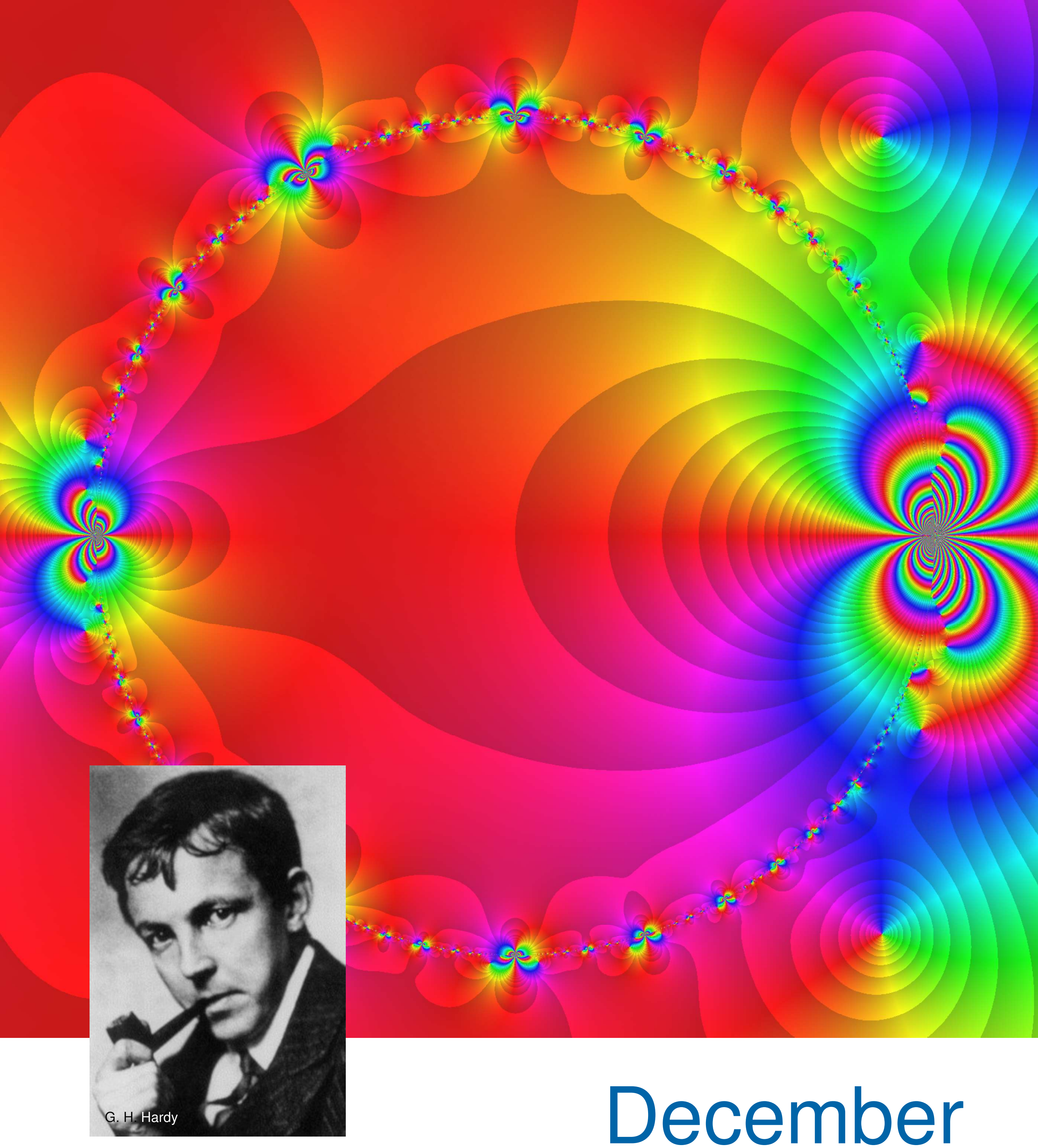

G. H. Hardy

December

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Theta Functions

A year ago in this calendar we portrayed Ramanujan and his mock theta functions. In his famous last letter to Hardy, these functions were introduced and motivated by the following "genuine" *theta function:*

$$g(q) = \sum_{n=0}^{\infty} p(n)q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2}.$$

Here $p(n)$ denotes the number of partitions of the positive integer n; that is, the number of different ways in which n may be written as a sum of positive integers, neglecting the order of the summations, e.g.

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1,$$

thus $p(5) = 7$. This month's phase portrait is of the *theta function* g. The poles of the summands of g, the n-th roots of unity for each positive integer n, form a dense set of singularities on the unit circle. Looking at the portrait, some singularities seem to be "worse" than others.

