Exercise Sheet 2
Topology & Continuity

The purpose of this example sheet is to become familiar with fundamental topological constructions: topology, closure, boundary, connectedness; continuous maps and its properties.

1. Easy: Check your understanding

Exercise 2.1.1. Show that the (topological) boundary of a set is a closed set.

Exercise 2.1.2. Let T_∞ be the collection of $X_{(a_1,a_2),r} = \{(x_1,x_2) \in \mathbb{R}^2 \colon \max(|x_1-a_1|, |x_2-a_2|) < r\}$ for all possible $r \in \mathbb{R}^+$ and $(a_1,a_2) \in \mathbb{R}^2$. Show that T_∞ is not a topology on $\mathbb{R}^2$.

Exercise 2.1.3. Let S be a circle. Is $\mathbb{R}^2 \setminus S$ a connected subset of $\mathbb{R}^2$? Justify your answer.

Exercise 2.1.4. Is the set $\{(\frac{1}{n}, 0), (0, \frac{1}{n}) \mid n \in \mathbb{N}\} \subset \mathbb{R}^2$ closed or open? Justify your answer.

Exercise 2.1.5. Consider the union of three disks on the plane

$$X := B_1((10,0)) \cap B_1((-10,0)) \cup B_1((0,10)) \subset \mathbb{R}^2$$

with the induced standard topology. Is $B_1((0,10))$ closed or open in X?

Exercise 2.1.6. Show that if A is a connected subset of a topological space X and $f\colon X \to Y$ is continuous, then $f(A)$ is connected in Y.

Exercise 2.1.7. Give an example of a closed A subset of a topological space X such that its image under a continuous map $f\colon X \to Y$ $f(A)$ is not closed in Y.

Exercise 2.1.8. A point $a \in A \subset \mathbb{R}^n$ is called *a limit point* of A, if for any open set $\mathbb{R}^n \supset U \ni a$ the intersection $A \cap U \neq \varnothing$. Show that the closure $\overline{A}$ is the union of A and its limit points.

Exercise 2.1.9. Show that $x \in \partial A \subset \mathbb{R}^n$ if and only if for any open set $U \ni x$ we have that $U \cap A \neq \varnothing$ and $U \cap (X \setminus A) \neq \varnothing$.

2. Harder: working-level knowledge

Exercise 2.2.1. Show that the Cartesian product of two topological spaces is a topological space.

Exercise 2.2.2. Show that a square $\{\square = (0,1) \times (0,1)\}$ is open with respect to the standard topology and write it as a union of open disks $\{x \in \mathbb{R}^2 \colon \|x-a\| < r\}$.

Definition. Given a set $A \subset \mathbb{R}^n$ and a point x, the quantity $d(x, A) = \inf_{a \in A} \|x - a\|$ is called the distance from x to A.

Exercise 2.2.3. Show that the function $f(x) := d(x, A)$ is continuous for any $A \subset \mathbb{R}^n$.

Exercise 2.2.4. Show that if the set A is closed, then $d(x, A) \neq 0$ for all $x \notin A$.

Exercise 2.2.5. Show that the topological definition of a continuous function on $\mathbb{R}^n$ and the $\varepsilon - \delta$ definition given in Analysis agree.

Exercise 2.2.6. Give an example of a continuous map between two topological spaces $f \colon X \to Y$ and a non-empty $A \subsetneq X$ such that $f(\partial A) \not\subset \partial(f(A))$.

3. Exam-level problems

Exercise 2.3.1. Give an example of a bijective continuous mapping between topological spaces such that the inverse is not continuous.

Exercise 2.3.2. We say that $x \in X$ is an *inner* point of X, if there exist an open set A such that $x \in A \subset X$. The *interior* is a union of all inner points. Show that the interior of A is the largest open set contained in A.

Exercise 2.3.3. Sketch the set $\{(x, \sin x^{-1}) \subset \mathbb{R}^2 \mid x \in (0, \infty)\}$ and describe its closure.

Exercise 2.3.4. Show that a set A in a topological space is closed if and only if $\partial A \subset A$.