Exercise Sheet 4
Connectedness and Path-Connectedness

1. Easy: check your understanding

Exercise 4.1.1. Using the fact that an interval is a connected set, show that any path-connected set is connected.

Exercise 4.1.2. Show that if two path-connected sets have non-empty intersection, then their union is a path-connected set.

Exercise 4.1.3. Show that the closure $\overline{A}$ of a connected set A is connected. Furthermore, show that any intermediate set $A \subset B \subset \overline{A}$ is also connected. *Hint*: Problem 2.1.8.

Exercise 4.1.4. Give an example of a path-connected set such that its closure is not path-connected.

Exercise 4.1.5. Consider the set $A = \{0\} \cup \{\frac{1}{n},\ n \in \mathbb{N}\}$. Show that the only connected subsets of A are individual points.

Exercise 4.1.6. Show that continuous image of a path-connected set is path-connected (i.e. if $A \subset X$ is path-connected and $f\colon X \to Y$ is continuous, then $f(A) \subset Y$ is also path-connected).

2. Harder: working-level knowledge

Exercise 4.2.1. Let γ be the infinite curve

$$\gamma(t) = \Big(\frac{1+t}{t}\cos t, \frac{1+t}{t}\sin t\Big) \subset \mathbb{R}^2, \qquad t > 0.$$

Let $\mathbb{S}^1$ be the unit circle. Sketch γ and show that $\gamma \cup \mathbb{S}^1$ is topologically connected.

Exercise 4.2.2. The *Infinite Broom* space is the set of intervals connecting the origin with $\big(1, \frac{1}{n}\big)$, for $n \in \mathbb{N}$ together with the point $(1, 0)$. Sketch the Infinite Broom and show that the it is a connected set

Exercise 4.2.3. Describe the closure of the Infinite Broom.

Exercise 4.2.4. The *Infinite Comb* space is the union of the interval $[0, 1]$ along the x-axis together with vertical line segments connecting $(1/n, 0)$ to $(1/n, 1)$ for $n \in \mathbb{N}$ and the single (red) point $(0, 1)$:

$$D = ([0, 1] \times \{0\}) \cup \bigcup_{n\in\mathbb{N}} (1/n \times [0, 1]) \cup (0, 1).$$

The y-axis strictly between 0 and 1 is not part of the comb. Sketch the Comb space and show that it is connected.

Exercise 4.2.5. Describe the closure of the Infinite Comb.

Exercise 4.2.6. Replicating the argument for topologist's sine curve, show that the Infinite Comb and the Infinite Broom are not path-connected.

3. Exam-level problems

Exercise 4.3.1. Show that the complement to (a) a line; (b) a circle; (c) a cylinder in $\mathbb{R}^3$ is a connected set.

Exercise 4.3.2. Consider the set A given by $A = \{0\} \cup \{1/n,\ n \in \mathbb{N}\}$. Describe all continuous maps $f\colon \mathbb{R} \to A$.

Exercise 4.3.3. Show that the graph of a continuous function $\mathbb{R} \to \mathbb{R}$ is a closed subset of $\mathbb{R}^2$. Deduce that a parabola is a closed subset of $\mathbb{R}^2$.

Exercise 4.3.4. Show that the direct product of two connected (path-connected) sets is connected (respectively, path-connected). Deduce that the torus is a path-connected set.