In what is considered one of the most important joint papers of Hardy and Ramanujan, these two mathematicians created the *"circle method"* to find an asymptotic formula for $p(n)$. At the time, the use of analytic methods to investigate algebraic or combinatorial functions was not yet very common. They started by applying Cauchy's integral formula

$$p(n) = \frac{1}{2\pi i} \int_\Gamma \frac{g(q)}{q^{n+1}} dq,$$

where Γ is a closed path encircling the origin exactly once in the counterclockwise direction and lying entirely inside the unit disk. Their novel idea was to push the path Γ towards the unit circle and approximate the effect of the most influential singularities. This method, now known as *Hardy-Ramanujan-Littlewood circle method,* was just the beginning. Hardy and Littlewood, as well as many other mathematicians, extended and applied it to other problems. In the case of our theta function, Hardy and Ramanujan used it to derive the asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}, \text{ as } n \to \infty.$$

Godfrey Harold Hardy (1877 – 1947)

was educated at Winchester and Trinity College in Cambridge. He excelled in his studies and was elected a fellow of Trinity. In 1919 he accepted the chair as Savilian Professor of Geometry in Oxford and became a fellow of New College. He spent a year at Princeton, in an exchange with Oswald Veblen who went to Oxford. In 1931 Hardy returned to Trinity in Cambridge as Sadleirian Professor, arguably the most prestigious mathematical chair in Great Britain, where he remained to his death.

Hardy had two outstanding mathematical collaborations. One of them, with J. E. Littlewood, lasted for thirty five years. Together they produced many very influential papers in function theory, inequalities, the Riemann zeta function, and other topics in analysis. The second collaboration was with Ramanujan. Hardy recognized the mathematical genius of Ramanujan immediately, invited him to Cambridge, mentored him, and together they produced beautiful mathematics until the untimely death of Ramanujan ended the collaboration.

In 1941, Hardy wrote *A mathematician's apology*, a description of a mathematician's thinking and an ode to pure mathematics. Hardy wrote: "The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colours or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics." To this day, this pamphlet remains an often quoted and classic text on the work of a mathematician.

W. Duke, Almost a Century of Answering the Question: What is a Mock Theta Function? AMS Notices 2014, Vol 61, No. 11, 1314–1320. J. J. O'Connor and E. F. Robertson, G. H. Hardy, MacTutor History of Mathematics, (University of St Andrews, Scotland, October 2003) http://www-history.mcs.st-andrews.ac.uk/Biographies/Hardy.html (accessed August, 2016).

October

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The Bergman Kernel (by Albrecht Böttcher)

The Bergman space $A^2(\mathbb{D})$ is the Hilbert space of all functions f that are analytic in the open unit disk $\mathbb{D}$ and for which $\|f\|^2 := \int_{\mathbf{D}} |f(z)|^2 \, dA(z) < \infty$, where $dA(z) = dx\,dy/\pi$ is normalized area measure. A Toeplitz equation in $A^2(\mathbb{D})$ is of the form

$$\big(T(a)f\big)(z) := \int_{\mathbb{D}} a(w)(1 - z\overline{w})^{-2} f(w)\, dA(w) = g(z), \quad z \in \mathbb{D}.$$

Here $g \in A^2(\mathbb{D})$ and $a \in L^\infty(\mathbb{D})$ are given and $f \in A^2(\mathbb{D})$ is sought. The functions

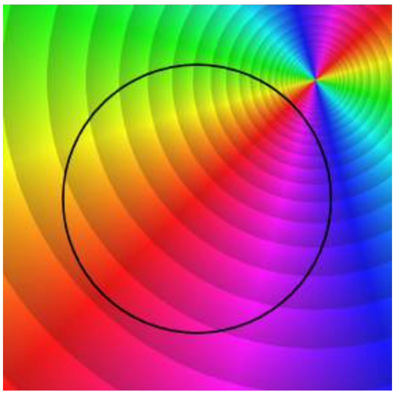
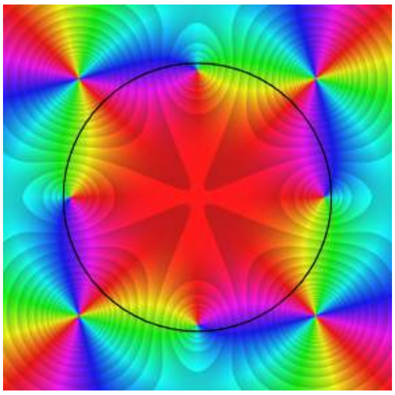
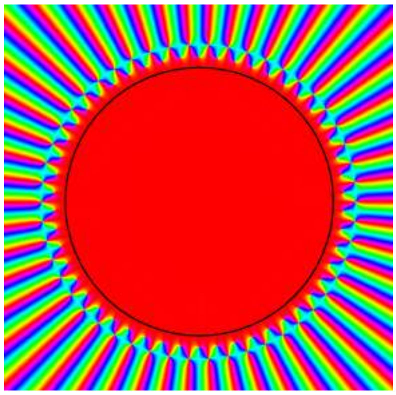
$$K_w(z) = (1 - z\overline{w})^{-2}, \quad w \in \mathbb{D},$$

are called *Bergman kernels.* They have the reproducing property: For every $f \in A^2(\mathbb{D})$ the scalar product of f with K_w is equal to the function value $f(w)$. The left picture below shows such a kernel.

