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Spherical Orbits and Abelian Ideals

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0. INTRODUCTION

Let G be a connected reductive complex algebraic group with Lie algebra Lie $G = \mathfrak{g}$. Let B be a Borel subgroup of G with unipotent radical B_u. We denote the Lie algebra of B and B_u by $\mathfrak{b}$ and $\mathfrak{b}_u$, respectively. The group B acts on any ideal of $\mathfrak{b}$ by means of the adjoint representation.

After a preliminary section we study a relationship between spherical nilpotent orbits and abelian ideals $\mathfrak{a}$ of $\mathfrak{b}$, using the structure theory for these orbits from [10]. The principal result of this section is that, for an abelian ideal $\mathfrak{a}$ of $\mathfrak{b}$, any nilpotent orbit meeting $\mathfrak{a}$ is a spherical G-variety, see Theorem 2.3. As a consequence of this we obtain a short conceptual proof of a finiteness theorem from [14]. Namely, for a parabolic subgroup P of G and an abelian ideal $\mathfrak{a}$ of $\mathfrak{p} = \mathrm{Lie}\, P$ in the nilpotent radical $\mathfrak{p}_u = \mathrm{Lie}\, P_u$, the group P operates on $\mathfrak{a}$ with finitely many orbits. The proof of this fact in [14] involved long and tedious case by case considerations. We also prove a partial converse to the result just mentioned. Following [4], we say that an ideal of $\mathfrak{b}$ is ad–*nilpotent* whenever it consists of nilpotent elements. In case G is simply laced, we show that an ad–nilpotent ideal $\mathfrak{c}$ of $\mathfrak{b}$ is abelian provided any nilpotent orbit meeting $\mathfrak{c}$ is spherical, see Proposition 2.7.

In Section 3 we consider some properties of ad–nilpotent ideals of $\mathfrak{b}$. In Theorem 3.2 we give a description of the normaliser of such ideals. This applies in particular to abelian ideals of $\mathfrak{b}$. A remarkable theorem of

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D. Peterson asserts that the number of abelian ideals of $\mathfrak{b}$ equals 2^r, where $r = \operatorname{rank} \mathfrak{g}$, see [8] or [4, Theorem 2.9]. We present an elementary proof of this fact in case $\mathfrak{g}$ is of type A_r or C_r. We also prove that the mapping $\mathfrak{a} \mapsto N_G(\mathfrak{a})$ is a one-to-one correspondence between the set of abelian ideals $\mathfrak{a}$ of $\mathfrak{b}$ and the set of standard parabolic subgroups of G for these two series of simple Lie algebras; this fails in all other instances, see Remark 3.4.

In Section 4 we study the set of maximal abelian ideals $\mathscr{A}_{max}$ of $\mathfrak{b}$. After recalling the classification of $\mathscr{A}_{max}$ from [14] we prove the existence of a canonical bijection between $\mathscr{A}_{max}$ and the set of long simple roots of $\mathfrak{g}$ in Theorem 4.3 and discuss some properties of this map. For instance, if σ is a long simple root and $\mathfrak{a}_\sigma \in \mathscr{A}_{max}$ is the corresponding maximal ideal, then the minimal number of generators of $\mathfrak{a}_\sigma$ viewed as a $\mathfrak{b}$-module is equal to the number of connected components of $\Delta \backslash \{\sigma\}$, where Δ denotes the Dynkin diagram of $\mathfrak{g}$.

1. PRELIMINARIES

1.1. We denote the Lie algebra of G by Lie G or $\mathfrak{g}$; likewise for subgroups of G. Let T be a fixed maximal torus in G and $\Psi = \Psi(G)$ the set of roots of G with respect to T and let $r = \dim T = \operatorname{rank} G$. Fix a Borel subgroup B of G containing T and let $\Pi = \{\sigma_1, \sigma_2, \ldots\}$ be the set of simple roots of Ψ defined by B such that the positive integral span of Π in Ψ is $\Psi^+ = \Psi(B)$. The highest (long) root in Ψ is denoted by $\varrho = \sum n_\sigma \sigma$ where the sum is taken over the simple roots Π. If all roots in Ψ are of the same length, they are all called *long*. A subset of Ψ^+ is an *ideal* in Ψ^+ provided it is closed under addition by elements from Ψ^+. As usual, we have the root space decomposition of $\mathfrak{g}$ relative to T,

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha .$$

For a T-stable subspace $\mathfrak{h}$ of $\mathfrak{g}$ we denote its set of roots with respect to T by $\Psi(\mathfrak{h})$. We may assume that each parabolic subgroup P of G considered contains B, i.e. is standard. For each $\alpha \in \Psi$, we choose a nonzero root vector e_α in $\mathfrak{g}_\alpha$.

Our basic reference concerning results on root system is [2]. Throughout, we use the labelling of the Dynkin diagram of G (i.e. of Π) as in [2]. We refer to [1] and [18] for terminology and standard results on algebraic groups.

1.2. Let $\mathfrak{g} = \bigoplus \mathfrak{g}(i)$ be any $\mathbb{Z}$-grading of $\mathfrak{g}$. The largest integer n so that $\mathfrak{g}(n) \neq \{0\}$ is called the *height* of the grading. In this context write $\Psi(i)$

instead of $\Psi(\mathfrak{g}(i))$ for each $i \in \mathbb{Z}$. It is well-known that $\mathfrak{g}(0)$ is reductive, for instance, see [18]. By $W(0)$ we denote the Weyl group of $\mathfrak{g}(0)$.

A grading is said to be *standard* if $\bigoplus_{i>0} \mathfrak{g}(i)$ is contained in $\mathfrak{b}_u$. Any choice of a standard parabolic subgroup P of G canonically defines a standard $\mathbb{Z}$-grading of $\mathfrak{g}$ as follows. Let $P = LP_u$ be the Levi decomposition of P with standard Levi subgroup L. Let $\Pi(L)$ be the set of simple roots of L. Define the function $d: \Psi \to \mathbb{Z}$ by setting $d(\sigma) := 0$ if σ is in $\Pi(L)$ and $d(\sigma) := 1$ if σ is in $\Pi \backslash \Pi(L)$, and extend d linearly to all of Ψ. Then for $i \neq 0$ we define $\mathfrak{g}(i) := \bigoplus_{d(\alpha)=i} \mathfrak{g}_\alpha$ and $\mathfrak{g}(0) := \mathfrak{t} \oplus \bigoplus_{d(\alpha)=0} \mathfrak{g}_\alpha$. Thus we have $\mathfrak{g} = \bigoplus_i \mathfrak{g}(i)$ and moreover, $\mathfrak{l} = \mathfrak{g}(0)$, $\mathfrak{p} = \bigoplus_{i \geqslant 0} \mathfrak{g}(i)$, and $\mathfrak{p}_u = \bigoplus_{i>0} \mathfrak{g}(i)$. Clearly, $d(\varrho) = \sum_{\sigma \in \Pi(L)} n_\sigma$ is the height of this grading.

2. ABELIAN IDEALS AND SPHERICAL ORBITS

A nilpotent orbit (conjugacy class) $\mathcal{O}$ in $\mathfrak{g}$ is said to be *spherical* whenever it is a spherical G-variety, that is B acts on it with an open orbit. Thus, by a fundamental theorem, due to M. Brion [3] and E. B. Vinberg [16] independently, B acts on $\mathcal{O}$ with a finite number of orbits. Since $\mathcal{O}$ is quasi-affine, it is spherical if and only if the algebra of polynomial functions $\mathbb{C}[\mathcal{O}]$ is a *multiplicity free* G-module [17].

The following characterisation of spherical nilpotent orbits can be found in [9, Section 3.1] and [10, Theorem 3.2].

