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MA222
Example Sheet 5
Normality and Compactness of Topological Spaces

Hand in solutions to the Problems P7, P10 and P11. Deadline: 2pm, Thursday 21st of February. We consider the space $\mathbb{R}^n$ with Euclidean topology, unless stated otherwise.

Problems P12–P14 are for *independent practice*.

P1. Decide whether the following subspaces of $\mathbb{R}$ or $\mathbb{R}^2$ are compact or not:

(1) $[0,1) \subset \mathbb{R}$, (2) $\mathbb{N} \subset \mathbb{Q} \subset \mathbb{R}$, (3) $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$, (4) $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<1\}$, (5) $\{(x,y) \in \mathbb{R}^2 \mid x^3+y^3=1\}$, (6) $\{(x,y) \in \mathbb{R}^2 \mid x \geq 1, 0 \leq y \leq 1/x\}$, (7) $(-\infty, 0] \subset \mathbb{R}$.

P2. Establish the following facts.

1. A finite set in any topological space is compact.
2. The discrete topology on a set X is compact if and only if X is finite.
3. Let X be uncountable. Consider a collection of subsets

$$\mathcal{T}_X = \{A \subset X \mid X \setminus A \text{ is countable }\} \cup \{\varnothing\}.$$

Show that $\mathcal{T}_X$ is a topology and that $(X, \mathcal{T}_X)$ is not compact.

P3. Show that any compact metric space has a countable dense subset.

P4. Show that every injective continuous map of $[0,1]$ to $\mathbb{R}^2$ is a homeomorphism of $[0,1]$ onto the image of $[0,1]$. Does the statement hold true for the open interval $(0,1)$?

P5. Let $\mathcal{U}$ be an open cover of a metric space (M,d). Consider a function

$$r(x) = \sup_{0<r\leq 1} \{r \mid B(x,r) \subset U \text{ for some } U \in \mathcal{U}\}$$

Is it continuous?

P6. Give an example of a non-Hausdorff topological space X and a sequence of non-empty compact sets $F_1 \supset F_2 \ldots \supset F_n \ldots$ such that $\bigcap_{j=1}^{\infty} F_j = \varnothing$. Hint: Consider $X = [0, +\infty)$ and a collection of infinite intervals $\mathcal{T}_X = \{(a, +\infty) \mid a \geq 0\} \cup \{\varnothing, [0, +\infty)\}$ as a topology.

P7. Let X be a compact Hausdorff space and let $f\colon X \to X$ be continuous. Show that there exists a non-empty subset $A \subset X$ such that $f(A) = A$. (Hint: Consider the sets $A_1 = X$, $A_n = f(A_{n-1})$ and $A = \bigcap_{n=1}^{\infty} A_n$.)

P8. Establish the following facts.

1. The set $\{(x, y) \in \mathbb{C}^2 \mid y = \alpha x + \beta\}$ is a nowhere dense set for any $\alpha, \beta \in \mathbb{C}$.
2. The set $\{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$ is meagre in $\mathbb{R}^2$ and is not homeomorphic to $\mathbb{R}$.

P9. Show that a metric space (M, d) is compact if and only if every continuous function $f\colon M \to \mathbb{R}$ is bounded. More precisely, establish the following.

1. If (M, d) is compact and $f\colon M \to \mathbb{R}$ is continuous, then for any sequence $\{x_n\} \subset M$ the sequence $\{f(x_n)\} \subset \mathbb{R}$ has a convergent subsequence. Deduce that $f(M)$ is bounded.
2. If (M, d) is not compact, then there exists an infinite set $\{x_n\} \subset M$ such that for some $\varepsilon_n > 0$ the closed balls $B(x_n, \varepsilon_n)$ are pairwise disjoint. Construct a non-bounded continuous function $f\colon M \to \mathbb{R}$ using the Tietze extension theorem.

Does the statement hold true for topological spaces?

P10. Let $C \subset \ell_\infty(\mathbb{C})$ be the subspace of convergent sequences. Show that the map

$$f\colon C \to \mathbb{C} \qquad f(\{x_n\}_{n=1}^\infty) = \lim_{n\to\infty} x_n$$

has a continuous extenstion to $\ell_\infty(\mathbb{C})$.

P11. Show that if $f\colon (X, \mathcal{T}_X) \to [0, 1]$ is a continuous function separating two points $x \neq y$ of a topological space X, i.e. $f(x) = 0$ and $f(y) = 1$, then $\overline{f^{-1}([0, \frac{1}{4}))} \cap \overline{f^{-1}((\frac{3}{4}, 1])} = \varnothing$.

☞ **P12.** Let (M, d) be a compact metric space and suppose that $f\colon M \to M$ is a continuous map such that $f(x) \neq x$ for any $x \in M$. Show that there exists $a > 0$ such that $d(f(x), x) > a$ for all $x \in M$.

☞ **P13.** Let X be a compact Hausdorff space, $A \subset X$ closed and $x \notin A$. Show that there is a compact set B with $x \in Int(B)$ such that $A \cap B = \varnothing$.

☞ **P14.** Let $I_n = (n, n+1)$ for all $n \in \mathbb{Z}$. Consider $X = \mathbb{R} \cup \{p_0, p_1\}$, where $p_0, p_1 \notin \mathbb{R}$ and $p_1 \neq p_0$. Define a collection of sets

$$\begin{aligned}\mathcal{T}_X = \{&U \subset X \mid U \subset \mathbb{R} \text{ is open }\}\\ &\cup \{U \subset X \mid U \cap \mathbb{R} \text{ is open, } p_0 \in U, \text{ and } I_n \subseteq U \text{ for all but finitely many } n \geq 0\}\\ &\quad \cup \{U \subset X \mid U \cap \mathbb{R} \text{ is open, } p_1 \in U, \text{ and } I_n \subseteq U \text{ for all but finitely many } n \leq 0\}\end{aligned}$$

Show that $(X, \mathcal{T}_X)$ is Hausdorff but not normal.