

# SOME AMAZING PROPERTIES OF SPHERICAL NILPOTENT ORBITS

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## INTRODUCTION

Let  $G$  be a simple algebraic group defined over an algebraically closed field  $\mathbb{k}$  of characteristic zero. Write  $\mathfrak{g}$  for its Lie algebra.

Let  $x \in \mathfrak{g}$  be a nilpotent element and  $G \cdot x \subset \mathfrak{g}$  the corresponding nilpotent orbit. The maximal number  $m$  such that  $(\operatorname{ad} x)^m \neq 0$  is called the *height* of  $x$  or of  $G \cdot x$ , denoted  $\operatorname{ht}(x)$ . Recall that an irreducible  $G$ -variety  $X$  is called  *$G$ -spherical* if a Borel subgroup of  $G$  has an open orbit in  $X$ . It was shown in [Pa1] that  $G \cdot x$  is  $G$ -spherical if and only if  $(\operatorname{ad} x)^4 = 0$ . This means that the spherical nilpotent orbits are precisely the orbits of height 2 and 3. Unfortunately, whereas my proof for the orbits of height 2 and height  $\geq 4$  was completely general, the argument for the orbits of height 3 explicitly used their classification. In this paper, we give a proof of sphericity that does not rely on the classification of nilpotent orbits, see Theorem 3.3. We begin with some properties of invariants of symplectic representations. For instance, we prove that (1) if  $H \subset Sp(V)$  is an irreducible representation without non-constant invariants, then  $H = Sp(V)$ , and (2) if  $H$  has non-constant invariants, then it has an invariant of degree 4. Applying these results to nilpotent orbits, we prove that the centraliser  $\mathfrak{z}_{\mathfrak{g}}(x)$  has a rather specific structure whenever  $\operatorname{ht}(x)$  is odd. From this description, we deduce a conceptual proof of sphericity in case  $\operatorname{ht}(x) = 3$ . As another application we compute the index of  $\mathfrak{z}_{\mathfrak{g}}(x)$ . It will be shown that  $\operatorname{ind} \mathfrak{z}_{\mathfrak{g}}(x) = \operatorname{rk} \mathfrak{g}$ , if  $\operatorname{ht}(x) = 3$ . This confirms Elashvili's conjecture for such  $x$  (see [Pa5, Sect. 3] about this conjecture). In Section 4, we prove that if  $\theta$ , the highest root of  $\mathfrak{g}$ , is fundamental, then  $\mathfrak{g}$  always has a specific orbit of height 3, which is denoted by  $\mathbb{O}$ . This orbit satisfies several arithmetical relations. Namely, if  $\mathfrak{g} = \bigoplus_{-3 \leq i \leq 3} \mathfrak{g}\langle i \rangle$  is the  $\mathbb{Z}$ -grading associated with an  $\mathfrak{sl}_2$ -triple containing  $e \in \mathbb{O}$ , then  $\dim \mathfrak{g}\langle 3 \rangle = 2$  and  $\dim \mathfrak{g}\langle 1 \rangle = 2 \dim \mathfrak{g}\langle 2 \rangle$ . Furthermore, the weighted Dynkin diagram of  $\mathbb{O}$  can explicitly be described. Let  $\beta$  be the unique simple root that is not orthogonal to  $\theta$  and let  $\{\alpha_i\}$  be all simple roots adjacent to  $\beta$  on the Dynkin diagram of  $\mathfrak{g}$ . Then one has to put '1' at all  $\alpha_i$ 's and '0' at all other simple roots (Theorem 4.5). It is curious that these properties of  $\mathbb{O}$  enables us to give an intrinsic construction of  $G_2$ -grading in each simple  $\mathfrak{g}$  whose highest root is fundamental.