THEOREM 2.1. *Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$. The following statements are equivalent:*

(i) *$\mathcal{O}$ is spherical;*

(ii) $(\operatorname{ad} x)^4 = 0$ *for every $x \in \mathcal{O}$;*

(iii) *$\mathcal{O}$ contains a representative of the form $e_{\alpha_1} + \cdots + e_{\alpha_t}$, where $\{\alpha_1, ..., \alpha_t\} \subseteq \Pi$ is a set of mutually orthogonal* simple *roots.*

It is not hard to prove that the number t in Theorem 2.1(iii) does not depend on the choice of a representative for $\mathcal{O}$. Also, the number of long and short roots among the α_i's is an invariant of the orbit. This property means that a minimal Levi subalgebra of $\mathfrak{g}$ meeting $\mathcal{O}$ is the sum of t copies of $\mathfrak{sl}_2$. This subalgebra is unique up to conjugation. If $\{\alpha_1, ..., \alpha_t\}$ consists of s short and l long roots, then we say that $\mathcal{O}$ *is of type* $s\widetilde{A}_1 + lA_1$. This notation is consistent with the one used for denoting nilpotent orbits in the exceptional Lie algebras [5, 6]. We also use this labelling for the classical Lie algebras.

The equivalence between parts (i) and (ii) of Theorem 2.1 is proved in [9, Section 3.1]. There it is shown *a priori* that whenever $(\operatorname{ad} x)^3 = 0$, then

$\mathcal{O}$ is spherical and also when $(\operatorname{ad} x)^4 \neq 0$, then $\mathcal{O}$ is not spherical. Case by case considerations are only required to show that $\mathcal{O}$ is spherical if $(\operatorname{ad} x)^4 = 0$ and $(\operatorname{ad} x)^3 \neq 0$ for every $x \in \mathcal{O}$.

Making use of Theorem 2.1, we set up a direct link between the abelian ideals in $\mathfrak{b}$ and spherical nilpotent orbits. It is easy to show that any abelian ideal $\mathfrak{a} \subset \mathfrak{b}$ contains no semisimple elements, that is $\mathfrak{a} \subset \mathfrak{b}_u$. Therefore, such an $\mathfrak{a}$ is completely determined by the corresponding subset $\Psi(\mathfrak{a})$ of Ψ.

PROPOSITION 2.2. *Let $\mathfrak{a}$ be an abelian ideal of $\mathfrak{b}$ and let $\mu_i \in \Psi(\mathfrak{a})$ for $i = 1, \ldots, 4$. Define the operator $\Upsilon\colon \mathfrak{g} \to \mathfrak{g}$ by $\Upsilon := \prod_{i=1}^{4} \operatorname{ad} e_{\mu_i}$. Then $\Upsilon \equiv 0$.*

Proof. Since $\mathfrak{a}$ is abelian, Υ does not depend on the ordering of the μ_i's.

1. We first show that Υ annihilates the lowest weight space of $\mathfrak{g}$, i.e., $\Upsilon e_{-\varrho} = 0$.

Assume this is not the case. Then $[e_{\mu_i}, e_{-\varrho}] \neq 0$ and hence $(\mu_i, \varrho) > 0$ for each i (since ϱ is long). More precisely, $(\mu_i, \varrho^\vee) = 2$ in case $\mu_i = \varrho$ and otherwise $(\mu_i, \varrho^\vee) = 1$. Since $\Upsilon e_{-\varrho} \in \mathfrak{g}_{-\varrho + \mu_1 + \cdots + \mu_4}$ and $(-\varrho + \mu_1 + \cdots + \mu_4, \varrho^\vee) \leqslant 2$, the only possibility is that $(\mu_i, \varrho^\vee) = 1$ for each $i = 1, \ldots, 4$ and therefore we have $-\varrho + \mu_1 + \cdots + \mu_4 = \varrho$; that is,

$$2\varrho = \mu_1 + \cdots + \mu_4. \tag{1}$$

Observe also that $\operatorname{ad} e_{\mu_i} \operatorname{ad} e_{\mu_j}(e_{-\varrho}) \neq 0$ for $i \neq j$ and, since $\mu_i + \mu_j$ is not a root, we have $\varrho - \mu_i - \mu_j \in \Psi$. It follows from (1) that

$$\sum_{1 \leqslant i < j \leqslant 4} (\varrho - \mu_i - \mu_j) = 6\varrho - 3(\mu_1 + \cdots + \mu_4) = 0.$$

Therefore, the set $\{\varrho - \mu_i - \mu_j\}_{i,j}$ contains a positive root. Without loss, we may suppose that $\varrho - \mu_1 - \mu_2 \in \Psi^+$. Then $\varrho - \mu_1 = (\varrho - \mu_1 - \mu_2) + \mu_2 \in \Psi(\mathfrak{a})$, since $\mathfrak{a}$ is an ideal in $\mathfrak{b}$. Thus both, μ_1 and $\varrho - \mu_1$ are in $\Psi(\mathfrak{a})$ contradicting the fact that $\mathfrak{a}$ is abelian. Consequently, we have $\Upsilon e_{-\varrho} = 0$, as claimed.

2. Here we show that $\Upsilon e_\gamma = 0$ for all remaining $\gamma \in \Psi \cup \{0\}$. (If $\gamma = 0$, then e_γ stands for an arbitrary element in $\mathfrak{t}$.) We argue by induction on the sum of the coefficients of the simple roots of the difference $\gamma - (-\varrho) = \sum_\sigma k_\sigma \sigma$ $(\sigma \in \Pi)$, i.e., on $\sum_\sigma k_\sigma$. The case when this sum is zero is just the one studied in part 1 above. Suppose that $e_\gamma = [e_\sigma, x]$, where $\sigma \in \Pi$ and either $x = e_{\gamma'}$ for some $\gamma' \in \Psi$ (such an equality exists provided $\gamma \neq -\varrho$), or, in the case $\gamma = \sigma$ is simple, we may choose a suitable element $x \in \mathfrak{t}$ that satisfies this relation. By Υ_i we denote the operator corresponding to the

quadruple of roots where μ_i is replaced by $\mu_i+\sigma$. (If $\mu_i+\sigma\notin\Psi$, then $\Upsilon_i\equiv 0$.) One checks that

$$\Upsilon e_\gamma=[e_\sigma,\Upsilon x]+\sum_{i=1}^{4}\Upsilon_i x.$$

By induction assumption for the operators Υ and Υ_i, we have $\Upsilon x=0$ and $\Upsilon_i x=0$. Thus $\Upsilon e_\gamma=0$, as desired. ∎

THEOREM 2.3. *If $\mathfrak{a}$ is an abelian ideal in $\mathfrak{b}$, then any G-orbit meeting $\mathfrak{a}$ is spherical and $G\cdot\mathfrak{a}$ is the closure of a spherical nilpotent orbit.*

Proof. If $x=\sum e_{\mu_i}\in\mathfrak{a}$, then $(\operatorname{ad}x)^4$ is the sum of operators of the form described in Proposition 2.2. Therefore, $(\operatorname{ad}x)^4=0$, and thus $G\cdot x$ is spherical, by Theorem 2.1. Because $G\cdot\mathfrak{a}$ is irreducible and the number of nilpotent orbits is finite, $G\cdot\mathfrak{a}$ is the closure of a single nilpotent orbit. ∎

COROLLARY 2.4. *Let $\mathfrak{a}$ be an abelian ideal in $\mathfrak{b}$. Then B has finitely many orbits in $\mathfrak{a}$.*

Proof. The desired finiteness follows readily from Theorem 2.3 and the finiteness property for spherical varieties. ∎

We obtain [14, Theorem 1.1] as an immediate consequence of Corollary 2.4:

COROLLARY 2.5. *Let P be a parabolic subgroup of G and let $\mathfrak{a}$ be an abelian ideal of $\mathfrak{p}$ in $\mathfrak{p}_u$. Then P acts on $\mathfrak{a}$ with finitely many orbits.*

Proof. Observe that $\mathfrak{a}\subseteq\mathfrak{p}_u\subseteq\mathfrak{b}_u$ is also an ideal of $\mathfrak{b}$. Thus, by Corollary 2.4, B acts on $\mathfrak{a}$ with a finite number of orbits and thus, so does P. ∎

Remarks 2.6. The particular case when $\mathfrak{a}$ is in the centre of $\mathfrak{p}_u$ is well-known. Then the action factors through a Levi subgroup of P. Here the finiteness follows from a result of E. B. Vinberg [15, Section 2] (see also V. G. Kac [7] or R. W. Richardson [12, Section 3]).

