Polina Vytnova

MA4K3

Example Sheet 3
Functionals and Operators on `Hardy Spaces`

Hand in solutions to five P-Problems of your choice.
Deadline: 2pm, Tuesday 5th of December.

E1. Let $U \subset \mathbb{H}^2$ be a closed subspace. Show that U is shift-invariant if and only if $\mathbb{H}^\infty U \subset U$.

E2. Let $z_0 \in \mathbb{D}$ and consider the Möbius transformation $\mu(z) = \frac{z-z_0}{1-\overline{z_0}z}$. Compute the pseudohyperbolic disk $K(z_0, r) := \mu(D(0, r))$, where $0 < r < 1$.

E3. Show that pseudohyperbolic distance is invariant with respect to Möbius transformations.

E4. Let $E \subset L_2(\partial\mathbb{D})$, with $zE \subset E$, $zE \neq E$. Show that there exists a measurable function φ (unique up to a constant) such that $|\varphi| = 1$ a.e. on $\partial\mathbb{D}$ and $E = \varphi\mathbb{H}^2$.

P1. Show that the multipliers $f_1, f_2 \in \mathcal{M}_{\mathbb{H}^2}$ are outer functions if and only if their product is an outer function.

P2. Find the smallest shift-invariant subspace U containing

$$(1)\ \frac{1}{2-z} \qquad (2)\ \left\{\sqrt[3]{1-z}, z^2\right\}$$

(3) a pair of Blaschke products for some $\{w_n\}_{n=0}^\infty$, $\{z_n\}_{n=0}^\infty \in \mathbb{D}$.

P3. Consider a function $f(z) = 1 - z^n$.

1. Find optimal approximants p_k^* to $1/f$.
2. Compute $\|p_{3n}^* f(z) - 1\|$.
3. Show that f is cyclic.

P4. Consider a sequence $\{z_k\}_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty}(1-|z_k|^2)\delta_{z_k}$ is a Carleson measure. Show that the following sum is finite:

$$\sum_{j=0}^{\infty} \frac{(1-|z_k|^2)(1-|z_j|^2)}{|1-\overline{z^k} z_j|^2}$$

P5. Let ρ be the pseudohyperbolic distance. Establish the inequality

$$\frac{\rho(z_0,z_2)-\rho(z_2,z_1)}{1-\rho(z_0,z_2)\rho(z_2,z_1)} \le \rho(z_0,z_1) \le \frac{\rho(z_0,z_2)+\rho(z_2,z_1)}{1+\rho(z_0,z_2)\rho(z_2,z_1)} \qquad \forall z_0,\, z_1,\, z_2 \in \mathbb{D},$$

and prove that ρ is a metric on $\mathbb{D}$.

P6. Give an example of sequence $W = \{w_n\}_{n=0}^{\infty} \in \mathbb{D}$ such that

1. W is a sequence of interpolation and for any $0 \le t < 2\pi$ there exists a subsequence w_{n_k} such that $\lim_{k\to\infty} w_{n_k} = e^{it}$;
2. W is a zero set for some $f \in \mathbb{H}^2$, but it is not a sequence of interpolation;
3. $\sum\limits_{k=0}^{\infty}(1-|w_k|^2)\delta_{w_k}$ is a Carleson measure, but W is not a sequence of interpolation.

P7. Which of the following sequences are the sequences of interpolation?

$$(1)\, 1-e^{-k} \quad (2)\, e^{ik}\left(1-\frac{1}{k+1}\right) \quad (3)\, 1-\frac{e^{ik}}{k+1} \quad (4)\, 2^{2ik}\left(1-\frac{1}{(k+1)^2}\right) \quad (5)\, e^{2ik}\left(1-2^{-k}\right).$$

P8. Let $E \subset L_2(\partial\mathbb{D})$ satisfy $zE = E$. Show that there exists a unique measurable set $A \subset \partial\mathbb{D}$ such that $E = \chi_A L_2(\mathbb{D}) = \{f \in L_2(\partial\mathbb{D}) : f = 0 \text{ a.e. outside } A\}$, where χ_A is the characteristic function of A.

P9. Let a sequence $\{z_n\}_{n=0}^{\infty} \in \mathbb{D}$ be such that for any $\{a_k\}_{k=0}^{\infty} \in \ell^{\infty}$ the interpolation problem $f(z_k) = a_k$ has a solution in H^{∞}. Show that

$$\sup_{\|a\|_\infty \le 1} \inf\{\|f\|_\infty \colon f \in \mathbb{H}^\infty,\, f(z_j) = a_j\, \forall j\}$$

is finite.

P10. Show that there exists $f \in \mathbb{H}^2$ such that

$$\sum_{k=1}^{\infty} \frac{|f(1-k^{-2})|^2}{k^2} = \infty$$

P11. Assume that a sequence $z_k \in \mathbb{D}$ satisfies $1 - |z_{n+1}| \leq c(1 - |z_n|)$ for some $c < 1$. Show that the measure μ defined by $\mu(z_k) = 1 - |z_k|^2$ is a Carleson measure.

P12. Construct a sequence $0 < z_1 < z_2 < \ldots$ in $\mathbb{D}$ such that the measure μ defined by $\mu(z_k) = 1 - |z_k|^2$ is not a Carleson measure.