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MA4K3

Example Sheet 1
Getting to know `Hardy Spaces`

Hand in solutions to at least five P-Problems of your choice. The mark will be a sum of the best five Solutions.

Deadline: 2pm, Thursday 26th of October.

Notation. $\mathbb{D}$ stands for the open unit disk in $\mathbb{C}$. $\mathcal{O}(\mathbb{D})$ stands for holomorphic functions on $\mathbb{D}$. $\mathbb{H}^2$ stands for the Hardy space of functions $f \in \mathcal{O}(\mathbb{D})$, such that $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ and $\sum\limits_{k=0}^{\infty} |a_k|^2$ is finite.

E1. Which of the following expressions define a holomorphic function $f\colon \mathbb{D} \to \mathbb{C}$?

(1) $f(x, y) = x^2 + iy^2$ (2) $f(x, y) = x^2 - y^2 + 2ixy$ (3) $f(x, y) = \sin(x^2 - y^2) + i\cos(xy)$

(4) $f(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{2^k}$ (5) $f(z) = \sum\limits_{k=0}^{\infty} i^k z^k$ (6) $f(z) = \sum\limits_{k=0}^{\infty} k! z^{k-2}$

E2. Which of the following subsets are conformally equivalent to the unit disk?

(1) $\mathbb{C} \setminus 0$ (2) $\mathbb{C} \setminus \mathbb{R}^-$ (3) $\mathbb{C} \setminus \partial\mathbb{D}$ (4) $\mathbb{D} \setminus \{0\}$ (5) $\{x + iy \colon x, y \in [0, 1]\}$

P1. Let $\{a_k\}$ be a sequence of complex numbers such that one of the following holds

$$(1)\ |a_k| \le \frac{1}{k+1}, \quad (2)\ \lim_{k\to\infty} a_k = 0, \quad (3)\ \lim_{k\to\infty} \frac{a_k}{k+1} = 0.$$

Does the formal series $\sum\limits_{k=0}^{\infty} a_k z^k$ define an $\mathbb{H}^2$ function?

P2. Show that $f(z) = (1 - z)^\alpha$ is an $\mathbb{H}^2$ function for all $\alpha > \frac{1}{2}$.

P3. Let $f \in \mathcal{O}(\mathbb{D})$. Assume that one of the following holds:

$$(1)\ \exists C > 0 \colon \forall z \in \mathbb{D} \colon |f(z)| \le C \quad (2)\ f' \in \mathbb{H}^2 \quad (3)\ \exists g \in \mathbb{H}^2 \colon f = g'.$$

Is f an $\mathbb{H}^2$ function?

P4. Given $p > 2$ construct a sequence $\{a_k\} \in \ell^p(\mathbb{C})$ such that $\sum\limits_{k=0}^{\infty} a_k z^k \notin \mathbb{H}^2$.

P5. Show that the monomials $\{z^k\}_{k=0}^{\infty}$ are orthogonal to each other in $\mathbb{H}^2$ and compute their norms. Show that $\mathbb{H}^2$ equipped with the inner product $\langle \cdot, \cdot \rangle$ is a Hilbert space over $\mathbb{C}$.

P6. Determine the type of singularities of the function $f(z)$ given by:

$$(1)\ \frac{1+z}{1-z} \qquad (2)\ \left(\log\left(\frac{1+z}{1-z}\right)\right)^{-1/4} \qquad (3)\ \exp\left(-\frac{1+z}{1-z}\right) \qquad (4)\ \frac{\sin(1-z^2)}{1-z}$$

P7. Study radial and nontangential limits of the functions from **P6**. (They may be finite, infinite, or may not exist.)

P8. Show that the weigthed Hardy space $\mathbb{H}^2_\omega$ contains all polynomials, and that the polynomials are dense. Show that not all functions in $\mathbb{H}^2_\omega$ are polynomials.

P9. Show that
(1) the Hardy space $\mathbb{H}^2$ contains the Dirichlet space
(2) the Bergman space $\mathcal{A}^2$ contains the Hardy space.

P10. Given a pair of weights ω, ω' satisfying $\omega_k \le C\omega'_k$ for some $C > 0$ and all k show that $\mathbb{H}^2_\omega \subset \mathbb{H}^2_{\omega'}$.

P11. Give an example of a function $f \in \mathcal{O}(\mathbb{D})$ for which the radial limit at 1 exists, but nontangential limit does not. Give an example of a function f for which nontangential limit at 1 exists, but $\lim_{z\to 1} f(z)$ does not.

P12. Show that for any $f \in \mathcal{O}(\mathbb{D})$ the following are equivalent:

1. $\lim_{r\to 1} \int_{\partial\mathbb{D}} |f(r\exp^{i\theta})|^2 dm(\theta)$ is finite;

2. $\sup_{0<r<1} \int_{\partial\mathbb{D}} |f(r\exp^{i\theta})|^2 dm(\theta)$ is finite.

P13. Let $\mathbb{H}^2_0 = \{f \in \mathbb{H}^2 \mid f(0) = 0\}$. Show that $L^2(\partial\mathbb{D}) = \mathbb{H}^2 \oplus \overline{\mathbb{H}^2_0}$.