Observe that for abelian P-invariant sub-factors in $\mathfrak{p}_u$, the analogous statement of Corollary 2.5 is false in general. Indeed, this fact is the basis for constructing entire families of parabolic subgroups which admit an infinite number of orbits on $\mathfrak{p}_u$, e.g., see [11] and [13]. Examples in this context also show that a parabolic subgroup may have an infinite number of orbits on ideals in $\mathfrak{p}_u$ of nilpotency class two.

Corollary 2.5 was first proved in [14] in a long case by case analysis. More specifically, it was shown in *loc. cit.* that for A a closed normal unipotent subgroup of P the number of P-orbits on A is finite provided A is abelian; the proof in *loc. cit.* is valid in arbitrary characteristic.

EXAMPLE. Abelian ideals of $\mathfrak{b}$ are readily constructed by means of gradings. Let $\mathfrak{g} = \bigoplus \mathfrak{g}(i)$ be a standard $\mathbb{Z}$-grading of $\mathfrak{g}$ of height d. Define $m := [d/2] + 1$ and set $\mathfrak{a} := \bigoplus_{i \geqslant m} \mathfrak{g}(i)$. Then $\mathfrak{a}$ is an abelian ideal of $\mathfrak{b}$. Obviously, m is the least possible value ensuring that $\bigoplus_{i \geqslant m} \mathfrak{g}(i)$ is abelian. Therefore, any nilpotent orbit in $\mathfrak{g}$ meeting $\mathfrak{a}$ is spherical. In the context of gradings this can be derived by a shorter argument than the one used in the proof of Proposition 2.2. For, let x be in $\mathfrak{a}$. As the components of x have degree at least m, we have

$$(\operatorname{ad} x)^4 \, \mathfrak{g}(j) \subseteq \bigoplus_{i \geqslant j+4m} \mathfrak{g}(i) = \{0\}$$

for each $j \in \mathbb{Z}$. Consequently, $(\operatorname{ad} x)^4 \equiv 0$ on all of $\mathfrak{g}$.

We close this section with a partial converse to Theorem 2.3.

PROPOSITION 2.7. *Suppose G is simply laced. Let $\mathfrak{c}$ be an* ad*–nilpotent ideal of $\mathfrak{b}$ such that any nilpotent orbit meeting $\mathfrak{c}$ is spherical. Then $\mathfrak{c}$ is abelian.*

Proof. Suppose $\mathfrak{c}$ is not abelian. Then there exist $\alpha, \beta, \gamma \in \Psi(\mathfrak{c})$ so that $[e_\alpha, e_\beta] = e_\gamma$. By the assumption on G, the roots α and β span a subsystem of Ψ of type A_2. Let H be the corresponding simple subgroup of G of type A_2. Then $x := e_\alpha + e_\beta$ is regular nilpotent in $\mathfrak{h}$. By direct matrix calculation, one obtains $(\operatorname{ad}_{\mathfrak{h}} x)^4 \not\equiv 0$. Consequently, $(\operatorname{ad} x)^4 \not\equiv 0$ on all of $\mathfrak{g}$. It follows from Theorem 2.1 that the corresponding nilpotent orbit in $\mathfrak{g}$ is not spherical, a contradiction. ▌

It is worth noting that Proposition 2.7 is false if G has two root lengths. For instance, let G be of type C_r $(r \geqslant 2)$ and let P be the stabiliser of the 1-dimensional space $\mathfrak{g}_\varrho$. Then P is parabolic and $\mathfrak{p}_u$ is the Heisenberg Lie algebra of dimension $2r-1$, which is not abelian. We have, however, $(\operatorname{ad} x)^4 = 0$ for all $x \in \mathfrak{p}_u$.

3. THE NORMALISER OF AN ABELIAN IDEAL AND PETERSON'S THEOREM

In his recent article [8], B. Kostant gives an account of a remarkable theorem of D. Peterson to the effect that the number of all abelian ideals of $\mathfrak{b}$ is equal to 2^r, see also [4, Theorem 2.9]. In this section we give an elementary proof of this equality in case $\mathfrak{g}$ is of type A_r or C_r. We show that for $\mathfrak{g}$ of type A_r or C_r the mapping $\mathfrak{a} \mapsto N_G(\mathfrak{a})$ establishes a one-to-one correspondence between $\mathscr{A}$, the set of abelian ideals of $\mathfrak{b}$, and the set of standard parabolic subgroups of G; thus in particular, $\# \mathscr{A} = 2^r$.

An ideal of $\mathfrak{b}$ is said to be ad–*nilpotent* whenever it contains no semisimple elements, or, equivalently, it is contained in $\mathfrak{b}_u$. Let $\mathfrak{c}$ be an ad–nilpotent ideal of $\mathfrak{b}$. A root $\gamma \in \Psi(\mathfrak{c})$ is said to be a *generator* of $\Psi(\mathfrak{c})$ (or of $\mathfrak{c}$), if $\gamma - \sigma \notin \Psi(\mathfrak{c})$ for all $\sigma \in \Pi$. The set of generators of $\mathfrak{c}$ is denoted by $\Gamma_{\mathfrak{c}}$. It is easily seen that $\Gamma_{\mathfrak{c}} = \Psi(\mathfrak{c}) \setminus (\Psi(\mathfrak{c}) + \Psi^+)$ and that the root vectors e_γ, $\gamma \in \Gamma_{\mathfrak{c}}$ form a minimal set of generators of the $\mathfrak{b}$-module $\mathfrak{c}$. Write $P_{\mathfrak{c}} := N_G(\mathfrak{c})$ for the normaliser of $\mathfrak{c}$ in G. Since $P_{\mathfrak{c}}$ contains B, it is a standard parabolic subgroup of G. So, in order to specify $P_{\mathfrak{c}}$, one merely has to indicate the simple roots of the standard Levi subgroup of $P_{\mathfrak{c}}$.

LEMMA 3.1. *Let $\beta \in \Psi^+$ and $\sigma, \sigma' \in \Pi$ with $\sigma \neq \sigma'$. Suppose $\beta - \sigma$, $\beta - \sigma' \in \Psi^+$. Then, either $\beta = \sigma + \sigma'$, or $\beta - \sigma - \sigma' \in \Psi^+$.*

Proof. Suppose $\beta \neq \sigma + \sigma'$. It is enough to prove that $(\beta - \sigma, \sigma') > 0$ or $(\beta - \sigma', \sigma) > 0$. Note that $(\sigma, \sigma') \leqslant 0$.

1. Assume that $\{\beta, \sigma, \sigma'\}$ contains a long root. If, say, β or σ is long, then $(\beta, \sigma) > 0$ and thus $(\beta - \sigma', \sigma) > 0$.

2. Assume that Ψ has roots of different lengths and that β, σ, and σ' are short. The presence of two distinct short simple roots already implies that $\mathfrak{g}$ is not of type G_2. Then the ratio of the squares of the different root lengths equals 2 and the hypothesis of the lemma implies that $(\beta, \sigma) \geqslant 0$ and $(\beta, \sigma') \geqslant 0$. Thus, if neither of the desired inequalities is satisfied, we obtain $(\beta, \sigma) = (\beta, \sigma') = (\sigma, \sigma') = 0$. Therefore, $(\beta - \sigma, \beta - \sigma') = (\beta, \beta) > 0$. Whence, $\sigma - \sigma' \in \Psi$, a contradiction. ∎

Next we present a characterisation of the normaliser of an arbitrary ad–nilpotent ideal of $\mathfrak{b}$.

