ON SPHERICAL IDEALS OF BOREL SUBALGEBRAS

DMITRI PANYUSHEV AND GERHARD RÖHRLE

ABSTRACT. The goal of this paper is to extend some previous results on abelian ideals of Borel subalgebras to so-called spherical ideals of $\mathfrak{b}$. These are ideals $\mathfrak{c}$ of $\mathfrak{b}$ such that their G-saturation $G\cdot\mathfrak{c}$ is a spherical G-variety. We classify all maximal spherical ideals of $\mathfrak{b}$ for all simple G.

1. INTRODUCTION

Let G be a connected reductive complex algebraic group with Lie algebra $\operatorname{Lie} G = \mathfrak{g}$. Let B be a Borel subgroup of G with unipotent radical B_u. We denote the Lie algebras of B and B_u by $\mathfrak{b}$ and $\mathfrak{b}_u$, respectively. The group B acts on any ideal of $\mathfrak{b}$ by means of the adjoint representation. There has been quite a lot of activity recently in the study of various aspects of ad-nilpotent ideals and, in particular, abelian ideals of $\mathfrak{b}$, for instance, see [5], [6], [9], [12], [14], and [16], and the additional references therein.

After a preliminary section, we recall our main finiteness results on abelian ideals from [14]. The goal of this paper is to extend these results to so-called *spherical* ideals of $\mathfrak{b}$ in the next section. These are ideals $\mathfrak{c}$ of $\mathfrak{b}$ such that their G-saturation $G\cdot\mathfrak{c}$ is a spherical G-variety. Our aim is to classify all maximal spherical ideals of $\mathfrak{b}$ for all simple G.

2. PRELIMINARIES

We denote the Lie algebra of G by $\operatorname{Lie} G$ or $\mathfrak{g}$; likewise for closed subgroups of G. Let T be a fixed maximal torus in G and $\Psi = \Psi(G)$ the set of roots of G with respect to T and let $r = \dim T = \operatorname{rank} G$. Fix a Borel subgroup B of G containing T and let $\Pi = \{\sigma_1, \dots, \sigma_r\}$ be the set of simple roots of Ψ defined by B such that the positive integral span of Π in Ψ is $\Psi^+ = \Psi(B)$. In case G is simple, the highest (long) root in Ψ is denoted by $\rho = \sum n_\sigma \sigma$, where the sum is taken over the simple roots Π. If all roots in Ψ are of the same length, they are all called *long*. A subset of Ψ^+ is an *ideal* in Ψ^+ provided it is closed under addition by elements from Ψ^+. As usual, we have the root space decomposition of $\mathfrak{g}$ relative to T,

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Psi} \mathfrak{g}_\alpha.$$

For a T-stable subspace $\mathfrak{h} \subset \mathfrak{g}$, we denote by $\Psi(\mathfrak{h})$ its set of roots with respect to T. Any parabolic subgroup P of G is assumed to be *standard*, i.e., it contains B. The notation $P = P(\sigma_{i_1}, \dots, \sigma_{i_t})$ means that $\sigma_{i_1}, \dots, \sigma_{i_t}$ are the simple roots of the standard Levi subgroup

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of this parabolic subgroup. Then the number t is called the *semisimple rank* of P, denoted $\operatorname{srk} P$. For each $\alpha \in \Psi$, we choose a nonzero root vector e_α in $\mathfrak{g}_\alpha$.