In Section 5, the problem of computing the algebra of covariants on a nilpotent orbit is

discussed. Let  $\mathfrak{g} = \bigoplus_i \mathfrak{g}\langle i \rangle$  be the  $\mathbb{Z}$ -grading associated with  $e$ . Set  $L = \exp \mathfrak{g}\langle 0 \rangle \subset G$ . Let  $\mathfrak{u}_- \oplus \mathfrak{t} \oplus \mathfrak{u}_+$  be a triangular decomposition of  $\mathfrak{g}$  such that  $\mathfrak{t} \subset \mathfrak{g}\langle 0 \rangle$  and  $\mathfrak{u}_+ \supset \mathfrak{g}\langle \geq 1 \rangle$ . Put  $U_- = \exp(\mathfrak{u}_-)$  and  $U(L)_- = \exp(\mathfrak{u}_- \cap \mathfrak{g}\langle 0 \rangle)$ . These are maximal unipotent subgroups in  $G$  and  $L$ , respectively. We show that there is an injective homomorphism  $\tau^\circ : \mathbb{k}[G \cdot e]^{U_-} \rightarrow \mathbb{k}[\mathfrak{g}\langle \geq 2 \rangle]^{U(L)_-}$ , which is birational, i.e., induces an isomorphism of the quotient fields (Theorem 5.3). This result can be restated in the “dominant” form, when the unipotent groups in question are replaced by the opposite ones and the algebra of functions on  $\mathfrak{g}\langle \geq 2 \rangle$  is replaced by the symmetric algebra. Namely, the natural homomorphism  $\hat{\tau}^\circ : \mathbb{k}[G \cdot e]^U \rightarrow \mathcal{S}(\mathfrak{g}\langle \geq 2 \rangle)^{U(L)}$  is injective and birational. A complete answer is obtained if  $\text{ht}(e) = 2$ . In this case, it is proved that  $\tau^\circ$  is an isomorphism and  $\mathbb{k}[\overline{G \cdot e}]$  is a free  $\mathbb{k}[\overline{G \cdot e}]^U$ -module, see Theorem 5.6. It is also shown how one can quickly determine the monoid of highest weights of all simple  $G$ -modules in the algebra of regular functions  $\mathbb{k}[\overline{G \cdot e}]$ . The case of height 3 is more complicated. However, we have a general conjecture describing the range of  $\tau^\circ$ , if  $G \cdot e$  is spherical, see (5.10). If true, this conjecture allows us to obtain a complete description of  $\mathbb{k}[G \cdot e]^U$ . The agreement of this description with known results for orbits in exceptional Lie algebras is in my view a significant evidence for the validity of Conjecture 5.10. One of the advantages of this method is that no computer calculations is needed. Furthermore, our approach coupled with results of [AHV] yields a description of the algebra of polynomial functions on the model orbit in  $E_8$  as *graded*  $G$ -module. Our computations for the spherical nilpotent orbits are gathered in two Tables in Section 6.

*Notation.* Algebraic groups are denoted by capital Latin characters and their Lie algebras are usually denoted by the corresponding small Gothic characters;  $H^0$  is the identity component of an algebraic group  $H$ . If  $H \hookrightarrow GL(V)$  and  $v \in V$ , then  $H \cdot v$  is the orbit and  $H_v$  is the isotropy group of  $v$ ;  $\mathfrak{h}_v$  is the stationary subalgebra of  $v$  in  $\mathfrak{h} = \text{Lie } H$ .

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## 1. ON INVARIANTS OF SYMPLECTIC REPRESENTATIONS

Let  $H$  be a reductive group with a maximal torus  $T_H$ . Fix also a triangular decomposition of the Lie algebra:  $\mathfrak{h} = \mathfrak{n}_+ \oplus \mathfrak{t}_H \oplus \mathfrak{n}_-$ . Write  $V_\lambda$  for the simple  $H$ -module with highest weight  $\lambda$ . We begin with recalling a standard fact on finite-dimensional representations.