THEOREM 3.2. *Let $\mathfrak{c}$ be an* ad–*nilpotent ideal of* $\mathfrak{b}$ *and $L_{\mathfrak{c}}$ the standard Levi subgroup of $P_{\mathfrak{c}}$. Then a simple root σ belongs to $\Pi(L_{\mathfrak{c}})$ if and only if $\gamma - \sigma$ is not a root for all $\gamma \in \Gamma_{\mathfrak{c}}$.*

Proof. By definition of $\Gamma_{\mathfrak{c}}$, the necessity of the given condition on the differences $\gamma - \sigma$ is obvious. The other implication can be restated in terms of $\Psi(\mathfrak{c})$ as follows:

> *Suppose $\beta \in \Psi(\mathfrak{c}) \setminus \Gamma_{\mathfrak{c}}$ and $\beta - \sigma \in \Psi^+ \setminus \Psi(\mathfrak{c})$. Then there exists a $\gamma \in \Gamma_{\mathfrak{c}}$ such that $\gamma - \sigma \in \Psi^+ \cup \{0\}$.*

Since $\beta \notin \Gamma_{\mathfrak{c}}$, there exists a $\sigma' \in \Pi$ such that $\beta_1 = \beta - \sigma' \in \Psi(\mathfrak{c})$. It follows from Lemma 3.1 that $\beta - \sigma - \sigma' \in \Psi^+ \cup \{0\}$. Clearly, $\beta - \sigma - \sigma' \notin \Psi(\mathfrak{c})$. Hence if $\beta_1 \in \Gamma_{\mathfrak{c}}$, then we are done. If not, we continue inductively with β_1 in place of β. Iterating this procedure, we eventually obtain a generator $\beta_k \in \Gamma_{\mathfrak{c}}$ with the desired property. ∎

THEOREM 3.3. *Let G be of type A_r or C_r. Then the map $\mathfrak{a} \mapsto P_{\mathfrak{a}}$ yields a one-to-one correspondence between the set of abelian ideals of $\mathfrak{b}$ and the set of standard parabolic subgroups of G. In particular, the number of abelian ideals of $\mathfrak{b}$ equals 2^r.*

Proof. (1) $\mathfrak{g} = \mathfrak{sl}_{r+1}$.

We assume that $\mathfrak{b}$ is the set of all upper-triangular $(r+1) \times (r+1)$ matrices. An arbitrary ad–nilpotent ideal in $\mathfrak{b}$ is then represented by a Young diagram above the main diagonal, as shown in Fig. 1. Such a diagram is completely determined by the coordinates of its southwest corners, say (i_1, j_1), ..., (i_k, j_k). Then we obviously have $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.

It is easy to see that the ideal in question is abelian if and only if the diagram fits in a rectangle of size $(r+1-i) \times i$ for some $i \in \{1, ..., r\}$. In terms of the indices of the corners this means that $i_k < j_1$. Consequently, there is a one-to-one correspondence between the abelian ideals of $\mathfrak{b}$ and the subsets of $\{1, 2, ..., r+1\}$ of *even* cardinality. The well-known equality

$$\sum_{k \geqslant 0} \binom{r+1}{2k} = 2^r$$

then proves Peterson's theorem in this instance. [It is easily seen that the number of maximal abelian ideals containing the given one equals $j_1 - i_k$. In Fig. 1 (where $r+1 = 12$, $i_k = 4$, and $j_1 = 6$) the dashed lines represent the two maximal abelian ideals containing the depicted one.]

The roots corresponding to the corners of the diagram are nothing but the generators of the ideal. We use the standard notation and numeration

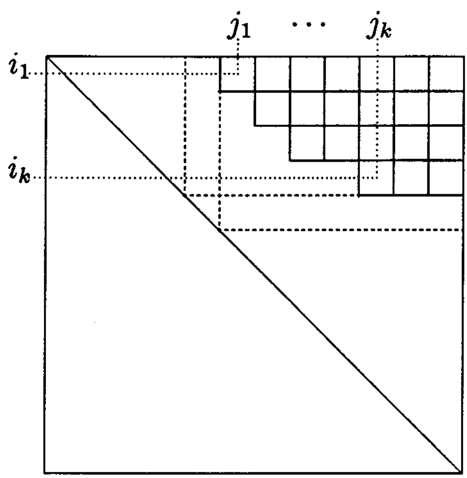

FIG. 1. An abelian ideal in A_r.

for the simple roots of $\mathfrak{sl}_{r+1}$ so that $\sigma_i = \varepsilon_i - \varepsilon_{i+1}$, for $i = 1, ..., r$. The root corresponding to (i, j) is $\sigma_{(i,j)} = \sigma_i + \cdots + \sigma_{j-1}$. Therefore, the only simple roots of $\mathfrak{g}$ that can be subtracted from $\sigma_{(i,j)}$ are σ_i and σ_{j-1}. Thus, it follows from Theorem 3.2 that, given an abelian ideal as above, the simple roots that do not belong to $\Pi(L_\mathfrak{a})$ have the indices $i_1, ..., i_k, j_1 - 1, ..., j_k - 1$. This determines the normaliser of the ideal. Note that precisely when $j_1 = i_k + 1$, an odd number of simple roots is excluded. Clearly, this procedure can be reversed so that we obtain a bijection. Formally, let $m_1 < \cdots < m_d$ be the indices of the simple roots in $\Pi \backslash \Pi(L_\mathfrak{a})$. Then the coordinates of the corners of the respective diagram are $(m_1, m_{[d/2]+1} + 1)$, $(m_2, m_{[d/2]+2} + 1)$, ..., and $(m_{[(d+1)/2]}, m_d + 1)$.

(2) $\mathfrak{g} = \mathfrak{sp}_{2r}$.

Choose a basis for a $2r$-dimensional vector space so that the skew-symmetric non-degenerate bilinear form has the matrix $\left(\begin{smallmatrix} 0 & J \\ -J & 0 \end{smallmatrix}\right)$, where J is the $r \times r$ matrix with 1's along the antidiagonal. Then $\mathfrak{b}$ is the set of all symplectic upper-triangular matrices and the unique maximal abelian ideal $\mathfrak{a}_{max}$ in $\mathfrak{b}$ is represented by the matrices of the form $\left(\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right)$, where M is any $r \times r$ matrix that is symmetric relative to the antidiagonal. Here $\Psi(\mathfrak{a}_{max}) = \{\varepsilon_i + \varepsilon_j \mid 1 \leqslant i, j \leqslant r\}$. It then follows that an arbitrary abelian ideal $\mathfrak{a}$ of $\mathfrak{b}$ is represented by a Young diagram that fits in the square of size r and is symmetric with respect to the antidiagonal, see Fig. 2. Notice that here j increases from right to left. Such a diagram is entirely determined by its corners on and above the antidiagonal. The coordinates (i_1, j_1), ..., (i_k, j_k) of these corners satisfy $i_1 < i_2 < \cdots < i_k \leqslant j_k < j_{k-1} < \cdots < j_1$. Hence a diagram with k corners determines a subset of $\{1, ..., r\}$ of cardinality

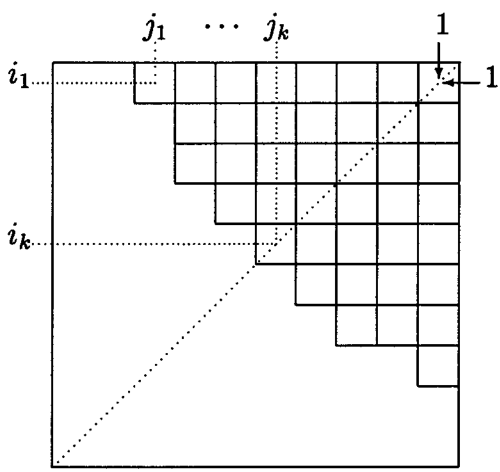

FIG. 2. An abelian ideal in C_r.

either $2k-1$ or $2k$. This establishes a one-to-one correspondence between the abelian ideals of $\mathfrak{b}$ and the subsets of $\{1, ..., r\}$, thereby proving Peterson's theorem in this instance.