Following Cellini and Papi [5], we say that an ideal of $\mathfrak{b}$ is ad-*nilpotent*, if it is contained in $\mathfrak{b}_u$. It is easily seen that any abelian ideal of $\mathfrak{b}$ is ad-nilpotent. Since we shall consider no other types of ideals, we usually write "$\mathfrak{b}$-ideal" or just "ideal" in place of "ad-nilpotent ideal of $\mathfrak{b}$". Given an ideal $\mathfrak{c} \subset \mathfrak{b}_u$ of $\mathfrak{b}$, we have $\mathfrak{c} = \oplus_{\alpha \in \Psi(\mathfrak{c})} \mathfrak{g}_\alpha$. In particular, $\mathfrak{c}$ is uniquely determined by its set of roots $\Psi(\mathfrak{c})$ which is an ideal in Ψ^+, where a subset Δ of Ψ^+ is called an *ideal* provided if $\alpha \in \Delta$, $\beta \in \Psi^+$ and $\alpha + \beta \in \Psi^+$, then $\alpha + \beta \in \Delta$.

Our basic reference concerning results on root systems is [2]. In case G is simple, we use the labelling of the Dynkin diagram of G (i.e. of Π) as in [2]. We refer to [1] and [19] for terminology and standard results on algebraic groups.

Suppose a connected algebraic group R acts morphically on an algebraic variety X. The *modality* of the action of R on X is defined as

$$\operatorname{mod}(R : X) := \max_{Z} \min_{z \in Z} \operatorname{codim}_Z R{\cdot}z,$$

where Z runs through all irreducible R-invariant subvarieties of X. This definition is due to E.B. Vinberg [17]. Note that $\operatorname{mod}(R : X) = 0$ if and only if R acts on X with a finite number of orbits.

3. Abelian Ideals and Spherical Orbits

A nilpotent G-orbit (conjugacy class) O in $\mathfrak{g}$ is called *spherical* whenever it is a spherical G-variety, that is, B acts on it with a dense orbit. By a fundamental theorem, due to M. Brion [3] and E.B. Vinberg [17], in that case B acts on O with finitely many orbits. Since O is quasi-affine, it is spherical if and only if the algebra of polynomial functions $\mathbb{C}[O]$ is a *multiplicity free* G-module [18].

The following characterisation of spherical nilpotent orbits is found in [10, §3.1] and [11, Thm. 3.2].

Theorem 3.1. *Let O be a nilpotent orbit in $\mathfrak{g}$. The following statements are equivalent:*

- (i) *O is spherical;*
- (ii) $(\operatorname{ad} x)^4 = 0$ *for any $x \in O$;*
- (iii) *O contains a representative of the form $e_{\alpha_1} + \cdots + e_{\alpha_t}$, where $\{\alpha_1, \ldots, \alpha_t\} \subseteq \Pi$ is a set of mutually orthogonal* simple *roots.*

The last property means that a minimal Levi subalgebra of $\mathfrak{g}$ meeting O is the sum of t copies of $\mathfrak{sl}_2$. Since such a Levi subalgebra is determined by the orbit up to conjugation, we see that the integer t in Theorem 3.1(iii) does not depend on the choice of a representative for O and, furthermore, the number of long and short roots among the α_i's is an invariant of the orbit.

The equivalence between parts (i) and (ii) of Theorem 3.1 is proved in [10, §3.1]. Some parts of that proof relied on the use of the classification of nilpotent orbits in simple Lie algebras. A classification-free proof is given in [13].

Making use of Theorem 3.1, we set up a direct link between the abelian ideals of $\mathfrak{b}$ and spherical nilpotent orbits. The following are proved in [14, Sect. 2]:

Theorem 3.2. *If $\mathfrak{a}$ is an abelian ideal in $\mathfrak{b}$, then $G{\cdot}\mathfrak{a}$ is the closure of a spherical nilpotent orbit. In particular, any G-orbit meeting $\mathfrak{a}$ is spherical.*

Corollary 3.3. *Let $\mathfrak{a}$ be an abelian ideal in $\mathfrak{b}$. Then B has finitely many orbits in $\mathfrak{a}$.*

Corollary 3.4. *Let P be a parabolic subgroup of G and let $\mathfrak{a}$ be an abelian ideal of $\mathfrak{p}$ in $\mathfrak{p}_u$. Then P acts on $\mathfrak{a}$ with finitely many orbits.*

It follows from the definition of modality and Corollary 3.3 that if $\mathfrak{a}$ is an abelian ideal of $\mathfrak{b}$, then $\operatorname{mod}(B:\mathfrak{a}) = 0$.