Equations with Hardy space Toeplitz operators can be solved via Wiener-Hopf factorization. This does not work on the Bergman space, and approximation methods are the only way we know to solve an integral equation in the Bergman space. One such method is *analytic element collocation:* Look for an approximate solution f_n as a linear combination $f_n = \sum_{j=1}^{n} x_j K_{z_j}$ of the “analytic elements” K_{z_j} and determine the coefficients x_j by requiring that

$$(T(a)f_n)(z_j) = g(z_j), \quad j = 1, \ldots, n,$$

which is a linear system of n equations for the n unknowns x_j. In joint work of Hartmut Wolf and the author, this method was shown to converge if a is continuous on $\overline{\mathbb{D}}$, the operator $T(a)$ is invertible, and, in the nth step, the points $z_1, \ldots, z_n$ are taken as the roots of $z^n - r^n$ for some fixed $r \in (0, 1)$.

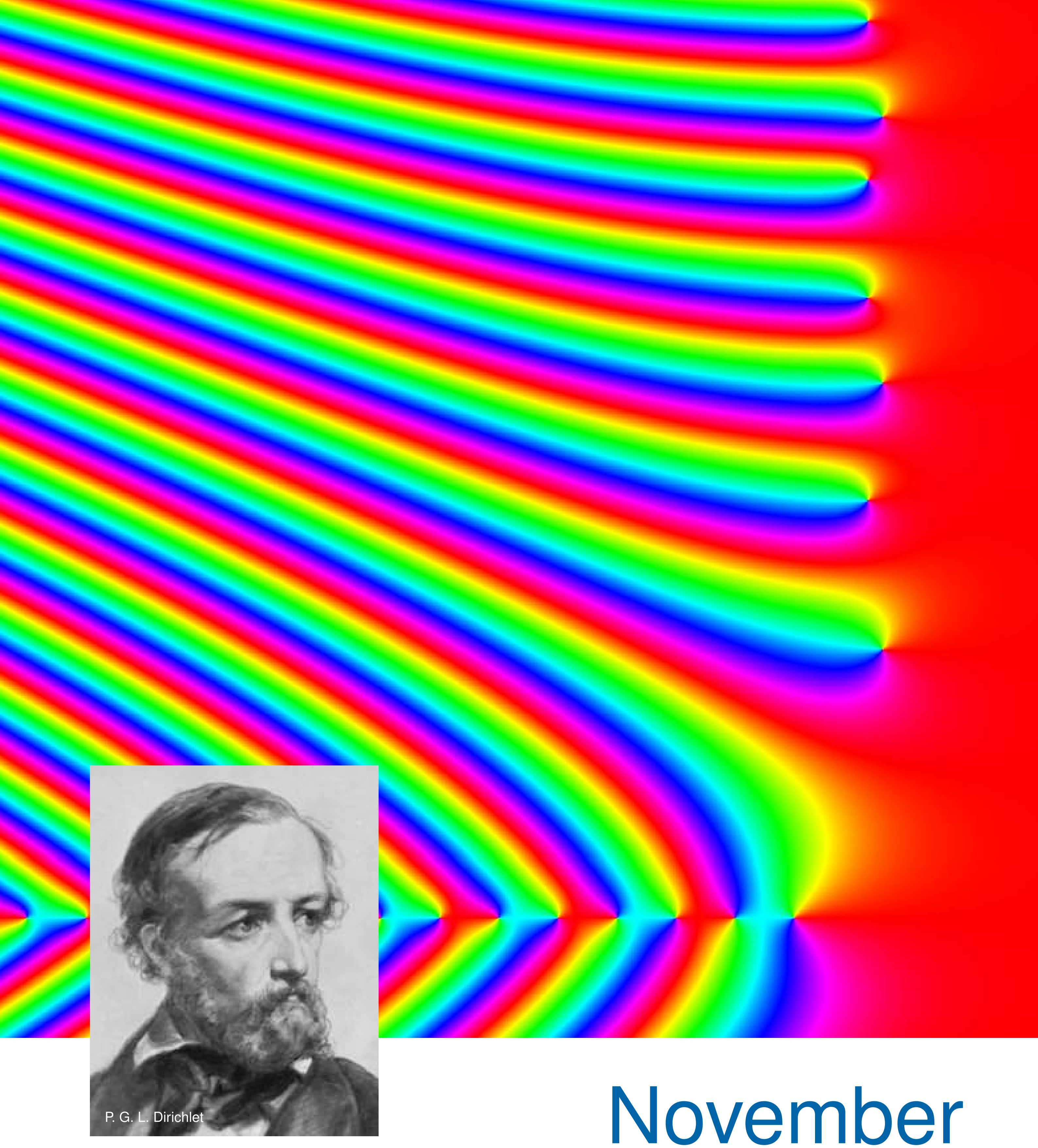
In the pictures in the middle and on the right we see the sum of 4 and 50 Bergman kernels. The picture of the month shows a linear combination of seven Bergman kernels.

