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MA448
Example Sheet 3
Isometries of $\mathbb{H}$ and $\mathbb{D}$; Discrete groups — I

In what follows I_γ stands for reflection with respect to geodesic γ and $\rho(x, y)$ stands for the hyperbolic distance between x and y.

P1. Show that:

1. If $g \in \mathrm{Isom}(\mathbb{H})$ is parabolic, then there exists two geodesics γ_1, γ_2 such that $\gamma_1 \cap \gamma_2 \in \partial\mathbb{H}$ and $g = I_{\gamma_1} I_{\gamma_2}$.
2. If $g \in \mathrm{Isom}(\mathbb{H})$ is elliptic, then there exists two geodesics γ_1, γ_2 such that $\gamma_1 \cap \gamma_2 \in \mathbb{H}$ and $g = I_{\gamma_1} I_{\gamma_2}$.
3. If $g \in \mathrm{Isom}(\mathbb{H})$ is hyperbolic, then there exists two geodesics γ_1, γ_2 such that $\gamma_1 \cap \gamma_2 = \varnothing$ and $g = I_{\gamma_1} I_{\gamma_2}$.

P2. Show that if $g \in \mathrm{Isom}(\mathbb{D})$ is elliptic with fixed point w and rotation angle φ then

$$\sinh \frac{1}{2}\rho(z, gz) = \sinh \rho(z, w) \left|\sin \frac{\varphi}{2}\right|, \quad \varphi \in [-\pi, \pi].$$

P3. Prove that $g \in \mathrm{Isom}(\mathbb{H})$ is elliptic, parabolic, or hyperbolic if the isometric circles for g and g^{-1} are intersecting, parallel, or disjoint, respectively.

P4. Let $g \in \mathrm{Isom}(\mathbb{H})$ and let $d = \inf_z \rho(z, gz)$. Show that g is hyperbolic if and only if $d > 0$. Show that g is elliptic if there exists $z \in \mathbb{H}$ such that $\rho(z, gz) = 0$ and g is parabolic if parabolic $d = 0$ but for all $z \in \mathbb{H}$ we have $\rho(z, gz) > 0$.

P5. Find all discrete subgroups of $GL(2, \mathbb{C})$ which contain only diagonal matrices.

P6. Prove that a discrete subgroup of $GL(2, \mathbb{C})$ is countable.

P7. Assume that a subgroup G of $GL(2, \mathbb{C})$ contains a discrete subgroup of finite index. Show that G is also discrete.

P8. Let $f, g \in \mathrm{Isom}(\hat{\mathbb{C}})$ Assume that g is loxodromic and suppose that g and f have exactly one fixed point in common. Show that the group $\langle f, g\rangle$ generated by f and g is not discrete.
Hint: consider the points $z, g^n z, f g^n z, g^{-n} f g^n z$ for large n.

P9. Let $n, k, m \in \mathbb{R}$ be such that $\frac{1}{n} + \frac{1}{k} + \frac{1}{m} < 1$. Show that the group generated by reflections with respect to the sides of the hyperbolic triangle with angles $\frac{\pi}{n}, \frac{\pi}{k}, \frac{\pi}{m}$ is discrete if and only if $n, k, m \in \mathbb{N}$.

P10. Let $P \subset \mathbb{D}$ be the regular hyperbolic octagon with angles $\frac{\pi}{4}$. Let g_1, g_2, g_3, g_4 be the hyperbolic transformations identifying the opposite sides of P. Show that the group $\langle g_1, g_2, g_3, g_4 \rangle$ generated by g_1, g_2, g_3, g_4 is discrete.

P11. Let $G \subset \mathrm{Isom}(\mathbb{H})$ be a subgroup and assume that G is acting properly discontinuously on $\mathbb{H}$. Let $g \in G$ and let $z \in \mathbb{H}$ be the fixed point of g. Show that there exists a neighbourhood U of z such that $U \setminus \{z\}$ doesn't contain any fixed points of elements of G.

P12. Let $\gamma_1, \gamma_2, \gamma_3$ be the geodesics in $\mathbb{H}$. Assume that γ_1 is represented by a half circle with end points $\{\pm 1\} \in \mathbb{R}$, γ_2 corresponds to the half circle with end points $\{0, 2\} \in \mathbb{R}$, and γ_3 has end points $\left\{\frac{1}{2}, \infty\right\}$. Let $B(z) = \frac{2(z+1)}{3}$. Show that the group $\langle I_{\gamma_1}, I_{\gamma_2}, I_{\gamma_3}, B \rangle$ is not discrete.