We should like to point out that if $\mathfrak{c}$ is an ad-nilpotent ideal of $\mathfrak{b}$ of nilpotency class at least 2, then there need not be any bound on the modality $\operatorname{mod}(B:\mathfrak{c})$ of the action of B on $\mathfrak{c}$, cf. [15]. In fact, examples in type A_r show that already for $\mathfrak{c}$ of nilpotency class 2 the values for $\operatorname{mod}(B:\mathfrak{c})$ may grow quadratically in the rank of G. For instance, let $\mathfrak{g}$ be of type A_r where $r = 3n-1$ for some $n \in \mathbb{N}$. Let $\mathfrak{c}$ be the ideal in $\mathfrak{b}$ generated by the two root spaces relative to the simple roots σ_n and σ_{2n}. Then $\mathfrak{c}$ is of nilpotency class 2. Using the bound from [15, Lemma 2.1], we obtain

$$\operatorname{mod}(B:\mathfrak{c}) \geq 2\dim\mathfrak{c} - \dim\mathfrak{b} - \dim[\mathfrak{c},\mathfrak{c}] = \frac{1}{18}(r^2-7r+10).$$

4. Spherical Ideals of $\mathfrak{b}$

A $\mathfrak{b}$-ideal $\mathfrak{c}$ is said to be *spherical*, if its G-saturation $G{\cdot}\mathfrak{c}$ is a spherical G-variety. Then clearly, by the discussion in the last section, we have $\operatorname{mod}(B:\mathfrak{c}) = 0$. The aim of this section is to find all spherical ad-nilpotent ideals of $\mathfrak{b}$ (or $\mathfrak{b}$-ideals) in case G is simple. In view of Theorem 3.2, we concentrate on the description of the non-abelian spherical ideals.

Proposition 4.1. *Let $\mathfrak{c}$ be an* ad*-nilpotent ideal of $\mathfrak{b}$.*

(i) *If $\mathfrak{g}$ is simply-laced and $[\mathfrak{c},\mathfrak{c}] \neq \{0\}$, then $\mathfrak{c}$ is not spherical.*
(ii) *If $\mathfrak{g}$ is doubly-laced and $[\mathfrak{c},[\mathfrak{c},\mathfrak{c}]] \neq \{0\}$, then $\mathfrak{c}$ is not spherical.*
(iii) *If $\mathfrak{g}$ is of type G_2, then any spherical ideal is abelian.*

Part (i) is already proved in [14, Prop. 2.7]. However, the proof for (i) and (ii) uses similar ideas, so that including our proof for (i) requires only a couple of extra lines.

Proof. We call a pair of roots $\{\gamma,\mu\} \in \Psi^+$ an *A_2-configuration* if $|\gamma| = |\mu|$ and $(\gamma,\mu) < 0$ and we call it a *C_2-configuration* if $|\gamma|^2 = 2|\mu|^2$ and $(\gamma,\mu) < 0$. To prove that an ideal $\mathfrak{c}$ is not spherical, it suffices to detect inside of $\Psi(\mathfrak{c})$ an A_2- or C_2-configuration $\{\gamma,\mu\}$. Indeed, in that case $x = e_\gamma + e_\mu \in \mathfrak{c}$ is a principal nilpotent element in a simple subalgebra of type A_2 or C_2. Denote this subalgebra by $\mathfrak{s}$. Then $(\operatorname{ad} x\,|_{\mathfrak{s}})^4 \neq 0$. Hence the orbit $G{\cdot}x$ is not spherical by Theorem 3.1 and so $G{\cdot}\mathfrak{c}$ is not spherical either.