The generator of $\Psi(\mathfrak{a})$ corresponding to the corner (i, j) is $\sigma_{(i,j)} := \varepsilon_i + \varepsilon_j$. The only simple roots of $\mathfrak{g}$ that can be subtracted from $\sigma_{(i,j)}$ are σ_i and σ_j. These two simple roots coincide only if $i = j$; this however may happen for at most one corner. Thus, according to Theorem 3.2, the simple roots of $\mathfrak{g}$ that do not belong to $\Pi(L_{\mathfrak{a}})$ have the indices $i_1, ..., i_k, j_k, ..., j_1$. Clearly, the procedure can be reversed, i.e., any subset of $\{1, ..., r\}$ uniquely determines a Young diagram of the required shape. For a subset of cardinality m, the resulting diagram has $[(m+1)/2]$ corners on and above the antidiagonal. ∎

Arguing in the same fashion as in Theorem 3.3, one can enumerate the abelian ideals of $\mathfrak{b}$ for $\mathfrak{g}$ of type B_r or D_r as well. However, we omit the arguments, as they are considerably less transparent.

Remark 3.4. The statement of Theorem 3.3 is not true in case G is not of type A_r or C_r. Here we present some counterexamples.

1. Let G be of type B_r for $r \geqslant 3$. Set $\beta = \sigma_1 + \cdots + \sigma_r$ and $\beta' = \beta + \sigma_r$. Consider the ideals $\mathfrak{a}$ and $\mathfrak{a}'$ of $\mathfrak{b}$ whose sets of generators are $\Gamma_{\mathfrak{a}} = \{\beta\}$ and $\Gamma_{\mathfrak{a}'} = \{\beta'\}$, respectively. Since $\mathrm{ht}(\varrho) = 2r-1$ and $\mathrm{ht}(\beta) = r$, both ideals are abelian. Using Theorem 3.2, it is easily seen that $P_{\mathfrak{a}} = P_{\mathfrak{a}'}$, this is the standard parabolic subgroup of G of semisimple rank $r-2$ whose simple roots have labels $2, ..., r-1$.

2. Let G be of type D_r for $r \geqslant 4$. Set $\beta = \sigma_1 + \cdots + \sigma_{r-1}$, $\gamma = \sigma_1 + \cdots + \sigma_{r-2} + \sigma_r$, and $\delta = \sigma_{r-2} + \sigma_{r-1} + \sigma_r$. Define two abelian ideals $\mathfrak{a}$ and $\mathfrak{a}'$ of $\mathfrak{b}$ with generating sets $\Gamma_{\mathfrak{a}} = \{\beta, \gamma, \delta\}$ and $\Gamma_{\mathfrak{a}'} = \{\beta, \delta\}$, respectively. One checks that $P_{\mathfrak{a}} = P_{\mathfrak{a}'}$, this is the standard parabolic subgroup of G with simple roots $\Pi \backslash \{\sigma_1, \sigma_{r-1}, \sigma_r\}$ (cf. Table I below).

3. Our counterexamples for the exceptional Lie algebras admit a uniform presentation. In each of these cases the highest root ϱ is fundamental, with a unique simple root σ^* such that $\varrho - \sigma^* \in \Psi^+$. Moreover, there exists a unique simple root α adjacent to σ^* in the Dynkin diagram of G. Let $\mathfrak{a}$ be the 2-dimensional abelian ideal $\mathfrak{g}_{\varrho-\sigma^*} \oplus \mathfrak{g}_{\varrho}$. Then $\Gamma_{\mathfrak{a}} = \{\varrho - \sigma^*\}$. Let $\mathfrak{a}'$ be the maximal abelian ideal attached to σ^*, according to the bijection of Theorem 4.3. Then $\Gamma_{\mathfrak{a}'}$ consists of a single root indicated below. Using Theorem 3.2, one finds that $P_{\mathfrak{a}} = P_{\mathfrak{a}'}$; this is the maximal parabolic subgroup of G with $\Pi(L) = \Pi \backslash \{\alpha\}$. The generators of the ideals $\mathfrak{a}$ and $\mathfrak{a}'$ and the simple root α are given as follows:

	E_6	E_7	E_8	F_4	G_2
$\mathfrak{a}$	$\begin{smallmatrix}12321\\1\end{smallmatrix}$	$\begin{smallmatrix}134321\\2\end{smallmatrix}$	$\begin{smallmatrix}2465431\\3\end{smallmatrix}$	1342	31
$\mathfrak{a}'$	$\begin{smallmatrix}01210\\1\end{smallmatrix}$	$\begin{smallmatrix}122100\\1\end{smallmatrix}$	$\begin{smallmatrix}0122221\\1\end{smallmatrix}$	1220	21
α	$\begin{smallmatrix}00100\\0\end{smallmatrix}$	$\begin{smallmatrix}010000\\0\end{smallmatrix}$	$\begin{smallmatrix}0000010\\0\end{smallmatrix}$	0100	10

4. MAXIMAL ABELIAN IDEALS

4.1. Throughout this section suppose that G is simple. We recall the classification of the maximal abelian ideals of $\mathfrak{b}$ from [14] and record it in Tables I and II below.

THEOREM 4.1. *Every maximal abelian ideal of* Lie $B = \mathfrak{b}$ *is listed in Tables* I *and* II.

The fact that each ideal $\mathfrak{a}$ listed in these tables is abelian follows from the observation that the sum of any two roots in $\Psi(\mathfrak{a})$ is not a root, because it exceeds ϱ in some coefficient. The fact that each of these ideals

TABLE I

The Maximal Abelian Ideals $\mathfrak{a}$ of Borel Subalgebras for Classical $\mathfrak{g}$

G	$\Gamma_\mathfrak{a}$	$P_\mathfrak{a}$	$\dim \mathfrak{a}$	$d_\mathfrak{a}$	$\sigma_\mathfrak{a}$	$\mathcal{O}_\mathfrak{a}$
A_r	$\sigma_i\ (1 \leqslant i \leqslant r)$	σ_i	$i(r-i+1)$	1	σ_i	$\min\{i, r-i+1\}\ \mathscr{A}_1$
B_r	σ_1	σ_1	$2r-1$	1	σ_1	$\tilde{A}_1$
	$\beta_i, \gamma_i\ (3 \leqslant i \leqslant r)$	σ_1, σ_i	$(4r+i^2-5i+2)/2$	3	σ_{i-1}	$\tilde{A}_1 + \left[\frac{i-1}{2}\right] A_1$
C_r	σ_r	σ_r	$(r^2+r)/2$	1	σ_r	$\left[\frac{r}{2}\right] \tilde{A}_1 + \left(r-2\left[\frac{r}{2}\right]\right) A_1$
D_r	σ_1	σ_1	$2r-2$	1	σ_1	$2A_1$
	σ_{r-1}/σ_r	σ_{r-1}/σ_r	$(r^2-r)/2$	1	σ_{r-1}/σ_r	$\left[\frac{r}{2}\right] A_1$
	β_i, γ_i $(3 \leqslant i \leqslant r-2)$	σ_1, σ_i	$(4r-5i+i^2)/2$	3	σ_{i-1}	$\left(1+\left[\frac{i+1}{2}\right]\right) A_1$
	β, γ, δ	$\sigma_1, \sigma_{r-1}, \sigma_r$	$(r^2-3r+6)/2$	3	σ_{r-2}	$\left(1+\left[\frac{r}{2}\right]\right) A_1$

