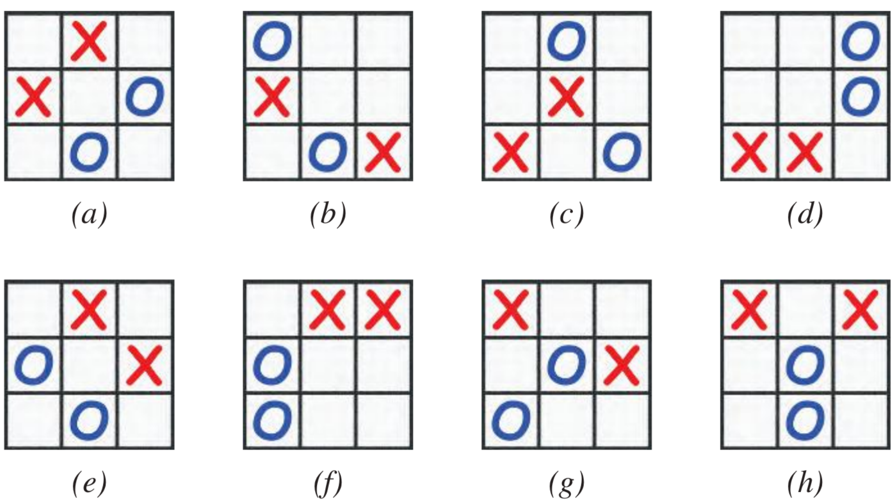


Figure 2: Identify equivalent positions in Torus Tic-Tac-Toe.

Exercise 5.1.2. Write down explicitly an injective map $\kappa\colon \square/_{(-1,y)\sim(1,y)} \to \mathbb{R}^3$ that maps the quotient space onto a cylinder.

Exercise 5.1.3. Show that the map $\mathsf{e}\colon \mathbb{R} \to \mathbb{S}^2$ given by $t \mapsto e^{2\pi t i}$ *descends* to the quotient space $\mathbb{R}/\mathbb{Z}$. In other words, show that for any two points in the same equivalence class $x_1, x_2 \in [x] \in \mathbb{R}/\mathbb{Z}$, their images are the same $\mathsf{e}(x_1) = \mathsf{e}(x_2)$. Deduce that $\mathbb{R}/\mathbb{Z}$ with the quotient topology is homeomorphic to a circle.

Exercise 5.1.4. Take a pair of paper strips and make two Möbius bands. Cut one along the middle line and the other one at a distance of one quarter of its width from the edge. Compare the surfaces you obtain with the answer to Exercise 5.3.1.

The difference you see (between Exercises 5.1.4 and 5.3.1) is that between different *embeddings* into 3-dimensional space. We shall learn more about it later.

Exercise 5.1.5. Glue a sphere from the square $\square$, i.e. write down explicitly an equivalence relation $\sim$ such that the quotient space $\square/\sim$ is homeomorphic to the sphere.

Note: It is not possible to make the homeomorphism differentiable.

Exercise 5.1.6. Find two points on the square that are mapped into points on

(a) a cylinder by κ from Exercise 5.1.2;

(b) a torus by $\tau^{a,b}$ from Exercise 5.2.1;

(c*) a Möbius band by μ^r from Exercise 5.2.2;

which are the furthest apart with respect to the Euclidean metric in $\mathbb{R}^3$.

Exercise 5.1.7. Find the points on the square that are mapped into points on

(a) a torus by $\tau^{a,b}$ from Exercise 5.2.1;

(b*) a Möbius band by μ^r from Exercise 5.2.2;

which are the closest and the furthest from the origin in $\mathbb{R}^3$.

2. Harder: working-level knowledge

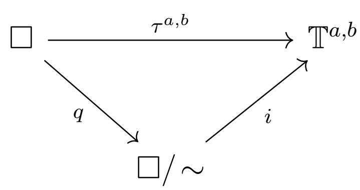
Exercise 5.2.1. Let $0 < a < b$. Show that the map $\tau^{a,b}\colon \square \to \mathbb{R}^3$ given by

$$\begin{aligned}
\tau_1^{a,b}(x_1, x_2) &= (a\cos(\pi x_1) + b)\cos(\pi x_2)\\
\tau_2^{a,b}(x_1, x_2) &= (a\cos(\pi x_1) + b)\sin(\pi x_2)\\
\tau_3^{a,b}(x_1, x_2) &= a\sin(\pi x_1)
\end{aligned}$$

is injective on the interior of $\square$.

Additionally, verify that $\tau^{a,b}(x_1, -1) = \tau^{a,b}(x_1, 1)$ and $\tau^{a,b}(-1, x_2) = \tau^{a,b}(1, x_2)$.