Stefan Bergman (1895 – 1977)

was born in Częstochowa, then Congress Poland and part of the Russian Empire. He received his Ph.D. from the University of Berlin in 1921 under the supervision of Richard von Mises. In 1922, he introduced the kernel that was later named after him. In 1933, Bergman was dismissed from his post at the University of Berlin because he was Jewish. He moved to Russia, then to Paris, and in 1939, he eventually emigrated to the United States. Bergman taught at Stanford University from 1952 until his retirement in 1972. He died in Palo Alto, California, aged 82.

Stephen Krantz reports the following anecdote: “Whenever someone proved a new theorem about the Bergman kernel or the Bergman metric, Bergman made a point of inviting the mathematician to his house for supper. Bergman and his wife were a gracious host and hostess and made their guest feel welcome. However, after supper the guest had to pay the piper by giving an impromptu lecture about the importance of the Bergman kernel.”

Steven G. Krantz, Mathematical anecdotes. *Math. Intelligencer* 12 (1990), no. 4, pp 32–38.



P. G. L. Dirichlet

November

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The Dirichlet η Function

The *Dirichlet eta function* is defined by the *Dirichlet series*:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots .$$

This series converges for all complex numbers s with positive real part. This should be compared to the *Riemann zeta function* defined by $\zeta(s) = \sum_{n=1}^{\infty}(1/n^s)$, which converges for all s with real part greater than 1. (See November 2011 of Complex Beauties.) The zeta function has a meromorphic continuation to all of $\mathbb{C}$ and a simple pole at $s = 1$. In fact, the two are connected via the formula

$$\eta(s) = (1 - 2^{1-s})\zeta(s). \tag{1}$$

From this we see that the zeros of the η function include the zeros of the ζ function. Riemann proved that there are no zeros of the zeta function in the set $\{s \in \mathbb{C} : \mathrm{Re}\,(s) > 1\}$ and he showed that the only zeros in the left half-plane are the negative even integers. Later, Hadamard (1896) and de la Vallée-Poussin (1896) proved independently that no zero of ζ lies on the line $\mathrm{Re}\,(s) = 1$. For the so-called *critical strip* $\{s \in \mathbb{C} : 0 < \mathrm{Re}\,(s) < 1\}$ Riemann showed that the zeros must be located symmetrically about $\mathrm{Re}\,(s) = 1/2$. Thus all non-trivial zeros of the Riemann zeta function must be in the inside of the critical strip. The *Riemann hypothesis* is a conjecture that says that the zeros of the Riemann zeta function occur at the negative even integers and complex numbers z for which the real part is equal to $1/2$. Indeed, in the picture of the month, we see zeros of the Dirichlet eta function lining up along the negative real axis (these are the *trivial zeros* of ζ) and two vertical lines consisting of complex numbers with real part equal to $1/2$ and 1. Besides the zeros of the zeta function, the Dirichlet eta function has other zeros where the factor $1 - 2^{1-s}$ in (1) vanishes: This happens at points $s = s_n = 1 + 2n\pi\mathrm{i}/\log 2$ for every nonzero integer n. Since these additional zeros are not inside the critical strip, the Riemann zeta and the Dirichlet eta functions have the same zeros in the critical strip.

Peter Gustav Lejeune Dirichlet (1805 – 1859)

was born in Düren, which was then part of the First French Empire. He attended gymnasium in Bonn and later in Cologne. Dirichlet went to Paris in 1822, where Fourier and Poisson worked. His first paper was recognized by Fourier and von Humboldt. In 1827, with Humboldt's support, he obtained a position at the University of Breslau, after earning his doctorate (honoris causa) from the University of Bonn. In 1828, he was promoted to ausserordentlicher Professor (senior lecturer) and later that year he took a position at the *Allgemeine Kriegsschule* or General Military School. While at the school, he also taught at the University of Berlin where, in 1839, he became ordentlicher Professor or full professor. Dirichlet married Rebecka Mendelssohn, a sister of Felix Mendelssohn-Bartholdy.

Dirichlet moved to Göttingen in 1855 as Gauss's successor. In 1858, he traveled to Switzerland to give a memorial speech about Gauss and it was there that he had a heart attack. He died one year later at the age of 54. Students in Dirichlet's lectures included Eisenstein, Kronecker, Riemann, and Dedekind. In 1856/57, he lectured on potential theory and these notes were published in 1876, but he is perhaps best known for his work in number theory. Dirichlet used the pigeonhole principle in the proof of a theorem in diophantine approximation, made important progress on Fermat's last theorem for the cases $n = 5$ and $n = 14$ (compare to the mathematician appearing in February of this year's Complex Beauties), studied the first boundary value problem, and his insights are credited with developing the definition of function that we use today. Dirichlet did not publish frequently; in fact, Gauss declared, "Gustav Lejeune Dirichlet's works are jewels, and jewels are not weighed with a grocer's scale."