TABLE II

The Maximal Abelian Ideals $\mathfrak{a}$ of Borel Subalgebras for Exceptional $\mathfrak{g}$

G	$\Gamma_{\mathfrak{a}}$	$P_{\mathfrak{a}}$	$\dim \mathfrak{a}$	$d_{\mathfrak{a}}$	$\sigma_{\mathfrak{a}}$	$\mathcal{O}_{\mathfrak{a}}$
E_6	σ_1/σ_6	σ_1/σ_6	16	1	σ_1/σ_6	$2A_1$
	$\genfrac{}{}{0pt}{}{01210}{1}$	σ_4	11	3	σ_2	$3A_1$
	$\genfrac{}{}{0pt}{}{11110}{0}, \genfrac{}{}{0pt}{}{01221}{1}$	σ_1, σ_5	13	3	σ_3	$3A_1$
	$\genfrac{}{}{0pt}{}{01111}{0}, \genfrac{}{}{0pt}{}{12210}{1}$	σ_3, σ_6	13	3	σ_5	$3A_1$
	$\genfrac{}{}{0pt}{}{11111}{0}, \genfrac{}{}{0pt}{}{01211}{1}, \genfrac{}{}{0pt}{}{11210}{1}$	$\sigma_1, \sigma_4, \sigma_6$	12	5	σ_4	$3A_1$
E_7	σ_7	σ_7	27	1	σ_7	$[3A_1]''$
	$\genfrac{}{}{0pt}{}{122100}{1}$	σ_3	17	3	σ_1	$[3A_1]'$
	$\genfrac{}{}{0pt}{}{012210}{1}$	σ_5	20	3	σ_2	$[3A_1]'$
	$\genfrac{}{}{0pt}{}{001111}{1}, \genfrac{}{}{0pt}{}{123210}{2}$	σ_2, σ_7	22	3	σ_6	$4A_1$
	$\genfrac{}{}{0pt}{}{012221}{1}, \genfrac{}{}{0pt}{}{122110}{1}$	σ_3, σ_6	18	5	σ_3	$[3A_1]'$
	$\genfrac{}{}{0pt}{}{012111}{1}, \genfrac{}{}{0pt}{}{123210}{1}$	σ_4, σ_7	20	5	σ_5	$4A_1$
	$\genfrac{}{}{0pt}{}{012211}{1}, \genfrac{}{}{0pt}{}{122210}{1}, \genfrac{}{}{0pt}{}{122111}{1}$	$\sigma_3, \sigma_5, \sigma_7$	19	7	σ_4	$4A_1$
E_8	$\genfrac{}{}{0pt}{}{0122221}{1}$	σ_7	29	3	σ_8	$3A_1$
	$\genfrac{}{}{0pt}{}{1232100}{2}$	σ_2	36	3	σ_1	$4A_1$
	$\genfrac{}{}{0pt}{}{1233210}{1}$	σ_5	34	5	σ_2	$4A_1$
	$\genfrac{}{}{0pt}{}{1122221}{1}, \genfrac{}{}{0pt}{}{2343210}{2}$	σ_1, σ_7	30	5	σ_7	$4A_1$
	$\genfrac{}{}{0pt}{}{1222221}{1}, \genfrac{}{}{0pt}{}{1343210}{2}$	σ_3, σ_7	31	7	σ_6	$4A_1$
	$\genfrac{}{}{0pt}{}{1233321}{1}, \genfrac{}{}{0pt}{}{1232210}{2}$	σ_2, σ_6	34	7	σ_3	$4A_1$
	$\genfrac{}{}{0pt}{}{1232221}{1}, \genfrac{}{}{0pt}{}{1243210}{2}$	σ_4, σ_7	32	9	σ_5	$4A_1$
	$\genfrac{}{}{0pt}{}{1233221}{1}, \genfrac{}{}{0pt}{}{1232221}{2}, \genfrac{}{}{0pt}{}{1233210}{2}$	$\sigma_2, \sigma_5, \sigma_7$	33	11	σ_4	$4A_1$
F_4	1220	σ_2	8	3	σ_1	$\tilde{A}_1 + A_1$
	1221, 0122	σ_2, σ_4	9	5	σ_2	$\tilde{A}_1 + A_1$
G_2	21	σ_1	3	3	σ_2	$\tilde{A}_1$

is maximal among the abelian ones and that this list is complete consists of a detailed case by case analysis.

The proof of Theorem 4.1 from [14], involving case by case considerations, is rather unsatisfactory. It would be very desirable to have a uniform proof of this result.

We are going to explain the various pieces of notation in Tables I and II associated to each maximal abelian ideal $\mathfrak{a}$ of $\mathfrak{b}$. In the second column we specify the set of generators $\Gamma_{\mathfrak{a}}$ for $\mathfrak{a}$. The simple roots σ_i are labeled as in [2]. We abbreviate some roots as follows: in type B_r set $\beta_i = \sigma_1 + \cdots + \sigma_i$ and $\gamma_i = \sigma_{i-1} + 2\sigma_i + \cdots + 2\sigma_r$, where $2 \leqslant i \leqslant r$. Similarly, for type D_r we define $\beta_i = \sigma_1 + \cdots + \sigma_i$ and $\gamma_i = \sigma_{i-1} + 2\sigma_i + \cdots + 2\sigma_{r-2} + \sigma_{r-1} + \sigma_r$ for $3 \leqslant i \leqslant r-2$, also $\beta = \beta_{r-2} + \sigma_{r-1}$, $\gamma = \beta_{r-2} + \sigma_r$, and $\delta = \sigma_{r-2} + \sigma_{r-1} + \sigma_r$.

The normalizer of $\mathfrak{a}$ in G is a parabolic subgroup of G, since it contains B. In the third column of the tables we indicate the standard Levi subgroup $L_{\mathfrak{a}}$ of $P_{\mathfrak{a}} := N_G(\mathfrak{a})$ by listing the complementary simple roots $\Pi \backslash \Pi(L_{\mathfrak{a}})$.

In the next two columns we list $\dim \mathfrak{a}$ and $d_{\mathfrak{a}} := d(\varrho)$, the height of the grading afforded by $P_{\mathfrak{a}}$, see (1.2), respectively.

It follows from Theorem 4.1 that the number of maximal abelian ideals of $\mathfrak{b}$ equals the number of long simple roots of G. In Section 4.2 we define a canonical bijection between these two sets. The simple root $\sigma_{\mathfrak{a}}$ corresponding to $\mathfrak{a}$ under this bijection is indicated in column 6 of the tables.

Since $\mathfrak{a}$ is an irreducible subvariety of $\mathfrak{b}_u$, there exists a unique nilpotent orbit $\mathcal{O}_{\mathfrak{a}}$ such that $\mathcal{O}_{\mathfrak{a}} \cap \mathfrak{a}$ is dense in $\mathfrak{a}$. In the last column of Table II we present the label of $\mathcal{O}_{\mathfrak{a}}$ following the labelling of the nilpotent classes according to E. B. Dynkin [6], see also [5].

Using the description of $P_{\mathfrak{a}}$ furnished in the third column in Tables I and II, the height $d_{\mathfrak{a}} = d(\varrho)$ of the grading afforded by $P_{\mathfrak{a}}$ is readily determined. Note that $d_{\mathfrak{a}}$ is always odd and for $m = [d_{\mathfrak{a}}/2] + 1$ we have $\mathfrak{a} = \bigoplus_{i \geqslant m} \mathfrak{g}(i)$. According to Theorem 2.3, the orbit $\mathcal{O}_{\mathfrak{a}}$ is always spherical. If the label of $\mathcal{O}_{\mathfrak{a}}$ is $s\tilde{A}_1 + lA_1$, then the sum $s + l$ is the number t from Theorem 2.1(iii). It is also possible to determine the labelling of the weighted Dynkin diagram defining $\mathcal{O}_{\mathfrak{a}}$.

