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MA222
Example Sheet 3
Getting to know Topological Spaces

Hand in solutions to the Problems P9 and P10. Deadline: 2pm, Thursday 7th of February. Given a set $A \subset X$ denote by $\overline{A}$ its closure, $\overline{A}^{\mathrm{o}}$ stands for the interior of the closure, and by $\partial_X A$ we understand the boundary of A in X.

Problems P11–P14 are for *independent practice*.

P1. Let (X, d) be a metric space with the discrete metric. Show that the induced topology consists of all the subsets of X.

P2. Show that for any set X we have that $\mathcal{T} = \{\varnothing, X\}$ is a topology. We call $\{\varnothing, X\}$ *the indiscrete* topology on X.

P3. Show that a pair of metrics generating the same topology are not necessarily equivalent. In other words, give an example of a pair of metrics d_1 and d_2 on $\mathbb{R}$ generating the same topology such that

$$\forall K > 0 \in \mathbb{R}\ \exists x, y \in \mathbb{R} : K d_1(x, y) \geq d_2(x, y) \text{ or } d_2(x, y) \geq K^{-1} d_1(x, y)$$

P4. Describe all open sets in the topology on $\mathbb{R}$ whose basis consists of intervals $(a, +\infty)$ where $a \in \mathbb{R}$.

P5. Decide whether the following statements hold for every subset A of every topological space X: (a) $\overline{A}^{\mathrm{o}} = A^{\mathrm{o}}$. (b) $\overline{\overline{A}^{\mathrm{o}}}^{\mathrm{o}} = \overline{A}^{\mathrm{o}}$. (c) $\overline{A} = A \cup \partial A$. (d) $A^{\mathrm{o}} = A \setminus \partial A$. (e) $\partial A = A \setminus A^{\mathrm{o}}$. (f) $\partial A = \overline{A} \setminus A^{\mathrm{o}}$.

P6. In $\mathbb{R}^2$ with the Euclidean topology find the interior, the closure and the boundary of

$$(1)\, \mathbb{R} \times \mathbb{Q}; \quad (2)\, \mathbb{Q} \times \mathbb{Q}; \quad (3)\, \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : x^2 + y^2 < 1\}; \quad (4)\, \mathbb{R}^2 \setminus \mathbb{Z}^2.$$

Also decide whether these sets are open, closed or neither, and consider the same questions about the first three sets in the space $\mathbb{R} \times \mathbb{Q}$.

P7. If S is a subspace of a topological space X and $H \subset S$, it is true that $\partial_S H = S \cap \partial_X H$? Also state and answer analogous questions for closure and interior.

P8. Let f be a continuous real-valued function on a topological space X. Is it necessarily true that $\partial\{x \in X : f(x) > 0\} = \{x \in X : f(x) = 0\}$?

$\boxed{\textbf{P9.}}$ Consider $\mathbb{R}$ with Zariski topology. Find the closure, the interior, and the boundary of a finite open interval $(a, b) \subset \mathbb{R}$.

$\boxed{\textbf{P10.}}$ Let $\mathcal{T}_\ell$, $\mathcal{T}_r$ be topologies on $\mathbb{R}$ with bases consisting of all half-open intervals $[a, b)$ and $(a, b]$, respectively.

1. Find the closure of $(0, 1)$ in $\mathcal{T}_\ell$ and in $\mathcal{T}_r$.

2. Among the three topologies $\mathcal{T}_\ell$, $\mathcal{T}_r$ and Euclidean, can you find the coarsest one and/or the finest one? (If $\mathcal{T} \subset \mathcal{T}'$, then $\mathcal{T}$ is *coarser* than $\mathcal{T}'$ and $\mathcal{T}'$ is *finer* than $\mathcal{T}$.)

3. Show that the subspace $\{(x, x) : x \in \mathbb{R}\}$ of $(\mathbb{R}, \mathcal{T}_\ell) \times (\mathbb{R}, \mathcal{T}_r)$ has the discrete topology; i.e. show that all the sets are open.

☞ **P11.** Let F be a finite set with more than one point. Show that

1. Given a metric d on F the induced topology is the discrete topology.

2. The indiscrete topology on F is not induced by any metric.

☞ **P12.** Establish the following facts.

1. There is no largest closed set contained in the unit interval $I = (0, 1)$ with the standard topology. In other words, show that for any closed subset $F \subsetneq I$ there is a closed $G \subset I$ such that $F \subsetneq G \subset I$.

2. Show that the smallest (by inclusion) open set containing a set $E \subset \mathbb{R}$ may not exist.

☞ **P13.** Show that a pair of norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on a vector space $\mathbb{V}$ generate the same topology if and only if

$$\exists K \in \mathbb{R}\ \forall x, y \in \mathbb{V} : K\|x - y\|_a \leq \|x - y\|_b \leq K^{-1}\|x - y\|_a.$$

☞ **P14.** Show that the set of prime numbers is infinite following H. Furstenberg[1].

Consider arithmetic progressions $S(a, b) = \{an + b \colon n \in \mathbb{Z}\}$ with $a \in \mathbb{Z} \setminus \{0\}$, $b \in \mathbb{Z}$. We say that $U \subset \mathbb{Z}$ is open if and only if $\forall x \in U\ \exists a \in \mathbb{Z} \setminus \{0\}$ such that $S(a, x) \subset U$. Establish the following facts.

1. Open sets together with the empty set satisfy the axioms of topology and thus define a topology τ.

2. There is no finite non-empty sets in τ.

3. Every basis set $S(a, b)$ is open and closed.

[1] There are many proofs that there exist an infinity of primes. This is a remarkable one published by Furstenberg in 1955 when he was still an undergraduate.

4.
$$\mathbb{Z} \setminus \bigcup_{p \text{ is a prime}} S(p, 0) = \{-1, 1\}.$$

Deduce that there are infinitely many prime numbers.