**1.1 Lemma.** *Let  $V_{\lambda_1}$  and  $V_{\lambda_2}$  be two simple  $H$ -modules. If  $\mu_i$  is an arbitrary weight of  $V_{\lambda_i}$  with respect to  $T_H$ , then  $\mu_1 + \mu_2$  is a weight of  $V_{\lambda_1 + \lambda_2}$ .*

We are going to prove a general result concerning symplectic representations without invariants. The reader may observe that the next proof can be made shorter, if one invokes the classification of simple algebraic groups. Since our intention is to argue in a classification-free way, we prefer to give a longer but “pure” argument.

**1.2 Theorem.** *Let  $H \rightarrow Sp(V)$  be a faithful symplectic representation of a reductive group  $H$ . Suppose  $\mathbb{k}[V]^H = \mathbb{k}$ . Then there exists a decomposition  $V = \bigoplus_{i=1}^l V_i$  such that  $H = \prod_{i=1}^l Sp(V_i)$ .*

*Proof.* 1°. Without loss of generality, we may assume that  $H$  is connected. Indeed, if the conclusion holds for  $H^0$ , then this automatically implies that  $H = H^0$ . Since  $(\mathcal{S}^2 V)^H = 0$  and  $\mathcal{S}^2 V \simeq \mathfrak{sp}(V)$ , we see that the centre of  $H$  is finite. Hence  $H$  is semisimple. Let  $\langle \cdot, \cdot \rangle$  denote a skew-symmetric  $H$ -invariant bilinear form on  $V$ .

2°. Assume first that  $V = V_\lambda$  is a *simple*  $H$ -module. Let  $\Delta$  be the root system with respect to  $T_H$  and  $\Delta^+$  the subset of positive roots corresponding to  $\mathfrak{n}_+$ . Let  $v_\lambda$  (resp.  $v_{-\lambda}$ ) be a highest (resp. lowest) weight vector in  $V_\lambda$ . Let  $L_\lambda$  denote the standard Levi subgroup in the parabolic subgroup that stabilises the line  $\mathbb{k}v_\lambda$ . Set  $L'_\lambda = L_\lambda \cap H_{v_\lambda}$ . It is a codimension-1 reductive subgroup of  $L_\lambda$ . Consider the  $H$ -orbit of  $w = v_\lambda + v_{-\lambda}$ . Because  $\mathbb{k}[V]^H = \mathbb{k}$ , each  $H$ -orbit is unstable, i.e., its closure contains the origin. In particular,  $\overline{H \cdot w} \ni 0$ . It is clear that  $L'_\lambda \subset H_w$ . By Luna's criterion [Lu, Cor. 2], we have

$$\overline{H \cdot w} \ni 0 \Leftrightarrow \overline{N_H(L'_\lambda) \cdot w} \ni 0.$$

As  $\text{rk } L'_\lambda = \text{rk } H - 1$ , there are only two possibilities for  $N_H(L'_\lambda)^0$ . It is either  $T_H$  or a subgroup of semisimple rank 1. Obviously,  $T_H \cdot w$  is closed. Therefore the second possibility must occur, i.e.,  $N_H(L'_\lambda)$  is locally isomorphic to  $SL_2 \times \{\text{torus}\}$ . Moreover, the difference  $\lambda - (-\lambda) = 2\lambda$  must be a root of  $SL_2$  and hence of  $H$ . Thus, our intermediate conclusion is: *If  $V_\lambda$  is a simple symplectic  $H$ -module without non-constant invariants, then  $2\lambda =: \alpha \in \Delta^+$ .*

