MAT3009 Manifolds and Topology
2024/25 Exam Program

In order to obtain a good mark, it is helpful *to understand* concepts listed in section Notions; to be able *to apply* results listed in section Theorems and *to provide examples* for statements and constructions listed in section Examples and Counterexamples.

> **MASS Sheets Warning**
> The MASS sheet should be treated as an incentive to lessen memorisation and focus instead on understanding the class materials and developing problem solving skills; solely relying on the MASS sheet is likely to lead to failing the exam.

You might be asked to give a definition of a notion, or to state a theorem and to provide a short proof, or to give an example of a certain construction with justification.

In what follows, all subsets of $\mathbb{R}^n$ are considered with the standard topology. A manifold stands for a smooth manifold.

1. Notions

1. Inclusion map;
2. Topological space, open set, closed set, closure and interior of a set;
3. Product topology;
4. Limit point of a set in a topological space, topological boundary;
5. Continuous functions, homeomorphisms;
6. Connected and path-connected sets;
7. Compactness in topological spaces;
8. Quotient topology;
9. Attachment map, gluing, connected sum;
10. Smooth function on *arbitrary* subset of $\mathbb{R}^n$;
11. Diffeomorphism between two subsets of $\mathbb{R}^n$;
12. k-dimensional smooth manifold, parametrisation, charts, atlas, submanifold;
13. Derivative of a map $\mathbb{R}^n \to \mathbb{R}^k$ as a linear mapping;
14. Tangent space to a manifold at a point;
15. Local diffeomorphism, immersion, canonical immersion;
16. Proper map, embedding;
17. Submersion, canonical submersion, regular value.
18. Independent system of functions at a point on a manifold.
19. Boundary of a manifold.

2. Propositions and Theorems

1. Product of two topological spaces is a topological space.
2. If X is compact, and $f\colon X \to \mathbb{R}^n$ is a continuous and one-to-one, then f is a homeomorphism.
3. Given a homeomorphism $\varphi\colon X \to Y$ and subset $A \subset X$; if A is open, closed, connected, or path-connected, then $\varphi(A)$ is also open, closed, connected, or path-connected, respectively. Furthermore, $\varphi(\partial(A)) = \partial(\varphi(A))$.