(i) If $[\mathfrak{c},\mathfrak{c}] \neq \{0\}$, then there exist $\alpha,\beta \in \Psi(\mathfrak{c})$ such that $\alpha+\beta \in \Psi$. In the simply-laced case, this means that $\{\alpha,\beta\}$ is an A_2-configuration, and we are done.

(ii) If $[[\mathfrak{c},[\mathfrak{c},\mathfrak{c}]] \neq \{0\}$, then there exist $\gamma_1,\gamma_2,\gamma_3 \in \Psi(\mathfrak{c})$ such that both $\gamma_1+\gamma_2$ and $\gamma_1+\gamma_2+\gamma_3$ are in Ψ. Now, considering the various possibilities for the lengths of the roots in question, we see that one can always detect an A_2- or C_2-configuration within $\Psi(\mathfrak{c})$:

(a) If γ_1,γ_2 are long, then $\{\gamma_1,\gamma_2\}$ is an A_2-configuration.

(b) If γ_1 is long and γ_2 is short, then $\{\gamma_1,\gamma_2\}$ is a C_2-configuration.

(c) If γ_1,γ_2 are short and $(\gamma_1,\gamma_2)<0$, then $\{\gamma_1,\gamma_2\}$ is an A_2-configuration.

(d) If γ_1,γ_2 are orthogonal short roots, then $\gamma_1+\gamma_2$ is long. Thus, if γ_3 is short (resp. long), then $\{\gamma_1+\gamma_2,\gamma_3\}$ is a C_2-configuration (resp. A_2-configuration).

(iii) Straightforward. □

Remark. The proof of Proposition 4.1 shows that a non-abelian ideal $\mathfrak{c}$ can only be spherical if all nonzero brackets in $[\mathfrak{c},\mathfrak{c}]$ correspond to orthogonal short roots, i.e., $[e_\alpha,e_\beta]\neq 0$ only if α,β are orthogonal and short.

Remark. Given an ideal $\mathfrak{c}\subset\mathfrak{b}_u$, we say that $\gamma\in\Psi(\mathfrak{c})$ is a *generator* if it cannot be written as a sum $\gamma=\mu+\sum\nu_i$, where $\mu\in\Psi(\mathfrak{c})$ and $\nu_i\in\Psi^+$. Alternatively, let $\tilde{\mathfrak{c}}$ be the unique T-stable complement to $[\mathfrak{b}_u,\mathfrak{c}]$ in $\mathfrak{c}$. Then γ is a generator of $\Psi(\mathfrak{c})$ if and only if it is a T-weight of $\tilde{\mathfrak{c}}$. We write $\Gamma(\mathfrak{c})$ for the set of generators. It is obvious from this description that $\Gamma=\{\gamma_1,\dots,\gamma_t\}\subset\Psi^+$ is a set of generators for some ad-nilpotent $\mathfrak{b}$-ideal if and only if $\gamma_i-\gamma_j$ is not a sum of positive roots for all pairs i,j, i.e., the elements of Γ are pairwise not comparable with respect to the standard partial ordering on Ψ^+. In the combinatorial language, this means that Γ is an *antichain* in Ψ^+. Since an ad-nilpotent ideal is completely determined by its set of generators, we write $\mathfrak{c}(\Gamma)$ for the ideal whose set of generators is an antichain $\Gamma\subset\Psi^+$.

Our method of classifying spherical ideals is that we first determine the ad-nilpotent ideals whose sets of roots do not contain A_2- or C_2-configurations (cf. proof of Proposition 4.1). Then we directly verify that all ideals obtained in this way are actually spherical.

Theorem 4.2. *Suppose $\mathfrak{g}$ is doubly-laced.*

(i) *Let $\mathfrak{g}$ be of type B_r ($r\geq 2$). Then there is a unique maximal non-abelian spherical $\mathfrak{b}$-ideal.*

(ii) *Let $\mathfrak{g}$ be of type C_r ($r\geq 2$). Then there are $r-1$ maximal non-abelian spherical $\mathfrak{b}$-ideals.*

(iii) *Let $\mathfrak{g}$ be of type F_4. Then there are two maximal non-abelian spherical $\mathfrak{b}$-ideals.*

The generators and dimensions of these ideals are listed in Table 1 below.