The previous conclusion shows that only one simple component of  $H$  can act non-trivially on  $V_\lambda$ . Hence  $H$  is a *simple* algebraic group. Since  $\alpha$  is a dominant root of  $H$ , it is either the highest (long) root or the highest short root. Assume  $\alpha$  is short, and let  $\beta$  be the highest root. Then  $(\alpha, \beta^\vee) = 1$  and hence  $(\lambda, \beta^\vee) = 1/2$ , which is impossible. Hence  $\alpha$  is the highest root in  $\Delta^+$ . As  $\lambda = \frac{1}{2}\alpha$ , we see that  $(\lambda, \nu^\vee) \leq 1$  for any  $\nu \in \Delta^+$ , i.e.,  $\lambda$  is minuscule. Therefore, all  $T_H$ -weights in  $V_\lambda$  are simple and nonzero. We have  $\mathfrak{h} = V_{2\lambda}$ , and the embedding of Lie algebras  $\mathfrak{h} \subset \mathfrak{sp}(V_\lambda)$  is nothing but the embedding  $V_{2\lambda} \subset \mathcal{S}^2 V_\lambda$ .

Let  $T$  be a maximal torus of  $Sp(V_\lambda)$  such that  $T_H \subset T$ . Let  $\{\pm \varepsilon_i \mid i = 1, \dots, n\}$  be the weights of  $T$  in  $V_\lambda$ . Set  $\mu_i = \varepsilon_i|_{T_H}$ . Since  $\lambda$  is minuscule, we have  $\mu_i \neq 0$  and  $\pm \mu_i \pm \mu_j \neq 0$  ( $i \neq j$ ). It follows from Lemma 1.1 that  $\pm \mu_i \pm \mu_j$  ( $i \neq j$ ) and  $\pm 2\mu_i$  are the nonzero weights of  $V_{2\lambda}$ , i.e., the roots of  $\mathfrak{h}$ . This means that, for any two different weights of  $V_\lambda$ , their difference is a root of  $\mathfrak{h}$ . It follows that  $\mathfrak{h} \cdot v_\lambda = V_\lambda$ , i.e. the  $H$ -orbit of highest weight vectors is dense in  $V_\lambda$ . The theory of orbits of highest weight vectors developed in [VP] says that in this situation  $\mathcal{S}^n V_\lambda$  is a simple  $H$ -module for any  $n$ . In particular,  $V_{2\lambda} = \mathcal{S}^2 V_\lambda$  and hence  $\mathfrak{h} = \mathfrak{sp}(V_\lambda)$ .

3°. Assume now that  $V$  is a *reducible*  $H$ -module, and let  $V = V_1 \oplus \dots \oplus V_l$  be a decomposition into the sum of irreducibles. For  $v = (v_1, v_2, \dots, v_l) \in V$ , define the  $H$ -invariant polynomial function  $f_{ij}$  by the formula  $f_{ij}(v) = \langle v_i, v_j \rangle$ . By the assumption,  $f_{ij} \equiv 0$ . Hence the restriction of  $\langle \cdot, \cdot \rangle$  to each  $V_i$  is non-degenerate and  $H \subset Sp(V_1) \times \dots \times Sp(V_l)$ . From the previous part of the proof, it follows that projection of  $H$  to the  $i$ -th factor equals  $Sp(V_i)$ .

If  $H$  has a simple factor that is embedded diagonally in the product  $Sp(V_i) \times Sp(V_j)$  (when  $\dim V_i = \dim V_j$ ), then  $\mathbb{k}[V_i \oplus V_j]^H \neq \mathbb{k}$ . Therefore  $H$  cannot have diagonally embedded factors and hence  $H = Sp(V_1) \times \dots \times Sp(V_l)$ .  $\square$

**1.3 Corollary** (of the Proof). *Let  $V_\lambda$  be a simple symplectic  $H$ -module. The following three conditions are equivalent: a)  $2\lambda$  is a root of  $\mathfrak{h}$ ; b) the orbit  $H \cdot (v_\lambda + v_{-\lambda})$  is unstable; c)  $H = Sp(V_\lambda)$ .*

This Corollary has a useful complement.