P. Gorkin and J. Smith, Dirichlet: his life, his principle, and his problem. *Math. Mag.* 78-4, 2005, pp. 283–296.

March

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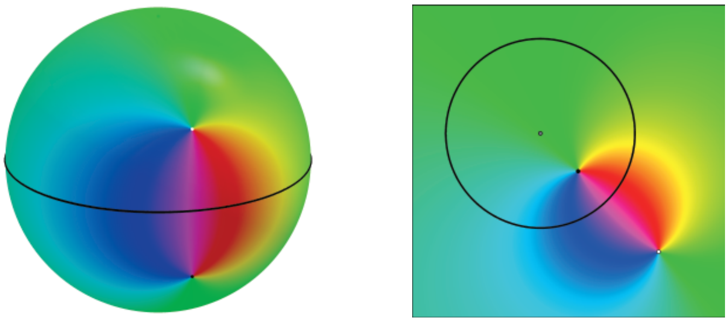
Blaschke Products

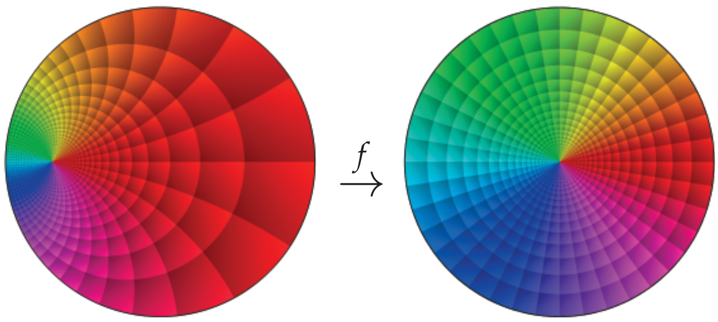
The quotients of two complex linear functions have interesting properties. Functions of the form

$$f(z) = c\,\frac{z - z_0}{1 - \overline{z}_0\, z}, \qquad |z_0| < 1,\ |c| = 1, \tag{1}$$

which are known as *Blaschke factors,* are of great importance. They have a unique zero at the point z_0 in the unit disc (the set of all points in the plane of distance less than 1 from the origin). The denominator vanishes at the point $1/\overline{z}_0$ and the function f has a pole at this point. On the Riemann sphere the points z_0 and $1/\overline{z}_0$ are symmetric with respect to the equator. In the plane this symmetry corresponds to a reflection with respect to the unit circle, also called an inversion.



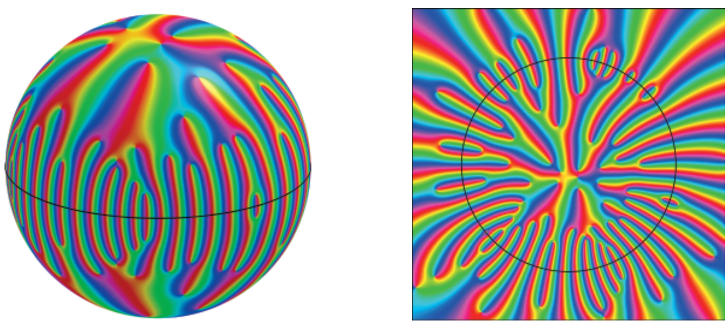
Blaschke factors are *self-mappings* of the unit disc $\mathbb{D}$: if the point z is in the unit disc then the *image point* $w = f(z)$ is in the unit disc and the converse also holds. In fact, every point w in $\mathbb{D}$ occurs exactly once as the image of a point in $\mathbb{D}$. As seen in the figure on the right, the point z_0 is mapped to the origin. For $z_0 = 0$ the function f reduces to a rotation about the origin.
The lines of the phase portraits emphasize the *conformality* of the mapping: The angle at which curves in the z-plane intersect is the same as the angle at which their image curves in the w-plane intersect.



A product of Blaschke factors with zeros at the points $z_1, \ldots, z_n$ of the unit disc,

$$f(z) = c\,\frac{z - z_1}{1 - \overline{z}_1\, z} \cdot \frac{z - z_2}{1 - \overline{z}_2\, z} \cdots \frac{z - z_n}{1 - \overline{z}_n\, z},$$

is called a (finite) *Blaschke product.* It also maps the unit disc $\mathbb{D}$ onto itself, but in this case every point w of $\mathbb{D}$ occurs exactly n times as the image of a point in $\mathbb{D}$. The picture of the month of March shows a Blaschke product with 50 zeros arranged in a regular pattern. The pictures to the right depict a Blaschke product on the Riemann sphere and in the complex plane which has 60 zeros arbitrarily distributed in $\mathbb{D}$.



Wilhelm Johann Eugen Blaschke (1885-1962)

was the son of a mathematics teacher in Graz. He decided early on to work in (differential) geometry and he studied with many of the leading experts of his time (Wirtinger, Bianchi, Klein, Hilbert, Runge, Study). After professorships in Prag, Leipzig, Königsberg, and Tübingen, he settled in Hamburg in 1919.

Wilhelm Blaschke is the author of influential books such as his “Lectures on Differential Geometry”. Because of his extensive travels and his numerous international contacts he initially opposed the Nazi regime. Later, he gave up his resistance and supported the political system.

