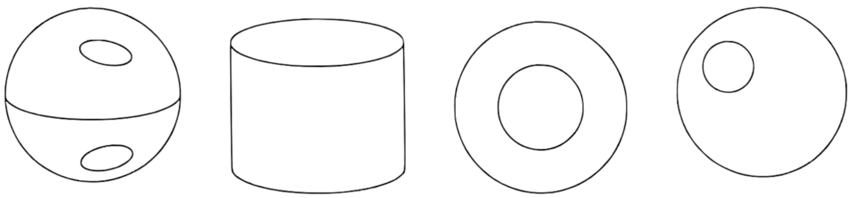
Exercise Sheet 3 — Unassessed Coursework 1
Homeomorphisms

Submit solutions to *four* problems of your choice from §1 and *three* problems of your choice from §2. All sets are considered with the standard topology induced from $\mathbb{R}^n$.

Easy: check your understanding

Exercise 3.1.1. Does there exist a homeomorphism $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{Q}) = f(\mathbb{R} \setminus \mathbb{Q})$? Justify your answer.

Exercise 3.1.2. Show that the following four surfaces are all homeomorphic to each other.



Exercise 3.1.3. Show that if $A, B \subset X$ are connected and $A \cap B \neq \varnothing$ then $A \cup B$ is also connected.

Exercise 3.1.4. Is the following set of nested circles a closed subset of $\mathbb{R}^2$? Justify your answer.

$$X = \bigcup_{n=3}^{\infty} \Big\{ (x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 2 - \frac{1}{n^2} \Big\}$$

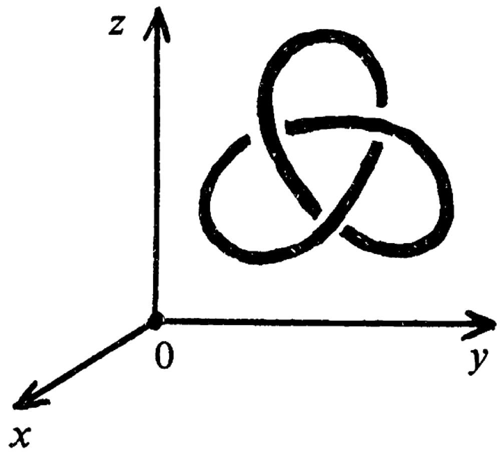
Exercise 3.1.5. Let X be a topological space. Show that if $A \subset X$ and $B \subset X$ are homeomorphic and there exists a point $a \in A$ such that the complement $A \setminus \{a\}$ is connected, then there exists a point $b \in B$ such that $B \setminus \{b\}$ is also connected.

Exercise 3.1.6. Assuming that the surfaces in Figure 1 are made from a very elastic material: their shape may be changed at will, you can bend, distort, stretch, and compress them as much as you like, but of course you may not tear them or glue parts of them together. Which pairs of these surfaces can then be turned into one another? Picture the deformations involved.

Exercise 3.1.7. A codling moth made a see-through hole in an apple. Which surface is the boundary of the apple homeomorphic to? What about two moths?

Harder: working-level knowledge

Exercise 3.2.1. Draw the three projections of the curve shown below onto 3 coordinate planes. The curve is called the trefoil knot.



Exercise 3.2.2. Using computer or otherwise, draw projection of the curve given by parametric equations

$$x(t) = (\cos t) \cdot (3\cos t + 2) \qquad y(t) = 5\cos t \cdot \sin t \qquad z(t) = (\sin t) \cdot (25\cos^2 t - 1)$$

onto the xy coordinate plane. Argue that it is homeomorphic to the trefoil knot.

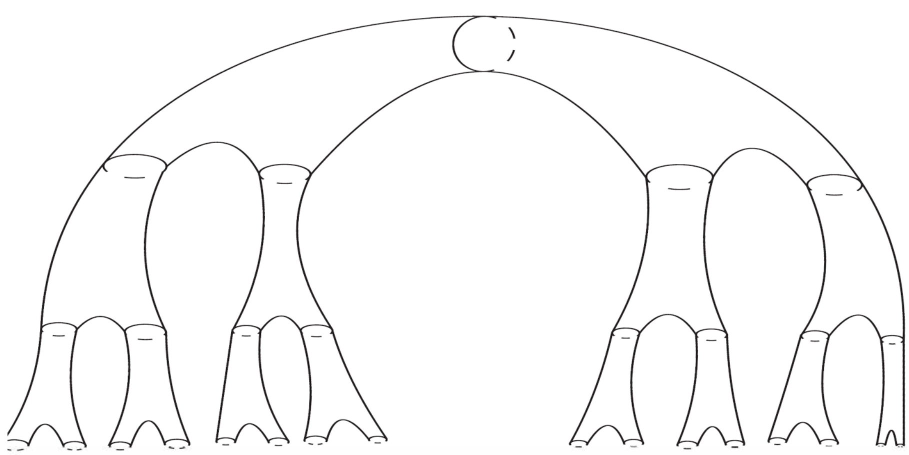
Exercise 3.2.3. Prove that the trefoil knot is homeomorphic to the circle.