**1.4 Proposition.** *Suppose  $H \rightarrow Sp(V_\lambda)$  is an irreducible symplectic representation. If  $\mathbb{k}[V_\lambda]^H \neq \mathbb{k}$ , then  $H$  has an invariant of degree 4.*

*Proof.* 1°. Let  $\Phi$  denote a symmetric non-degenerate  $H$ -invariant bilinear form on  $\mathfrak{h}$  and  $\langle, \rangle$  a symplectic form on  $V_\lambda$ . Consider the bilinear mapping  $\tilde{\psi} : V_\lambda \times V_\lambda \rightarrow \mathfrak{h}$  which is defined by  $\Phi(\tilde{\psi}(v, w), s) = \langle s \cdot v, w \rangle$ ,  $s \in \mathfrak{h}$ ,  $v, w \in V_\lambda$ . We also need the quadratic mapping  $\psi : V_\lambda \rightarrow \mathfrak{h}$ ,  $\psi(v) := \tilde{\psi}(v, v)$ . Define the  $H$ -invariant polynomial of degree 4 by  $F(v) = \Phi(\psi(v), \psi(v))$ . The problem is to prove that  $F \neq 0$ . Actually, we show that  $F(v_\lambda + v_{-\lambda}) \neq 0$ .

2°. By Corollary 1.3, we have  $2\lambda \notin \Delta$ . Therefore  $\mathfrak{h} \cdot v_\lambda \not\cong v_{-\lambda}$ . Then  $\Phi(\psi(v_\lambda), s) = \langle s \cdot v_\lambda, v_\lambda \rangle = 0$  for any  $s \in \mathfrak{h}$ . Hence  $\psi(v_\lambda) = 0$ , and similarly  $\psi(v_{-\lambda}) = 0$ . It then follows from bilinearity that  $\psi(v_\lambda + v_{-\lambda}) = 2\tilde{\psi}(v_\lambda, v_{-\lambda})$ . Next,  $\Phi(\tilde{\psi}(v_\lambda, v_{-\lambda}), \mathfrak{n}_+) = \langle \mathfrak{n}_+ \cdot v_\lambda, v_{-\lambda} \rangle = 0$ , and likewise for  $\mathfrak{n}_-$ . Hence  $\tilde{\psi}(v_\lambda, v_{-\lambda}) \in \mathfrak{t}_H$ .

3°. Without loss of generality assume that  $\langle v_\lambda, v_{-\lambda} \rangle = 1$ . Then  $\Phi(\tilde{\psi}(v_\lambda, v_{-\lambda}), s) = (d\lambda)(s)$  for any  $s \in \mathfrak{t}_H$ . It follows that  $\tilde{\psi}(v_\lambda, v_{-\lambda}) \neq 0$  and furthermore  $\tilde{\psi}(v_\lambda, v_{-\lambda})$  is orthogonal to  $\text{Ker}(d\lambda) \subset \mathfrak{t}_H$  with respect to  $\Phi$ . Since  $\lambda$  is an algebraic character of  $T_H$ , the restriction of  $\Phi$  to  $\text{Ker}(d\lambda)$  is non-degenerate, i.e.,  $\tilde{\psi}(v_\lambda, v_{-\lambda}) \notin \text{Ker}(d\lambda)$ . Therefore

$$0 \neq d\lambda(\tilde{\psi}(v_\lambda, v_{-\lambda})) = \Phi(\tilde{\psi}(v_\lambda, v_{-\lambda}), \tilde{\psi}(v_\lambda, v_{-\lambda})) = \frac{1}{4}F(v_\lambda + v_{-\lambda}).$$

$\square$

**Remarks.** 1. Using Luna's criterion [Lu, Cor. 1], one can prove that if  $H \neq Sp(V_\lambda)$ , then the orbit  $H \cdot (v_\lambda + v_{-\lambda})$  is closed.

2.  $\psi : V_\lambda \rightarrow \mathfrak{h} \simeq \mathfrak{h}^*$  is the moment mapping associated with the symplectic  $H$ -variety  $V_\lambda$ . Similarly, regarding  $V_\lambda \times V_\lambda$  as the cotangent bundle of  $V_\lambda$ , one sees that  $\tilde{\psi}$  is the moment mapping for this  $H$ -variety.