4.2. By Theorem 4.1, the number of maximal abelian ideals equals the number of long simple roots of $\mathfrak{g}$. This numerical coincidence suggests that there should exist a canonical one-to-one correspondence between these two sets. We show that this correspondence can be obtained in an axiomatic way. It is presented in column 6 of Tables I and II.

Let $\Delta := \Delta(\mathfrak{g})$ be the Dynkin diagram of $\mathfrak{g}$. We identify the nodes of Δ with the simple roots Π of $\mathfrak{g}$ and write Δ^{σ} for the Dynkin diagram which is obtained from Δ by removing $\sigma \in \Pi$ together with the edges linked to it. By $\pi_0(\Delta^{\sigma})$ we denote the set of connected components of Δ^{σ} and by $\Delta^{\sigma} =$

$\bigcup_c \Delta_c^\sigma$ for $c \in \pi_0(\Delta^\sigma)$ the decomposition of Δ^σ into its components. We write Ψ_c^σ for the root system corresponding to Δ_c^σ and ϱ_c^σ for the highest (long) root in Ψ_c^σ for each c. Observe that if we consider the standard grading of $\mathfrak{g}$ corresponding to $\sigma \in \Pi$, then, using the previous notation, we have $\Psi(0) = \bigsqcup_c \Psi_c^\sigma$.

Let Π_ℓ denote the set of long simple roots and $\mathscr{A}_{max}$ the set of all maximal abelian ideals in $\mathfrak{b}$. Associated with any $\mathfrak{a} \in \mathscr{A}_{max}$, we have the following data: the set of generators $\Gamma_\mathfrak{a} \subset \Psi^+$ and the height $d_\mathfrak{a}$ of the grading determined by $P_\mathfrak{a} = N_G(\mathfrak{a})$. The following observation giving a more precise form for the equality $\Pi_\ell = \mathscr{A}_{max}$ is indicative for our construction. Recall the decomposition of ϱ as the sum of simple roots $\varrho = \sum n_\sigma \sigma$ from (1.1). The number of times a fixed integer occurs as the value for $d_\mathfrak{a}$, as $\mathfrak{a}$ varies over $\mathscr{A}_{max}$, equals the number of times it occurs as the expression $2n_\sigma - 1$, as σ runs through Π_ℓ. Therefore, it is just to require that the sought after bijection

$$\psi\colon \Pi_\ell \to \mathscr{A}_{max}, \qquad \sigma \mapsto \psi(\sigma) =: \mathfrak{a}_\sigma,$$

does satisfy the condition $d_{\mathfrak{a}_\sigma} = 2n_\sigma - 1$ for each $\sigma \in \Pi_\ell$.

Ideally, starting with a long simple root, an explicit *a priori* procedure should yield the corresponding maximal abelian ideal. Indeed, we are able to state such a construction when $n_\sigma \leqslant 2$. It is worth noting that this is sufficient to cover all classical instances.

The case when $n_\sigma = 1$ is straightforward. Here the simple root σ (which is always long) determines a grading $\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$, and we merely set $\mathfrak{a}_\sigma = \mathfrak{g}(1)$. It is easily seen that $\mathfrak{g}(1)$ is a maximal abelian ideal. Notice that in this case $\mathfrak{a}_\sigma$ is the nilpotent radical of the parabolic subalgebra corresponding to σ and $\Gamma_{\mathfrak{a}_\sigma} = \{\sigma\}$.

The case $n_\sigma = 2$ is the subject of the following theorem.

THEOREM 4.2. *Let $\sigma \in \Pi_\ell$ such that $n_\sigma = 2$. Let $\mathfrak{g} = \bigoplus_{i=-2}^{2} \mathfrak{g}(i)$ be the corresponding $\mathbb{Z}$-grading. Let e_γ be a highest weight vector in the* $\mathfrak{g}(0)$*-module* $\mathfrak{g}(1)$. *Then we have*

(i) $\mathfrak{a}_\sigma := [e_\gamma, \mathfrak{g}(0)] \oplus \mathfrak{g}(2)$ *is an abelian ideal in* $\mathfrak{b}$;

(ii) $\Gamma_{\mathfrak{a}_\sigma} = \{\gamma - \varrho_c^\sigma \mid c \in \pi_0(\Delta^\sigma)\}$; *in particular,* $\#\Gamma_{\mathfrak{a}_\sigma} = \#\pi_0(\Delta^\sigma)$;

(iii) $\mathfrak{a}_\sigma$ *is maximal and* $d_{\mathfrak{a}_\sigma} = 3$.

Proof. (i) Notice that σ and γ are the lowest and highest weight in the $\mathfrak{g}(0)$-module $\mathfrak{g}(1)$, respectively. It follows that γ is $W(0)$-conjugate to σ and therefore γ is long.

It is easily seen that $\mathfrak{a}_\sigma$ is an ideal of $\mathfrak{b}$ in $\mathfrak{b}_u$. Set $V = [e_\gamma, \mathfrak{g}(0)]$. Clearly, $\mathfrak{a}_\sigma$ is abelian if and only if $[V, V] = \{0\}$; that is, if $\mu_1, \mu_2 \in \Psi(V)$, then $\mu_1 + \mu_2$ is not a root. By the definition of V, we have $\mu_i = \gamma - \beta_i$ for some $\beta_i \in \Psi(0)^+ \cup \{0\}$, $i = 1, 2$. We distinguish various possibilities for β_1 and β_2.

(a) $\beta_1 \neq 0$, $\beta_2 = 0$:

Since γ is long and $\gamma \neq \beta_1$, we have $(\gamma, \gamma) > (\gamma, \beta_1)$. Therefore, $(\gamma, \gamma - \beta_1) > 0$ and hence $\gamma + (\gamma - \beta_1) \notin \Psi$.

(b) $\beta_1 \neq 0$, $\beta_2 \neq 0$:

Since γ is long, the condition $\gamma - \beta_i \in \Psi$ means that $(\gamma, \beta_i) > 0$ and then $(\gamma, \beta_i) = \frac{1}{2}(\gamma, \gamma)$, $i = 1, 2$. Therefore, we have

$$(*) \quad (\gamma - \beta_1, \beta_2) = \tfrac{1}{2}(\gamma, \gamma) - (\beta_1, \beta_2) \geqslant 0, \text{ since } \beta_1 \neq \beta_2, \text{ and}$$

$$(**) \quad (\gamma - \beta_1, \gamma - \beta_2) = (\beta_1, \beta_2).$$

(b_1) At least one of β_1 and β_2, say β_2, is long.

Then $\gamma - \beta_2$ is long as well. Since $\gamma - \beta_1 \in \Psi(1)$ and $\beta_2 \in \Psi(0)^+$, we have $\gamma - \beta_1 \neq \beta_2$ and hence $(\gamma - \beta_1, \beta_2) < (\beta_2, \beta_2) = (\gamma, \gamma)$. It then follows from the equality in $(*)$ that $(\beta_1, \beta_2) > -\frac{1}{2}(\gamma, \gamma)$ and, consequently, $(\beta_1, \beta_2) \geqslant 0$. Now using $(**)$, we obtain $(\gamma - \beta_1) + (\gamma - \beta_2) \notin \Psi$, since $\gamma - \beta_2$ is long.

(b_2) Both β_1 and β_2 are short.

Then $|(\beta_1, \beta_2)| \leqslant \frac{1}{2}(\beta_1, \beta_1) < \frac{1}{2}(\gamma, \gamma)$ and $(*)$ shows that $(\gamma - \beta_1, \beta_2) > 0$. Therefore, $\gamma - \beta_1 - \beta_2$ is a root in $\Psi(1)$. Since γ is long and $(\gamma, \beta_i) = \frac{1}{2}(\gamma, \gamma)$ for $i = 1, 2$, we conclude that $(\gamma - \beta_1 - \beta_2, \gamma) = 0$ and therefore, $(\gamma - \beta_1 - \beta_2) + \gamma \notin \Psi$.