Remark. The image of $\tau^{a,b}$ is a torus and we shall denote it $\mathbb{T}^{a,b}$. Let $\sim$ be the equivalence relation identifying the opposite points on the square boundary, i.e., $(x_1, -1) \sim (x_1, 1)$ and $(-1, x_2) \sim (1, x_2)$. The Exercise implies that the map $\tau^{a,b}$ *descends* to the Quotient space $\square/\sim$; moreover there exist homeomorphisms i so that the diagram is commutative.

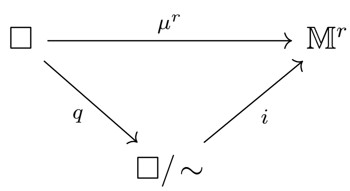


Exercise 5.2.2. Let $r > 2$. Show that the map $\mu^r \colon \square \to \mathbb{R}^3$ given by

$$\begin{aligned} \mu_1^r(x_1, x_2) &= -\Big(r - \frac{x_2}{2}\sin\frac{\pi x_1}{2}\Big)\sin(\pi x_1) \\ \mu_2^r(x_1, x_2) &= \Big(r - \frac{x_2}{2}\sin\frac{\pi x_1}{2}\Big)\cos(\pi x_1) \\ \mu_3^r(x_1, x_2) &= \frac{x_2}{2}\cos\frac{\pi x_1}{2}. \end{aligned}$$

is injective on the interior of the square. Verify that $\mu_r(-1, x_2) = \mu_r(1, -x_2)$.

Remark. The image of μ^r is a Möbius band and we shall denote it $\mathbb{M}^r$. Let $\sim$ be the equivalence relation identifying two sides of the boundary with a half-twist, i.e., $(-1, x_2) \sim (1, -x_2)$. The Exercise implies that the map μ^r *descends* onto the Quotient space $\square/\sim$; moreover there exists a homeomorphism i such that the diagram is commutative.



Exercise 5.2.3. What surface does one obtain by gluing together (three pairs of) opposite sides of a regular hexagon? Draw the sides of the hexagon on a more familiar net. *Hint*: you can make extra cuts of the hexagon if necessary, marking sides to remember to re-attach.

Exercise 5.2.4. (a) What curve does one obtain by attaching two disjoint intervals to each other along the boundary? More formally, let $I_1 = [0, 1]$, $I_2 = [2, 3]$, and an attachment map $f \colon \partial I_1 \to \partial I_2$ given by $f(0) = 2$, $f(1) = 3$. What curve is $(I_1 \sqcup I_2)/_{x\sim f(x)}$ homeomorphic to?

(b) What surface does one obtain by attaching two disjoint disks to each other along the boundary? Formally speaking, let $B_1 = \{z \in \mathbb{C} \colon |z - 2| \le 1\}$ and $B_2 = \{z \in \mathbb{C} \colon |z + 2| \le 1\}$, and take an attachment map $f \colon \partial B_1 \to \partial B_2$ given by $f(z) = z - 4$. What surface is the quotient space $(B_1 \sqcup B_2)/_{z\sim f(z)}$ homeomorphic to?

Exercise 5.2.5. In Torus Tic-Tac-Toe, how many essentially different opening moves does the first player have? How many different responses does their opponent have? Is either player guaranteed to win, assuming optimal play?

Exercise 5.2.6. How many points can you select on **(a)** a sphere; **(b)** a cylinder; **(c)** a torus; **(d)** a Möbius band; such that every pair is connected by a continuous path not intersecting other such paths? (Paths cannot traverse other selected points on the way either.)

3. Exam-level questions

Exercise 5.3.1. Let $\gamma^0, \gamma^1 \subset \mathbb{M}^r$ be two curves on the Möbius band given by

$$\gamma^0_1(t) = r\cos(t), \quad \gamma^0_2(t) = r\sin(t), \quad \gamma^0_3(t) = 0, \qquad 0 \le t \le 2\pi; \text{ and}$$
$$\gamma^1_1(t) = \Big(r \pm \frac{1}{8}\cos\frac{t}{2}\Big)\cos(t), \quad \gamma^1_2(t) = \Big(r \pm \frac{1}{8}\cos\frac{t}{2}\Big)\sin(t), \quad \gamma^1_3(t) = \pm\frac{1}{8}\sin\frac{t}{2}, \qquad 0 \le t \le 2\pi.$$

Describe the complements $\mathbb{M}^r \setminus \gamma^0$ and $\mathbb{M}^r \setminus \gamma^1$.

Exercise 5.3.2. Let $\gamma \subset \mathbb{T}^{1,4}$ be the curve given by

$$\gamma_1(t) = (\cos(\pi t)+4)\cos(3\pi t), \quad \gamma_2(t) = (\cos(\pi t)+4)\sin(3\pi t), \quad \gamma_3(t) = \sin(\pi t), \qquad -1 \le t \le 1.$$

Is the complement $\mathbb{T}^{1,4} \setminus \gamma$ a connected subset of $\mathbb{T}^{1,4}$? Justify your answer.