## 2. PROPERTIES OF THE $\mathbb{Z}$ -GRADING ASSOCIATED WITH AN $\mathfrak{sl}_2$ -TRIPLE

Let  $\mathfrak{g}$  be a simple Lie algebra with a fixed triangular decomposition  $\mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{t} \oplus \mathfrak{u}_+$  and  $\Delta$  the corresponding root system. The roots of  $\mathfrak{u}_+$  are positive. Write  $\Delta_+$  (resp.  $\Pi$ ) for the set of positive (resp. simple) roots;  $\theta$  is the highest root in  $\Delta^+$ . The fundamental weight corresponding to the  $i$ -th simple root is denoted by  $\varphi_i$ . The Killing form on  $\mathfrak{g}$  is denoted by  $\Phi$ , and the induced bilinear form on  $\mathfrak{t}_\mathbb{Q}^*$  is denoted by  $(, )$ . For  $x \in \mathfrak{g}$ , let  $Z_G(x)$  and  $\mathfrak{z}_\mathfrak{g}(x)$

denote the centralisers in  $G$  and  $\mathfrak{g}$ , respectively. If  $M$  is a subset of  $\mathfrak{g}$ , then  $\mathfrak{z}_M(x) = \mathfrak{z}_{\mathfrak{g}}(x) \cap M$ . Let  $\mathcal{N} \subset \mathfrak{g}$  be the nilpotent cone. By the Morozov-Jacobson theorem, each nonzero element  $e \in \mathcal{N}$  can be included in an  $\mathfrak{sl}_2$ -triple  $\{e, h, f\}$  (i.e.,  $[e, f] = h$ ,  $[h, e] = 2e$ ,  $[h, f] = -2f$ ). The semisimple element  $h$ , which is called a *characteristic* of  $e$ , determines a  $\mathbb{Z}$ -grading in  $\mathfrak{g}$ :

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}\langle i \rangle,$$

where  $\mathfrak{g}\langle i \rangle = \{x \in \mathfrak{g} \mid [h, x] = ix\}$ . Set  $\mathfrak{g}\langle \geq j \rangle = \bigoplus_{i \geq j} \mathfrak{g}\langle i \rangle$ . We also write  $\mathfrak{l}$  for  $\mathfrak{g}\langle 0 \rangle$  and  $\mathfrak{p}$  for  $\mathfrak{g}\langle \geq 0 \rangle$ . Since all characteristics of  $e$  are  $Z_G(e)$ -conjugate, the properties of this  $\mathbb{Z}$ -grading do not depend on a particular choice of  $h$ .

The orbit  $G \cdot h$  contains a unique element  $h_+$  such that  $h_+ \in \mathfrak{t}$  and  $\alpha(h_+) \geq 0$  for all  $\alpha \in \Pi$ . The Dynkin diagram of  $\mathfrak{g}$  equipped with the numerical labels  $\alpha_i(h_+)$ ,  $\alpha_i \in \Pi$ , at the corresponding nodes is called the *weighted Dynkin diagram* of  $e$ . After Dynkin, it is known (see [Dy, 8.1, 8.3]) that

$$(a) \alpha_i(h_+) \in \{0, 1, 2\};$$

(b)  $\mathfrak{sl}_2$ -triples  $\{e, h, f\}$  and  $\{e', h', f'\}$  are  $G$ -conjugate if and only if  $h$  and  $h'$  are  $G$ -conjugate if and only if their weighted Dynkin diagrams coincide.

The following facts on the structure of  $Z_G(e) \subset G$  and  $\mathfrak{z}_{\mathfrak{g}}(e) \subset \mathfrak{g}$  are standard, see [SS, ch. III] or [CM].