June

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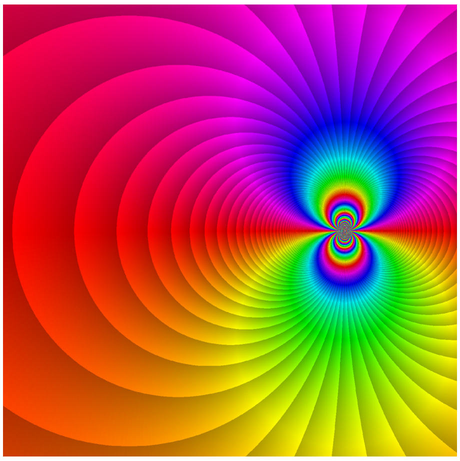
Singular Inner Functions

In complex analysis the Hardy spaces, H^p (for $1 \le p \le \infty$), are classes of analytic functions on the unit disk $|z| < 1$ that are important in analysis, control theory and scattering theory. They were introduced by F. Riesz who named them after G. H. Hardy, because of Hardy's 1915 paper *On the mean of the modulus of an analytic function*.

Arne Beurling showed that every function f in H^p has a factorization as $f = IG$ where I is an *inner function;* that is, a function I analytic on $|z| < 1$, mapping the unit disk to itself, with the property that for almost every θ, the values $I(re^{i\theta})$ for $0 < r < 1$ tend to a value of modulus one as r approaches 1, and G is called an *outer function.* Beurling was able to use inner functions to characterize the (closed) subspaces M of H^2 that are invariant under multiplication by the function $g(z) = z$; that is, M has the property that whenever $f \in M$, then $gf \in M$.

What do inner functions look like? Every such inner function I is either determined by its zeros, an inner function with no zeros, or a product of such functions. An inner function with no zeros is said to be a singular inner function, and one of them appears in the inset on the right:



$$S(z) = \exp\left(\frac{z+1}{z-1}\right),$$

which is called the *atomic singular inner function* and is the simplest singular inner function. This function is continuous everywhere except at the point $z = 1$, which is the point where the lines of constant moduli intersect. Looking at the point $z = 1$ in the figure we see that S has constant modulus on circles that are tangent to the unit circle at $z = 1$ and has constant argument on the circles of radius r centered at the points $(1, r)$. As a singular inner function, S is never zero but it assumes every other value in $|z| < 1$ infinitely often.

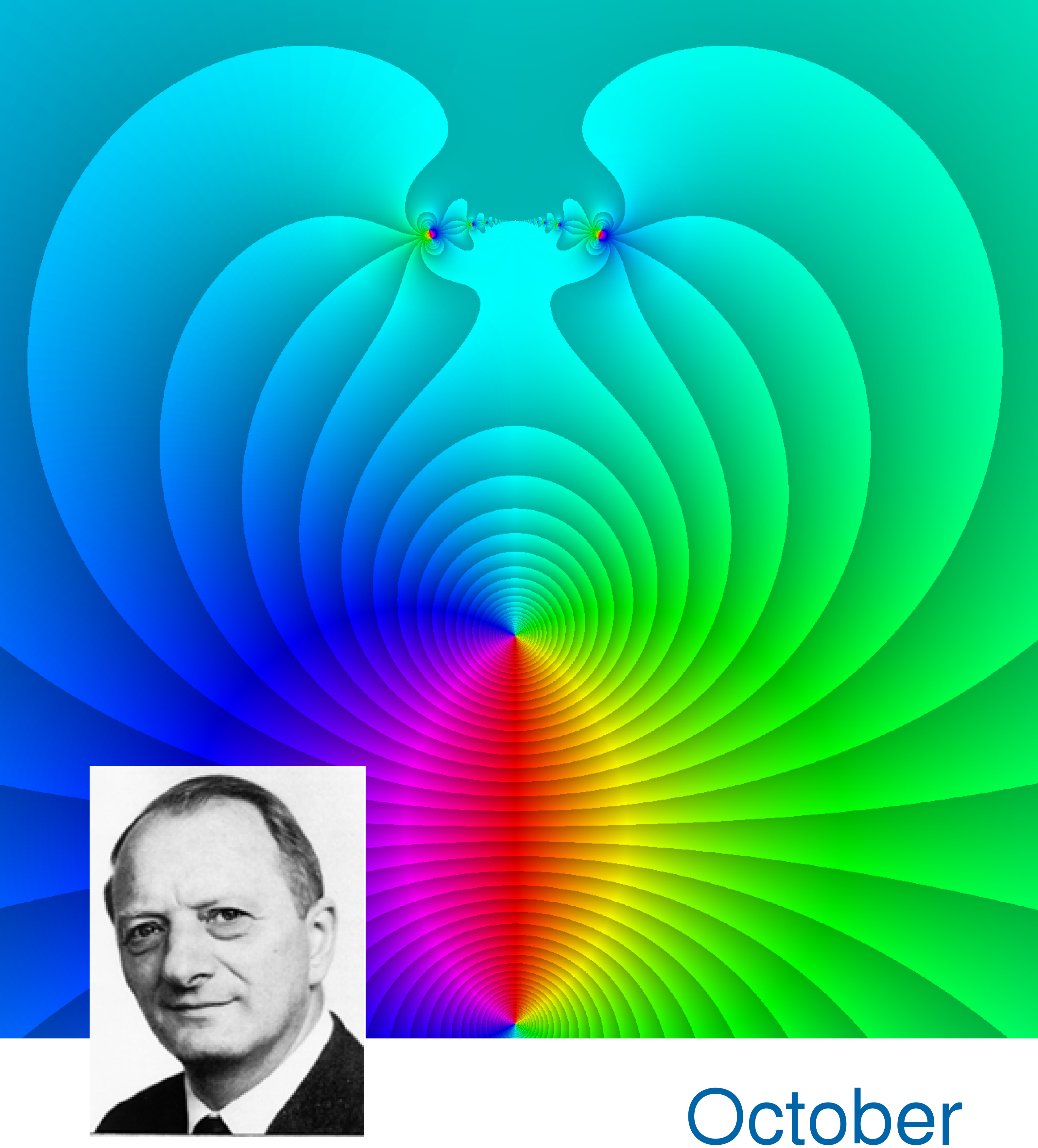
The function featured this month is a singular inner function with (precisely) five points of discontinuity. They appear at the five points on the unit circle at which $w^5 = 1$, points called the fifth roots of unity.