(ii) Using the notation of part (i), we have $\Psi(\mathfrak{a}_\sigma) = \Psi(V) \cup \Psi(2)$. First we show that none of the generators in $\Gamma_{\mathfrak{a}_\sigma}$ lies in $\Psi(2)$. For this end, it suffices to show that the lowest weight δ in $\Psi(2)$, is not a generator. (Recall that $\mathfrak{g}(2)$ is an irreducible $\mathfrak{g}(0)$-module and therefore δ is uniquely determined in $\Psi(2)$.) Since $\delta - \sigma$ is a root (in $\Psi(1)$), it is enough to show that it lies in $\Psi(V)$. Because γ is the highest weight in $\Psi(1)$, we see that $\gamma + \sigma$ is a root, and hence $(\gamma, \sigma) < 0$. Since $\Psi(3)$ is empty, $\gamma + \delta$ is not a root. Thus, $(\gamma, \delta) \geqslant 0$ and then $(\gamma, \delta - \sigma) > 0$. This implies that $\gamma - (\delta - \sigma)$ is a root lying in $\Psi(0)^+$. By the very construction of V, this means $\delta - \sigma \in \Psi(V)$, as desired.

Now we consider the elements of $\Psi(V)$. Let w_0 be the longest element in $W(0)$. Then $w_0(\varrho_c^\sigma) = -\varrho_c^\sigma$ for each $c \in \pi_0(\Delta^\sigma)$ and $w_0(\sigma) = \gamma$. Since ϱ_c^σ is the highest root in Ψ_c^σ (but not in Ψ), we have $\varrho_c^\sigma + \sigma \in \Psi$. Hence $w_0(\varrho_c^\sigma + \sigma) = \gamma - \varrho_c^\sigma$ is also a root. According to the construction of part (i), the corresponding root space lies in $\mathfrak{a}_\sigma$. Moreover, since $\{\varrho_c^\sigma\}_c$ are clearly the maximal possible elements of $\Psi(0)^+$ that can be subtracted from γ, i.e., that $\{\gamma - \varrho_c^\sigma\}_c$ are the elements of $\Psi(V)$ of minimal height, we obtain $\{\gamma - \varrho_c^\sigma\}_c \subseteq \Gamma_{\mathfrak{a}_\sigma}$. On the other hand, suppose $\gamma - \mu \in \Psi(V)$, where $\mu \in \Psi(0)^+ \setminus \{\varrho_c^\sigma\}_c$. Then $\mu \in \Psi_c^\sigma$ for some $c \in \pi_0(\Delta^\sigma)$ and hence $\varrho_c^\sigma - \mu$ is a sum of positive roots from Ψ_c^σ and so is $(\gamma - \mu) - (\gamma - \varrho_c^\sigma) = \varrho_c^\sigma - \mu$. Therefore, $\gamma - \mu \notin \Gamma_{\mathfrak{a}_\sigma}$.

(iii) Using the information in Tables I and II this is readily verified. ∎

Utilising Proposition 4.2, we can describe the map ψ in all classical cases. In the following theorem we axiomatise the properties of this mapping. Whenever ϱ is fundamental (this refers to all simple Lie algebras except for those of type A_r and C_r) there is a unique simple root σ^* such that $(\varrho, \sigma^*) \neq 0$, see [2]. Observe that σ^* is always long.

THEOREM 4.3. *There is a unique bijection* $\psi\colon \Pi_\ell \to \mathscr{A}_{max}$ $(\mathfrak{a}_\sigma := \psi(\sigma))$ *satisfying the following conditions:*

1. $d_{\mathfrak{a}_\sigma} = 2n_\sigma - 1$.
2. *If* $n_\sigma = 1$, *then* $\Gamma_{\mathfrak{a}_\sigma} = \{\sigma\}$.
3. *If* $n_\sigma = 2$, *then* $\mathfrak{a}_\sigma$ *is defined as in Proposition* 4.2.
4. $\#\Gamma_{\mathfrak{a}_\sigma} = \#\pi_0(\Delta^\sigma)$ *provided* $\mathfrak{g}$ *is not of type* A_r.
5. *Suppose* ϱ *is fundamental. Then for any sequence* $(\sigma^*, \alpha, \beta, \ldots)$ *of simple roots, adjacent in* Δ (*and mutually distinct*), *we have* $\dim \mathfrak{a}_{\sigma^*} < \dim \mathfrak{a}_\alpha < \dim \mathfrak{a}_\beta < \cdots$.

Proof. The proof consists of a case by case argument. One only needs to exploit the second and fifth columns in Tables I and II. The resulting correspondence is presented in Figures 3 and 4 where we label each node $\sigma \in \Pi_\ell$ with $\dim \mathfrak{a}_\sigma$. ∎

Observe that conditions 4 and 5 follow from the first three for B_r, D_r, E_6, and F_4. In fact, condition 5 is required only to construct ψ for E_7 and E_8.

In the diagrams in Figures 3 and 4 the marked node indicates the one corresponding to the simple root σ^*. Because there is a unique long simple root in C_r and G_2, these cases are omitted.

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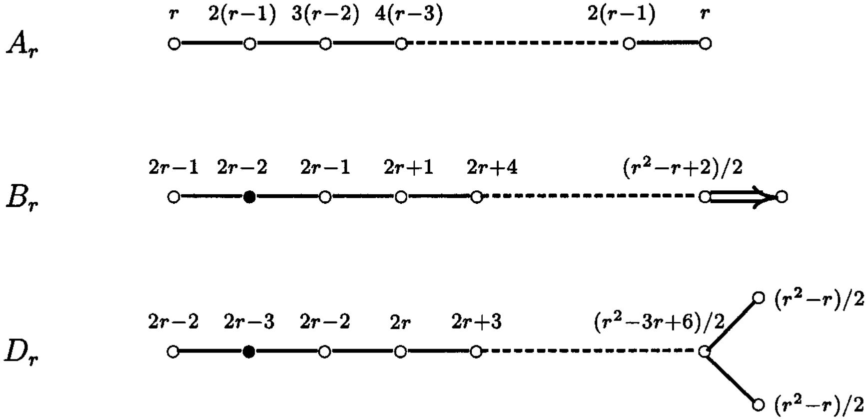

FIG. 3. The function $\sigma \mapsto \dim \mathfrak{a}_\sigma$ in the classical cases.

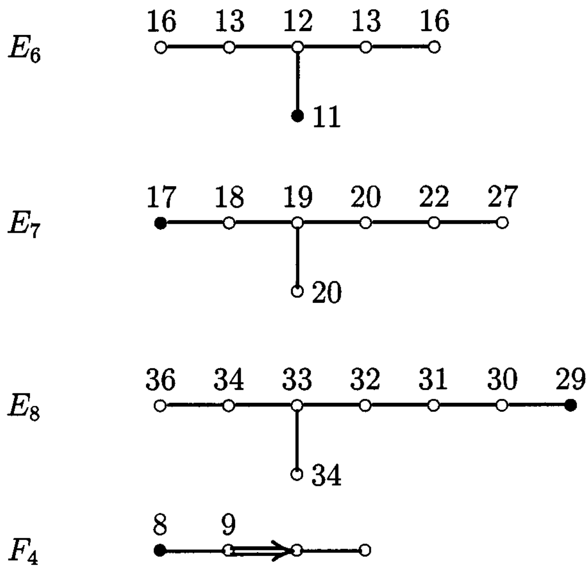

FIG. 4. The function $\sigma \mapsto \dim \mathfrak{a}_\sigma$ in the exceptional cases.

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