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MA222
Example Sheet 8
Path-Connected and Complete Spaces

Hand in solutions to the Problems P3, P8 and P10. Deadline: 2pm, Thursday 14th of March. We consider the space $\mathbb{R}^n$ with the Euclidean topology, unless stated otherwise.

Problems P11–P14 are for *independent practice*.

P1. Let $\mathbb{N}$ be equipped with the metric

$$d(m,n) = \begin{cases} 0, & \text{if } m = n, \\ 1 - \frac{1}{100(m+n)}, & \text{if } m \neq n \end{cases}$$

Show that $(\mathbb{N}, d)$ is a complete metric space. Define $S_n = \{m \in \mathbb{N} \mid m > n\}$. Prove that $\{S_n\}_{n=1}^{\infty}$ is a nested sequence of closed sets whose intersection is empty.

P2. Let (X,d) be a metric space, and $d(A,B) > 0$ for any pair of disjoint closed sets $A, B \subset X$. Prove that (X,d) is complete.

$\boxed{\textbf{P3.}}$ Show that the open interval $(-1,1) \subset \mathbb{R}$ is not complete with respect to the topology induced by Euclidean metric. Show that the metric $d_{\text{hyp}}(a,b) = \left|\int_a^b \frac{dt}{1-t^2}\right|$ induces the same topology on $(-1,1)$, and that $(-1,1)$ is complete with respect to d_{hyp}.

P4. Consider topological space $(\mathbb{R}, \mathcal{T})$ with the basis of intervals $[a,b)$. Establish the following facts.

1. The topology $\mathcal{T}$ is Hausdorff
2. The connected components of $(\mathbb{R}, \mathcal{T})$ are the one point sets $\{x\}$.
3. The intervals $[a,b]$ with $a < b$ are not compact in $\mathcal{T}$.
4. The space $\mathbb{R}^2$ with the product topology has a countable dense subset.
5. The subspace topology on $\{(x,-x) \mid x \in \mathbb{R}\}$ induced by the product topology is discrete.

P5. Decide which of the following metric spaces are path-connected and which are complete. Consider the same question for a closed unit ball in each of the spaces.

$$(1)\ (\ell_\infty(\mathbb{C}), \|\cdot\|_\infty) \qquad (2)\ (\ell_1(\mathbb{C}), \|\cdot\|_\infty) \qquad (3)\ (\ell_2(\mathbb{C}), \|\cdot\|_\infty)$$
$$(4)\ (\ell_1(\mathbb{C}), \|\cdot\|_1) \qquad (5)\ (\ell_1(\mathbb{C}), \|\cdot\|_2) \qquad (6)\ (\ell_2(\mathbb{C}), \|\cdot\|_2)$$

P7. Let $C \subset \ell_\infty(\mathbb{C})$ be the subset of convergent subsequences. Is it true that C complete?

P6. Show that the space of continuous functions $C([a, b])$ is complete with respect to the metric induced by the norm $\|\cdot\|_\infty$ but not complete with respect to the topology induced by $\|\cdot\|_1$ or $\|\cdot\|_2$.

$\boxed{\textbf{P8.}}$ Consider the space of sequences $X = \{0, 1\}^{\mathbb{N}}$ with the Hausdorff distance. Establish the following facts.

1. X is a complete compact metric space.
2. The single point sets are not open in X.
3. Connected components of X are the single point sets.
4. $X \times X$ with the product topology is homeomorphic to X.

P9. Let X be a vector space equipped with a norm $\|\cdot\|$. Show that X is complete with respect to the metric induced by the norm $\|\cdot\|$ if and only if for any sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $\sum_{j=1}^{\infty} \|x_j\|$ is finite we have that there exist $\tilde{x} = \sum_{j=1}^{\infty} x_n \in X$.

$\boxed{\textbf{P10.}}$ Show that the set of polynomials isn't complete in $C([a, b])$ with respect to $\|\cdot\|_\infty$. Is it path-connected?

☞ **P11.** Prove that a metric space is complete if and only if the intersection of every nested sequence of non-empty closed balls with radii tending to zero is non-empty.

☞ **P12.** Find the connected components of a closed unit ball in the following spaces. (Hint: first check if any of the balls is path-connected).

1. The spaces of p-adic rationals and p-adic integers $\mathbb{Z}_p$ with respect to p-adic norm.
2. The space of subsets of a finite set with the Hamming distance.

☞ **P13.** Establish the following facts.

1. If $(X, \mathcal{T}_X)$ is a path-connected topological space and $\sim$ is an equivalence relation on X, then $X/\!\sim$ with the quotient topology is path-connected.
2. If $A \subset (X, \mathcal{T}_X) \times (Y, \mathcal{T}_Y)$ with the product topology is path-connected, then the projection $\pi_X(A) \subset X$ is connected.
3. If $(X, \mathcal{T}_X)$ is a path-connected topological space and $E \subset X$, then E with the subspace topology may not be path-connected.

☞ **P14.** Find the connected components of $\{0\} \cup \left\{\frac{1}{n^2} \mid n \in \mathbb{Z} \setminus \{0\}\right\}$. Which are open in the subspace topology and which are not? Consider the same question with the finite complement topology.