Arne Carl-August Beurling (1905 – 1986)

was born in Göteborg, Sweden. Beurling was a professor in Uppsala from 1937 until 1954, spending 1948-49 at Harvard. In 1954 he emigrated to the United States and became a professor at the Institute for Advanced Studies in Princeton, New Jersey. During World War II, Beurling worked on cracking the German codes and, according to David Kahn's book *Codebreakers*, "Quite possibly the finest feat of cryptanalysis performed by the Swedes, was Arne Beurling's solution of the German Siemens machine."

Lars Ahlfors and Lennart Carleson began their paper "Arne Beurling in memoriam" with the words, "Arne Beurling's legacy will influence mathematicians for many years to come, maybe even for generations." According to Carleson, Beurling's thesis provided a proof of the Denjoy conjecture about asymptotic values of an entire function but rather than publish it, Beurling went crocodile hunting with his father. Ahlfors became the first to publish a proof. Beurling's thesis was, however, published in 1933 and recognized as a program for research in function theory. He worked in harmonic analysis, complex analysis and potential theory.

Beurling was awarded the first Celsius Gold Medal in mathematics in 1961, by the Vetenskapssocieteten in Uppsala, he was elected to the Royal Swedish Academy of Sciences, the Finnish Academy of Sciences, the Royal Physiographical Society in Lund, Sweden, and the Danish Academy of Sciences. He was elected a Fellow of the American Academy of Arts and Sciences in 1970.



October

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Frostman's Theorem

In June we saw that a bounded analytic function f can always be written as a product IG, where I is called an inner function and G an outer function. The inner function broke down into a function determined by its zeros and one with no zeros at all. In June we focused on the function with no zeros. This time, we look at the function determined by its zeros. If our bounded analytic function vanishes at one or more points, its zero sequence, $\{a_n\}$ with $|a_n| < 1$, satisfies a condition known as the Blaschke condition,

$$\sum_n (1 - |a_n|) < \infty.$$

This condition ensures that the infinite product

$$\prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$$

converges for all z with $|z| < 1$. (If some $a_n = 0$, we interpret $|a_n|/a_n = 1$.) This infinite product, and rotations of such products, define non-zero functions known as Blaschke products. Simply stated, finite Blaschke products map the unit disk to itself and the circle to the circle; infinite Blaschke products almost do so. More precisely, letting B denote an infinite Blaschke product, at almost every point of the unit circle, $e^{i\theta}$, the limit $B(re^{i\theta}) \to \gamma$ for some $|\gamma| = 1$ as $r \to 1$ and $0 < r < 1$. Because of the special form of Blaschke products, they are easy to study – but their behavior isn't always simple. Composing on the left with a disk automorphism

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a} z}, \text{ where } a \in \mathbb{D},$$

keeps you in the class of inner functions, but it can throw you out of the class of Blaschke products. In his paper, *Sur les produits de Blaschke*, which linked function theory and potential theory, Frostman showed that $\varphi_a \circ B$ is a Blaschke product for "almost every" $a \in \mathbb{D}$. In the accompanying figure, we have taken an atomic singular inner function

$$S(z) = \exp\left(\frac{-1 + iz}{1 + iz}\right)$$

and composed it on the left with φ_a for $a = .727$. For this S and every $a \neq 0$, it turns out that $\varphi_a \circ S$ is a Blaschke product – a function determined by its zeros, which are located at the infinitely many points where $S(z) = a$. But for $a = 0$ the function $\varphi_a \circ S$ has no zeros at all.

Otto Frostman (1907 – 1977)

Otto Frostman received his first degree in mathematics from Lund University in Sweden, where he pursued graduate studies under the younger of the two Riesz brothers, Marcel Riesz. In 1935 he defended his thesis *Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions*, in which he extended Gauss's ideas on potential theory to kernels of a very general type. Frostman maintained his interest in potential theory until his death.