4. Graph of a continuous function $f\colon \mathbb{R}^k \to \mathbb{R}^n$ is a closed subset of $\mathbb{R}^{k+n}$.
5. If subsets A and B of a topological space are homeomorphic, then for any finite number of points $b_1, \dots, b_n$ there exist a finite number of points $a_1, \dots, a_n$ such that $B \setminus \{b_1, \dots, b_n\}$ and $A \setminus \{a_1, \dots, a_n\}$ are also homeomorphic.
6. If A is a compact subset of a topological space and $f\colon X \to Y$ is continuous, then $f(A)$ is compact.
7. Open subset of a manifold (with respect to the induced topology) is a submanifold.
8. If X is a smooth k-manifold with parametrisation (U_j, φ_j) and Y is a smooth n-manifold with parametrisation (V_i, ψ_i), then $X \times Y$ is a smooth $n+k$-manifold with parametrisation $(U_j \times V_i, \varphi_j \times \psi_i)$.
9. Classification of 1-dimensional manifolds.
10. For any sets $X, Y, Z \subset \mathbb{R}^n$ and any smooth maps $f\colon X \to Y$, $g\colon Y \to Z$ the composition $h\colon X \to Z$ given by $h = g \circ f$ is also smooth.
11. Given $k < n$, any k-dimensional linear subspace of $\mathbb{R}^n$ is a smooth k-manifold.
12. Given manifolds X and Y, the projection map $\pi\colon X \times Y \to X$ given by $\pi(x, y) = x$ is a smooth map.
13. Given two manifolds X and Y, the graph of smooth function $f\colon X \to Y$, that is the set $G(f) = \{(x, f(x)) \mid x \in X\}$, is a smooth manifold.
14. Given a smooth manifold X, the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is a smooth manifold diffeomorphic to X. $T_x(\Delta) = T_xX \times T_xX$.
15. The dimension of a tangent space to a manifold at a point is equal to the dimension of the manifold.
16. The chain rule for smooth maps between smooth manifolds.
17. The Inverse Function Theorem.
18. An embedding $f\colon X \to Y$ maps X diffeomorphically onto a submanifold of Y.
19. If f and g are immersions, then $f \times g$ and $f \circ g$ are also immersions. If $f\colon X \to Y$ is an immersion and $\dim X = \dim Y$, then f is a local diffeomorphism.
20. Local Submersion Theorem.
21. Preimage Theorem.
22. If the smooth real-valued functions $g_1, \dots, g_k$ on a smooth manifold X are independent at each point where they all vanish, the set of common zeros is a submanifold of X, and its dimension is $\dim X - l$.
23. If X is compact and Y is connected, then every submersion $f\colon X \to Y$ is surjective.
24. If y is a regular value of a smooth map $f\colon X \to Y$, then the preimage submanifold $f^{-1}(y)$ can be cut by a system of independent functions. Every submanifold is *locally* cut by a system of independent functions.
25. Let $Z = f^{-1}(y)$ be the preimage of a regular value under a smooth map $f\colon X \to Y$. Then the kernel of the derivative $df_x\colon T_xX \to T_{f(x)}Y$ at any point $x \in Z$ is the tangent space T_xZ.
26. The product of a manifold without boundary X and a manifold with boudary Y is another manifold with boundary, $\partial(X \times Y) = X \times \partial Y$ and $\dim(X \times Y) = \dim X + \dim Y$.
27. If $f\colon X \to Y$ is a diffeomorphsim of manifolds with boundary, then the restriction onto the boundary $f\colon \partial X \to \partial Y$ is also a diffeomorphism.

3. Basic constructions, Examples and Counterexamples

1. Given a continuous function $\varphi\colon X \to Y$ and subset $A \subset X$; if A is closed, then $\varphi(A)$ need not to be closed; if A is open, $\varphi(A)$ need not to be open; $\varphi(\partial A)$ does not need to belong to $\partial(\varphi(A))$; if A is connected, $\varphi(A)$ is connected; if A is path-connected, $\varphi(A)$ is path-connected.
2. The closure of a connected set is connected; the closure of a path-connected set need not to be path-connected; the closure of a manifold need not to be a manifold.
3. A subset $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded. However, there exist subsets $X \subsetneq \mathbb{R}^n$ such that $Y \subsetneq X$ is closed and bounded in induced topology, but not compact.
4. If $f\colon X \to Y$ is an immersion, then $f(X)$ does not need to be a submanifold of Y.
5. Spaces: sphere, cylinder, Möbius band, torus, Klein bottle, projective space, infinite broom, infinite comb, topologists sine curve: topological properties, net, manifold structure (for smooth manifolds).

Essential materials: exercise sheets and solution sheets.

Recommended reading: relevant concepts are covered, for instance, in the following books, although they contain by far more than we considered.

1. "Introduction to metric and topological space" Wilson Sutherland. Chapters depends on the edition; in 2009 edition they are §7 Topological spaces, §8 Continuity in topological spaces, §9 Some concepts in topological spaces, §10 Subspaces and product spaces, §12 Connected spaces, §13 Compact spaces, §15 Quotient spaces and surfaces. We covered, however, these concepts in much less depth.
2. "The shape of space" by Jeffrey Weeks; Chapter 1, §2 Gluing, §3 Vocabulary.
3. "Differential Topology" by Victor Guillemin and Alan Pollack. Chapter 1 "Manifolds and Smooth maps", §1 Definitions and §2 Derivatives and Tangents, §3 The inverse function theorem and immersions, §4 Submersions; Chapter 2 §1 Manifolds with boundary.