Proof. For B_r and C_r, we use the standard expression of roots in terms of the fundamental dominant weights $\varepsilon_1,\dots,\varepsilon_r$. For instance, $\sigma_1=\varepsilon_1-\varepsilon_2$ for both B_r and C_r, while $\sigma_r=\varepsilon_r$ for B_r and $\sigma_r=2\varepsilon_r$ for C_r.

Let $\mathfrak{c}$ be a spherical $\mathfrak{b}$-ideal in $\mathfrak{g}$.

(i) Recall that the positive roots of B_r are $\varepsilon_i\pm\varepsilon_j$ ($1\leq i<j\leq r$) and ε_i ($1\leq i\leq r$).

(a) Suppose $\varepsilon_i-\varepsilon_j\in\Psi(\mathfrak{c})$. Then we also have $\varepsilon_1+\varepsilon_j\in\Psi(\mathfrak{c})$, since $(\varepsilon_1+\varepsilon_j)-(\varepsilon_i-\varepsilon_j)$ is a sum of positive roots. Because $\{\varepsilon_i-\varepsilon_j,\varepsilon_1+\varepsilon_j\}$ is an A_2-configuration if $i>1$, we conclude that $\varepsilon_i-\varepsilon_j$ is a root of a spherical ideal only if $i=1$.

(b) Suppose $\varepsilon_1-\varepsilon_j\in\Psi(\mathfrak{c})$ for some $j\geq 2$, and j is the minimal such possible index. If $j=2$, then $\varepsilon_2+\varepsilon_3\notin\Psi(\mathfrak{c})$, since $\{\varepsilon_1-\varepsilon_2,\varepsilon_2+\varepsilon_3\}$ is an A_2-configuration. Hence we obtain the maximal abelian ideal with the generator $\varepsilon_1-\varepsilon_2$. Thus, we may assume that $j>2$.

- Since $\{\varepsilon_1-\varepsilon_j,\varepsilon_2+\varepsilon_j\}$ is an A_2-configuration, we have $\varepsilon_2+\varepsilon_j\not\in\Psi(\mathfrak{c})$. This implies that the root $\varepsilon_k+\varepsilon_l$ belongs to $\Psi(\mathfrak{c})$ only if $k=1$ (and l is arbitrary), or $k,l<j$.
- Assume now $\varepsilon_k\in\Psi(\mathfrak{c})$ for $k>1$. Then $\varepsilon_2\in\Psi(\mathfrak{c})$ and hence $\varepsilon_2+\varepsilon_j\in\Psi(\mathfrak{c})$ for $j>2$. This yields in $\Psi(\mathfrak{c})$ either the C_2-configuration $\{\varepsilon_1-\varepsilon_2,\varepsilon_2\}$ (for $j=2$), or the A_2-configuration $\{\varepsilon_1-\varepsilon_j,\varepsilon_2+\varepsilon_j\}$ (for $j>2$). This contradiction shows that $\varepsilon_k\not\in\Psi(\mathfrak{c})$, if $k\ge 2$.

Thus, we conclude that $\Psi(\mathfrak{c})$ is contained in the set of roots of the form $\varepsilon_1-\varepsilon_k$ ($k\ge j$), ε_1, $\varepsilon_1+\varepsilon_l$ ($2\le l\le r$), and $\varepsilon_k+\varepsilon_l$ ($1<k<l<j$). Clearly, such an ideal $\mathfrak{c}$ is abelian.