Exercise 3.2.4. Is the set

$$A = \left\{(x, y) \in \mathbb{R}^2 \colon x > 0, y = \cos\frac{1}{x}\right\} \cup \left\{(0, 0)\right\}$$

a topologically connected subset of $\mathbb{R}^2$? Justify your answer.

Exercise 3.2.5. Show that the surface pictured below is homeomorphic to a plane $\mathbb{R}^2$ with 15 points removed. Picture proof accepted. Boundary circles are not included.



Exercise 3.2.6. Let $S^2 \subset \mathbb{R}^3$ be a sphere. Define

$$X = \left\{(x, y) \in \mathbb{R}^2 \colon (x-2)^2 + y^2 = 1\right\} \cup \left\{(x, y) \in \mathbb{R}^2 \colon (x+2)^2 + y^2 = 1\right\}$$

Let $f \colon S^2 \to X$ be a continuous map. What are the maximal sets with respect to set inclusion that $f(S^2)$ can be equal to?

Exercise 3.2.7. Let $X \subset \mathbb{R}^n$ be connected and let $\varphi \colon X \to \mathbb{Z}$ be continuous. Show that it is the constant map.

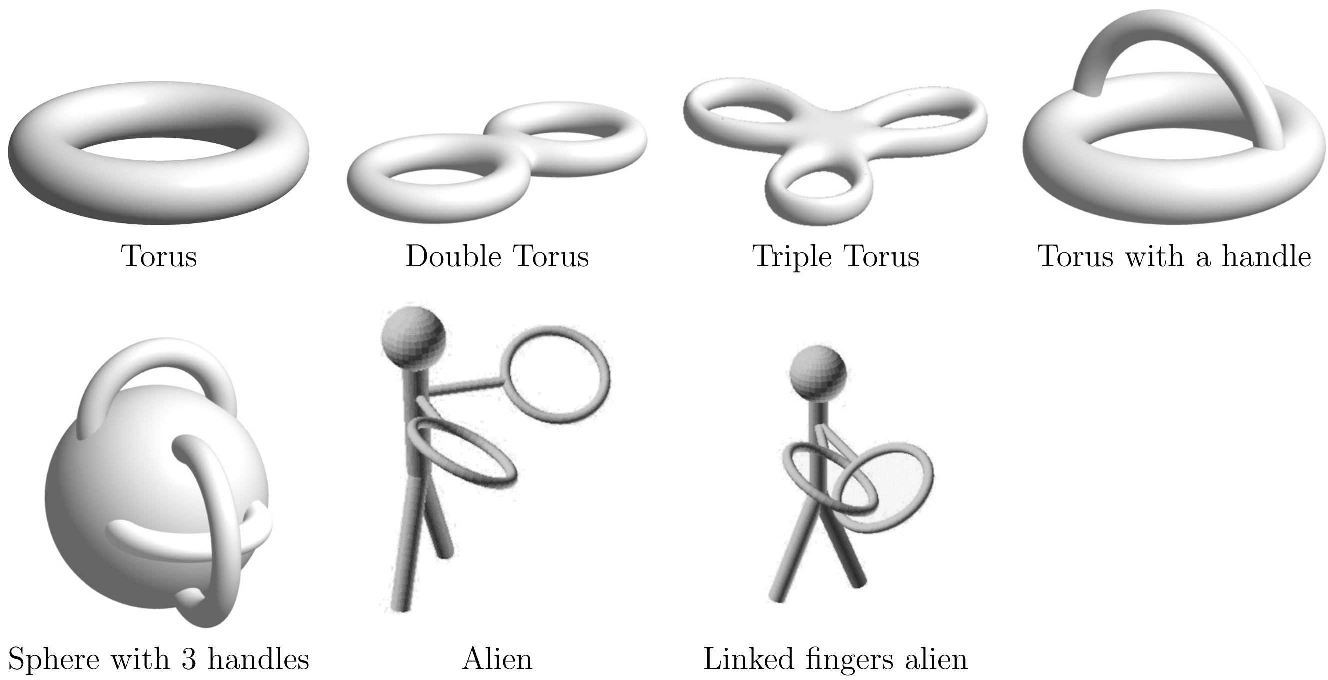


Figure 1: Surfaces for Problem 3.1.6.