**2.1 Proposition.** *Let  $L$  (resp.  $P$ ) be the connected subgroup of  $G$  with Lie algebra  $\mathfrak{l}$  (resp.  $\mathfrak{p}$ ), and put  $K = Z_G(e) \cap L$ . Then*

- (i)  $K = Z_G(e) \cap Z_G(f) = Z_G(f) \cap L$ , and it is a maximal reductive subgroup in both  $Z_G(e)$  and  $Z_G(f)$ ;  $Z_G(e) \subset P$ ;
- (ii) the Lie algebra  $\mathfrak{z}_{\mathfrak{g}}(e)$  (resp.  $\mathfrak{z}_{\mathfrak{g}}(f)$ ) is positively (resp. negatively) graded;  $\mathfrak{z}_{\mathfrak{g}}(e) = \bigoplus_{i \geq 0} \mathfrak{z}_{\mathfrak{g}}(e)\langle i \rangle$ , where  $\mathfrak{z}_{\mathfrak{g}}(e)\langle i \rangle = \mathfrak{z}_{\mathfrak{g}}(e) \cap \mathfrak{g}\langle i \rangle$ , and likewise for  $\mathfrak{z}_{\mathfrak{g}}(f)$ ;
- (iii) for any  $i$ , there are  $K$ -stable decompositions:

$$\mathfrak{g}\langle i \rangle = \mathfrak{z}_{\mathfrak{g}}(e)\langle i \rangle \oplus [f, \mathfrak{g}\langle i+2 \rangle] \quad \mathfrak{g}\langle i \rangle = \mathfrak{z}_{\mathfrak{g}}(f)\langle i \rangle \oplus [e, \mathfrak{g}\langle i-2 \rangle].$$

In particular,  $\text{ad } e : \mathfrak{g}\langle i-2 \rangle \rightarrow \mathfrak{g}\langle i \rangle$  is injective for  $i \leq 1$  and surjective for  $i \geq 1$ ;

- (iv)  $(\text{ad } e)^i : \mathfrak{g}\langle -i \rangle \rightarrow \mathfrak{g}\langle i \rangle$  is one-to-one;
- (v)  $\dim \mathfrak{z}_{\mathfrak{g}}(e) = \dim \mathfrak{g}\langle 0 \rangle + \dim \mathfrak{g}\langle 1 \rangle$ .

It follows that the reductive group  $K$  is the centraliser of the  $\mathfrak{sl}_2$ -triple  $\{e, h, f\}$ . Notice that  $K$  can be disconnected and  $\mathfrak{z}_{\mathfrak{g}}(e)\langle 0 \rangle = \mathfrak{z}_{\mathfrak{g}}(f)\langle 0 \rangle = \mathfrak{k} = \text{Lie } K$ . By [Pa3, 1.2],

$\mathfrak{g}\langle i \rangle$  is an orthogonal (resp. symplectic)  $K$ -module if  $i$  is even (resp. odd).

In particular,  $\dim \mathfrak{g}\langle i \rangle$  is even whenever  $i$  is odd. Let  $S$  be a generic stabiliser for the orthogonal  $K$ -module  $\mathfrak{g}\langle 2 \rangle$ . By a result of Luna, it is again a reductive group. Formulas for the complexity and rank of  $G \cdot e$  (see [Pa1, 2.3] or [Pa3, 4.2]) exploit the above  $\mathbb{Z}$ -grading and the groups  $K$  and  $S$ . We recall it in Section 3.

The integer  $\max\{i \in \mathbb{N} \mid \mathfrak{g}\langle i \rangle \neq 0\}$  is called the *height* of  $e$  or of the orbit  $G \cdot e$  and is denoted by  $\text{ht}(e)$ . It is clear that  $\text{ht}(e) \geq 2$  for any  $e \neq 0$ , and  $\text{ht}(e) = m$  if and only if  $(\text{ad } e)^m \neq 0$

and  $(\operatorname{ad} e)^{m+1} = 0$ . See [Pa3, Sect. 2] for some results concerning the height.