(c) It remains to consider the case when $\Psi(\mathfrak{c})$ does not contain roots of the form $\varepsilon_i-\varepsilon_j$. Here all roots remaining at our disposal form an ideal of Ψ^+, say $\Psi(\mathfrak{c}')$, whose unique generator is $\varepsilon_r=\sigma_r$. Using the canonical matrix realization of $\mathfrak{so}_{2r+1}$, we see that $x^3=0$ and rank $(x^2)\le 1$ for any matrix $x\in\mathfrak{c}'\subset\mathfrak{so}_{2r+1}$. It then follows from [10, 4.3] that $G{\cdot}x$ is spherical.

Thus, we have found a unique maximal spherical non-abelian ideal. It is worth noting that $\tilde{\mathfrak{c}}$ contains one of the maximal abelian ideals, namely the ideal whose set of generators is $\{\varepsilon_1,\varepsilon_{r-1}+\varepsilon_r\}$.

(ii) The argument for C_r is similar to that for B_r, so that we omit the details. Recall that the positive roots of C_r are $\varepsilon_i\pm\varepsilon_j$ ($1\le i<j\le r$) and $2\varepsilon_i$ ($1\le i\le r$).

(a) Again, if $\varepsilon_i-\varepsilon_j\in\Psi(\mathfrak{c})$ and $i>1$, then $\mathfrak{c}$ cannot be spherical.

(b) If $\varepsilon_1-\varepsilon_j\in\Psi(\mathfrak{c})$ for some $j\ge 2$, and j is the minimal possible index, then $\Psi(\mathfrak{c})$ can only contain the roots $\varepsilon_1-\varepsilon_k$ ($j\le k$), $\varepsilon_1+\varepsilon_l$ ($1\le l\le r$), $\varepsilon_k+\varepsilon_l$ ($1<k,l<j$). This yields $r-1$ maximal possibilities for $\Psi(\mathfrak{c})$. Namely, we get the ideals $\mathfrak{c}_j$ generated by the roots $\varepsilon_1-\varepsilon_j,2\varepsilon_{j-1}$, if $3\le j\le r$; and by $\varepsilon_1-\varepsilon_2$, if $j=2$.
Using the canonical matrix realization of $\mathfrak{sp}_{2r}$, we see that $x^2=0$ for any matrix $x\in\mathfrak{c}_j\subset\mathfrak{sp}_{2r}$. It then follows from [10, 4.2] that $G{\cdot}x$ is spherical.

(c) Notice that if $\Psi(\mathfrak{c})$ contains no roots of the form $\varepsilon_i-\varepsilon_j$, then we obtain the unique maximal abelian ideal, which is generated by $\sigma_r=2\varepsilon_r$.

(iii) We denote a root $\sum_{i=1}^4 n_i\sigma_i$ in the root system of type F_4 simply by $(n_1\,n_2\,n_3\,n_4)$. Working directly within the root system of type F_4, one easily finds that there are two maximal non-abelian ideals without A_2- or C_2-configurations. These are $\mathfrak{c}_1=\mathfrak{c}(\{(1111),(0122)\})$ and $\mathfrak{c}_2=\mathfrak{c}(\{(0121)\})$. Here is a sketch of the argument.

- If $(1120)\in\Psi(\mathfrak{c})$, then $\Psi(\mathfrak{c})$ contains also the A_2-configuration (1122), (1220);
- If $(0111)\in\Psi(\mathfrak{c})$, then $\Psi(\mathfrak{c})$ contains also the A_2-configuration (1111), (0121);
- If $(1220)\in\Psi(\mathfrak{c})$, then either $\mathfrak{c}$ is abelian, or $\Psi(\mathfrak{c})$ contains the A_2-configuration (1122), (1220).

Thus, any ideal without A_2-configurations is contained in the ideal with generators (1111), (0121). Furthermore, these two roots cannot occur together. This leads to the above two possibilities.