Frostman received a docent position at Lund upon completion of his thesis, but no permanent position became available and so he worked as a teacher for ten years in Halmstad and Lund. In 1952, he became professor of mathematics at what is now Stockholm University and he remained there until retiring in 1973. He also worked with the International Mathematical Union, serving as secretary from 1971 to 1974, and was director of the Mittag-Leffler Institute for fifteen years. He was elected to the Royal Academy of Science in 1952.

J. E. Littlewood

April

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Littlewood Polynomials

A polynomial $P(z) = a_n z^n + \ldots + a_1 z + a_0$ with all coefficients a_k equal to $+1$ or -1 is called a *Littlewood polynomial.* An example with all coefficients equal to 1 is

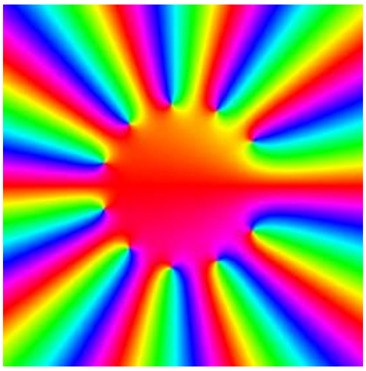
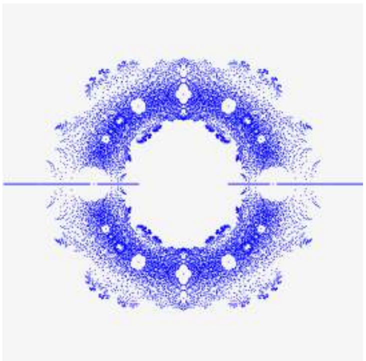
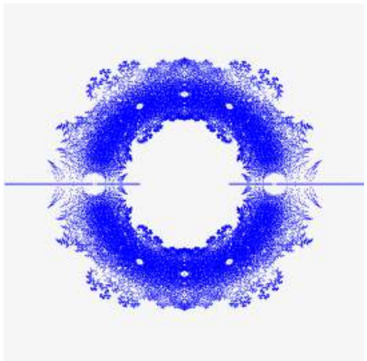
$$P(z) = z^n + \ldots + z + 1 = \frac{z^{n+1} - 1}{z - 1},$$

and its zeros are the $(n+1)$th roots of unity, except for 1. These all lie on the unit circle (see the figure below on the left). We will now give an argument that the zeros of an arbitrary Littlewood polynomial P are close to the unit circle; in fact, they are in the annulus $1/2 < |z| < 2$. So suppose that z is a zero of P that lies inside the unit circle. Since $P(z) = 0$ we conclude that

$$1 = |a_0| = |a_1 z + a_2 z^2 + \ldots + a_n z^n| \leq |z| + |z|^2 + \ldots + |z|^n = |z| \frac{1 - |z|^n}{1 - |z|} < \frac{|z|}{1 - |z|}.$$

Thus $1 - |z| < |z|$ or, equivalently, $|z| > 1/2$. A similar argument is used if z is outside the unit circle. This month's title page shows a Littlewood polynomial of degree 50 with randomly chosen coefficients. In the 1960s, J. L. Littlewood initiated an investigation of these polynomials. We now know a lot about them. For example, P. Borwein and J. Erdélyi showed that inside any polygon with vertices on the unit circle there are at most $c\sqrt{n}$ zeros, where c is a constant that depends only on the polygon.

The set of all zeros of Littlewood polynomials is also interesting and not fully understood. The middle figure below shows the set of all zeros of all Littlewood polynomials of degree eleven. The figure below on the right shows all zeros of all Littlewood polynomials of degree at most twelve.

John Edensor Littlewood (1885 – 1977)

was born in Rochester (in the southeast of England). He spent a part of his childhood (1892–1900) in South Africa, where his father taught mathematics. After his return to England, he was able to improve his mathematics education at St. Paul's School in London. He then entered Trinity College in Cambridge and scored best in his age group in the Tripos exams. He began his mathematical research under the guidance of E.W. Barnes (see July 2017) who gave him the Riemann conjecture to work on (see this November). Littlewood discovered the connection to the prime number theorem (which had long been known in continental Europe, showing the isolation of the British mathematicians). Later Littlewood said that it is possible to learn a lot from a problem that is out of reach.

About 1910 his long and fruitful collaboration with G. H. Hardy began (see December 2017). The two of them dominated British mathematics in the first half of the 20th century. Some mathematicians believed that Littlewood was a pseudonym for Hardy. Other important collaborators were M. Cartwright (see April 2016), S. Ramanujan (see December 2016 and July 2013) and R. Paley. Littlewood was athletic and remained mathematically active into old age. Throughout his life he battled depression and it was only from 1957 on that he was able to control it with the help of medication. He received many honors and awards (among them the Sylvester and the De Morgan medal) and he was elected to many academies.

P. Borwein and T. Erdélyi, On the zeros of polynomials with restricted coefficients, Illinois J. Math. **41** (1997), no. 4, 667-675.
R. Reyna and St. Damelin, On the structure of the Littlewood polynomials and their zero sets. arXiv:1504.08058 [math.CV]