Polina Vytnova

MA222

Example Sheet 2
Convergence and Continuity

Hand in solutions to the Problems P10 and P11. Deadline: 2pm, Thursday 31st of January. Let (M_1, d_1) and (M_2, d_2) be a pair of metric spaces. Let $\|\cdot\|$ stand for the Euclidean norm.

P1. Show that $U \subset M_1$ is open if and only if for every sequence $\{x_k\} \subset M_1$ such that $x_k \to x \in U$ there exists $K \in \mathbb{N}$ so that $x_k \in U$ for all $k \geq K$.

P2. Show that a function $f\colon (M_1, d_1) \to (M_2, d_2)$ is continuous if and only if preimage of any closed set is a closed set.

P3. Show that for a continuous function $f\colon (M_1, d_1) \to (M_2, d_2)$ the image of any closed set is closed.

P4. Describe all continuous functions

$$\begin{array}{ll} (1)\ (\mathbb{R}, \|\cdot\|) \to (\mathbb{Z}, \|\cdot\|); & (3)\ ((0,1), \|\cdot\|) \to ((0,1),\ \text{discrete}); \\ (2)\ (\mathbb{Z}, \|\cdot\|) \to (\mathbb{R}, \|\cdot\|); & (4)\ ((0,1),\ \text{discrete}) \to ((0,1), \|\cdot\|). \end{array}$$

P5. Consider a function $f : (M_1, d_1) \to (M_2, d_2)$. Which of the following of the following two implications valid?

1. If f is continuous, then $f(U) \subset M_2$ is open for every open $U \subset M_1$;
2. If $f(U) \subset M_2$ is open for every open $U \subset M_1$, then f is continuous.

P6. Give an example of a continuous function $f\colon (\mathbb{R}, \|\cdot\|) \to (\mathbb{R}, \|\cdot\|)$ which is not Lipschitz continuous.

P7. Show that the product of two Lipschitz continuous (real-valued) functions on a metric space need not be Lipschitz continuous.

P8. Let α be a homeomorphism $\alpha\colon (M_1, d_1) \to (M_2, d_2)$. Assume that following properties hold true in M_1.

1. every continuous function $f\colon M_1 \to \mathbb{R}$ is bounded;
2. $(\forall x \in M_1)(\exists y \in M_1)\colon d_1(x, y) > 0$;
3. $(\forall x \in M_1)(\exists y \in M_1)\colon d_1(x, y) > 1$.

Decide which of them hold true for M_2.

P9. Consider a sequence $\{x_k\}_{k=0}^{\infty} \subset \mathbb{R}^n$

1. Assume that $x_k \to 0$ with respect to ℓ_1 norm. Is it true that $x_k \to 0$ with respect to (a) ℓ_2 norm? (b) ℓ_∞ norm?
2. Assume that $x_k \to 0$ with respect to ℓ_2 norm. Is it true that $x_k \to 0$ with respect to (a) ℓ_1 norm? (b) ℓ_∞ norm?

P10. Let $J_{\alpha,\beta}$ be the identity map on the space of continuous functions

$$J_{\alpha,\beta}\colon\ (C([0,1]), \|\cdot\|_\alpha) \to (C([0,1]), \|\cdot\|_\beta)\,; \qquad J_{\alpha,\beta}\colon f \mapsto f.$$

1. Show that $J_{\infty,1}$ and $J_{\infty,2}$ are continuous but $J_{1,\infty}$ and $J_{2,\infty}$ are not.
2. Using the Cauchy—Schwarz inequality $|\langle f,g\rangle| \le \|f\|_2 \cdot \|g\|_2$ with $g=1$, or otherwise, show that $J_{2,1}$ is continuous. Show that $J_{1,2}$ is not. [Hint: Consider functions of the form $f_{R,K}(x) = K\max\{0, 1-Rx\}$.]

P11. Establish the following facts.

1. With respect to the discrete metric d, all subsets are both open and closed.
2. Let V be a vector space over $\mathbb{R}$ and let $|\cdot|$ be a norm. Define a metric $d(x,y) = |x-y|$. Then the one point sets $\{x\}$ are not open with respect to d.
3. The discrete metric on the vector space V cannot be derived from a norm on V.

P12. Show that every sequence in $(\mathbb{Z}, p\text{-adic metric})$ has a convergent subsequence.

P13*. This is a variation of **P10**. Let $\xi^n \in \ell_1(\mathbb{R})$ be a sequence.

1. Show that $\xi^n \in \ell_2(\mathbb{R})$.
2. Assume that ξ^n converges to the zero sequence with respect to ℓ_1 norm:
$$\|\xi_k^n\|_1 = \sum_{k=1}^{\infty} |\xi_k^n| \to 0 \text{ as } n \to \infty.$$
Is it true that ξ^n converges with respect to ℓ_2 norm?
3. Assume that ξ^n converges to the zero sequence with respect to ℓ_2 norm.
$$\|\xi_k^n\|_2 = \sum_{k=1}^{\infty} |\xi_k^n|^2 \to 0 \text{ as } n \to \infty.$$
Is it true that ξ^n converges with respect to ℓ_1 norm?