Next, we show that there are subgroups $H_1, H_2 \subset G$ and an abelian ideal $\mathfrak{a}$ such that $\mathfrak{c}_i = H_i{\cdot}\mathfrak{a}$. (That is, each $\mathfrak{c}_i$ is a "partial saturation" of an abelian ideal.) This obviously means that $G{\cdot}\mathfrak{c}_i$ contains the same dense G-orbit as $G{\cdot}\mathfrak{a}$. Since this orbit is spherical by Theorem 3.2, we conclude that $\mathfrak{c}_i$ is spherical, too. In both cases, the subgroup H_i is a standard parabolic subgroup of G. Namely, we take

$$\Gamma(\mathfrak{a}) = \{(0122), (1221)\},\ H_1 = P(\sigma_2, \sigma_3)\ ,\ \text{and}\ H_2 = P(\sigma_1, \sigma_2)\ .$$

It is straightforward to verify that $\mathfrak{a}$ is a maximal abelian ideal, $\dim \mathfrak{a} = 9$, and $N_G(\mathfrak{a}) = P(\sigma_1, \sigma_3)$. Then considering the roots involved shows that $\mathfrak{c}_i = H_i{\cdot}\mathfrak{a}$ in both cases.

It follows from this construction that, similarly to the case of B_r, both $\mathfrak{c}_1$ and $\mathfrak{c}_2$ contain the abelian ideal $\mathfrak{a}$ defined above. □

In Table 1, we list all the maximal non-abelian spherical ideals (i.e., in the non-simply laced cases).

G	$\Gamma(\mathfrak{c})$	$\dim \mathfrak{c}$
B_r	ε_r	$r(r+1)/2$
C_r	$\varepsilon_1 - \varepsilon_2$	$2r-1$
	$\varepsilon_1 - \varepsilon_{k+1}, 2\varepsilon_k\ (2 \le k \le r-1)$	$2r + (k^2 - 3k)/2$
F_4	(1111), (0122)	11
	(0121)	11

TABLE 1. The maximal spherical non-abelian ideals of $\mathfrak{b}$ in type B_r, C_r, and F_4.

Remark. Clearly, if $\mathfrak{c}$ is a spherical ideal of $\mathfrak{b}$, then we have $\mathrm{mod}(B : \mathfrak{c}) = 0$. Sphericity, however, is not necessary for the resulting modality zero statement, as many examples show, e.g., see the results in [7] and [8].

Corollary 4.3. *Suppose G is simple and not of type G_2. The number of maximal abelian ideals of $\mathfrak{b}$ plus the number of maximal non-abelian spherical ideals of $\mathfrak{b}$ equals* $\mathrm{rank}\, \mathfrak{g}$.

Proof. This follows immediately from the classification of the maximal abelian ideals of $\mathfrak{b}$ in [16] and the above classification of the maximal spherical non-abelian ideals. □

Remarks. 1. As indicated in the proof of Theorem 4.2, it happens for B_r and F_4 that a maximal abelian ideal is properly contained in a maximal spherical non-abelian ideal. Therefore, the number of maximal spherical ideals is equal to the rank of $\mathfrak{g}$ if and only if $\mathfrak{g}$ is either simply-laced or of type C_r.

2. Thanks to a theorem of D. Peterson, for G simple, the number of abelian ideals of $\mathfrak{b}$ is 2^r, cf. [9], [5, Thm. 2.9]. There is no analogous uniform expression for the number of spherical ideals. For, this number is equal to 2^r in the simply-laced cases and also for G_2, see Proposition 4.1. In type B_r this number equals $2^r + 2^{r-2}$ and in type C_r it is $2^r + 2^{r-1} - 1$. In type F_4 this number equals 21.

3. The theory of $\mathfrak{sl}_2$-triples readily implies that any nilpotent orbit intersects some ad-nilpotent ideal in a dense subset. For, let e be a nilpotent element in $\mathfrak{g}$ and let h be the semisimple element of an $\mathfrak{sl}_2$-triple containing e. Consider the $\mathbb{Z}$-grading of $\mathfrak{g} = \oplus\mathfrak{g}(i)$ afforded by h. Then $G \cdot e$ meets the ideal $\oplus_{i\geq 2}\mathfrak{g}(i)$ of $\mathfrak{b}$ in a dense subset, e.g., see [4, Sect. 5.7]. In particular, any spherical nilpotent orbit intersects some spherical ideal of $\mathfrak{b}$ in a dense subset.

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References

[1] A. Borel, *Linear Algebraic Groups*, Graduate Texts in Mathematics, **126**, Springer-Verlag (1991).

[2] N. Bourbaki, *Groupes et algèbres de Lie, Chapitres 4,5 et 6*, Hermann, Paris, (1975).

[3] M. Brion, *Quelques propriétés des espaces homogénes sphériques*, Man. Math. **55** (1986), 191–198.

[4] R.W. Carter, *Finite Groups of Lie Type: Conjugacy Classes and Complex Characters*, Chichester etc., Wiley (1985).

[5] P. Cellini, P. Papi, ad-*nilpotent ideals of a Borel subalgebra*, J. Algebra **225** (2000), no. 1, 130–141.

[6] ______, ad-*nilpotent ideals of a Borel subalgebra*, II, J. Algebra **258** (2002), no. 1, 112–121.

[7] L. Hille, G. Röhrle, *A classification of parabolic subgroups of classical groups with a finite number of orbits on the unipotent radical*, Transformation Groups, **4(1)** (1999), 35–52.

[8] U. Jürgens, G. Röhrle, MOP – *Algorithmic Modality Analysis for Parabolic Group actions*, Experimental Math., **11**, (2002) no. 1, 57–67.

[9] B. Kostant, *The set of abelian ideals of a Borel subalgebra, Cartan decompositions, and discrete series representations*, Internat. Math. Res. Notices, no. 5, (1998), 225–252.

[10] D. Panyushev, *Complexity and nilpotent orbits*, Manuscripta Math. **83**(3-4) (1994), 223–237.

[11] ______, *On spherical nilpotent orbits and beyond*, Annales de l'Institut Fourier, **49**(5) (1999), 1453–1476.

[12] ______, *Abelian Ideals of a Borel Subalgebra and Long Positive Roots*, IMRN, (2003), no. 35, 1889–1913.

[13] ______, *Some amazing properties of spherical nilpotent orbits*, Math. Z., **245**(2003), 557–580.

[14] D. Panyushev, G. Röhrle, *Spherical Orbits and Abelian Ideals*, Adv. Math., **159**(2001), 229–246.

[15] G. Röhrle, *A note on the modality of parabolic subgroups*, Indag. Math. N.S. **8 (4)**, (1997), 549–559.

[16] ______, *On Normal Abelian Subgroups of Parabolic groups*, Annales de l'Institut Fourier, **48**(5) (1998), 1455–1482.

[17] E. B. Vinberg, *Complexity of actions of reductive groups*, Funkt. Analiz i Prilozhen. **20**(1986), № 1, 1–13 (Russian). English translation: Funct. Anal. Appl. **20**, (1986), 1–11.

[18] E. B. Vinberg, B. N. Kimel'feld, *Homogeneous domains on flag manifolds and spherical subgroups of semisimple Lie groups*, Funkt. Analiz i Prilozhen. **12**(1978), № 3, 12–19 (Russian). English translation: Funct. Anal. Appl. **12**, (1978), 168–174.

[19] E. B. Vinberg, A. L. Onishchik, "Seminar on Lie groups and algebraic groups", Moskva: "Nauka" 1988 (Russian). English translation: A. L. Onishchik and E. B. Vinberg: "Lie groups and algebraic groups". Berlin Heidelberg New York: Springer 1990.

INDEPENDENT UNIVERSITY OF MOSCOW, BOL'SHOI VLASEVSKII PER. 11, MOSCOW 121002, RUSSIA
E-mail address: panyush@mccme.ru

SCHOOL OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF BIRMINGHAM, BIRMINGHAM B15 2TT, UK
E-mail address: ger@for.mat.bham.ac.uk