Whenever we discuss in the sequel a  $\mathbb{Z}$ -grading associated with some  $e \in \mathcal{N}$ , this means that  $e$  is regarded as member of an  $\mathfrak{sl}_2$ -triple and the grading is determined by the corresponding semisimple element.

**2.2 Proposition.** *Suppose  $\operatorname{ht}(e)$  is odd, say  $2d + 1$ . Then  $K$  has an open orbit in  $\mathfrak{g}\langle 2d+1 \rangle$ .*

*Proof.* By a result of Vinberg [Vi, 2.6],  $L$  has finitely many orbits in each  $\mathfrak{g}\langle i \rangle$ ,  $i \neq 0$ . Consider the periodic grading of  $\mathfrak{g}$  that is obtained by assembling together the spaces  $\mathfrak{g}\langle i \rangle$  modulo  $2d+3$ . It is formally defined by the inner automorphism  $\vartheta = \operatorname{Ad}(\exp(\frac{2\pi\sqrt{-1}}{2d+3}h))$ . Let  $\zeta$  be a primitive root of unity, of degree  $2d+3$ . Then  $\mathfrak{g} = \bigoplus_{i=0}^{2d+2} \mathfrak{g}_i$ , where  $\mathfrak{g}_i$  is the eigenspace of  $\vartheta$  corresponding to the eigenvalue  $\zeta^i$ . For a suitable choice of  $\zeta$ , we have  $\mathfrak{g}_0 = \mathfrak{g}\langle 0 \rangle$ ,  $\mathfrak{g}_1 = \mathfrak{g}\langle 1 \rangle$ , and  $\mathfrak{g}_2 = \mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle -2d-1 \rangle$ . Therefore  $L = G_0$  has an open orbit in  $\mathfrak{g}_1$ . It follows from Vinberg's argument (see [Vi, 3.1]) that the same holds for any  $\mathfrak{g}_i$  such that  $\gcd(i, 2d+3) = 1$ . In particular,  $L$  has an open orbit in  $\mathfrak{g}_2$ . By the definition of  $K$ , it is equivalent to the fact that  $K$  has an open orbit in  $\mathfrak{g}\langle -2d-1 \rangle$ . Since  $K$  is reductive, the same holds for  $\mathfrak{g}\langle 2d+1 \rangle$ .  $\square$

The symbol  $\odot$  is used below for almost direct products of algebraic groups. This means that one has a direct sum for the corresponding Lie algebras.

**2.3 Proposition.** *Suppose  $\operatorname{ht}(e) = 2d + 1$ . Then*

1.  $\mathfrak{g}\langle 2d+1 \rangle = \bigoplus_{i=1}^s V_i$  and  $K = \prod_{i=1}^s \operatorname{Sp}(V_i) \odot K_1$ , where the reductive group  $K_1$  acts trivially on  $\mathfrak{g}\langle 2d+1 \rangle$ .
2.  $L = \operatorname{GL}(\mathfrak{g}\langle 2d+1 \rangle) \odot L_1$ , where the reductive group  $L_1$  acts trivially on  $\mathfrak{g}\langle 2d+1 \rangle$ . In particular,  $\mathfrak{g}\langle 2d+1 \rangle$  is a two-orbit  $L$ -module.

*Proof.* 1. Since  $\mathfrak{g}\langle 2d+1 \rangle$  is a symplectic  $K$ -module, this is a straightforward consequence of Theorem 1.2 and Proposition 2.2.

2. Let  $\mathcal{O}$  be the open  $K$ -orbit in  $\mathfrak{g}\langle 2d+1 \rangle$ . It follows from part 1 that its complement has exactly  $s$  irreducible components:

$$(2.4) \quad \mathfrak{g}\langle 2d+1 \rangle \setminus \mathcal{O} = \bigcup_{j=1}^s \left( \bigoplus_{i \neq j} V_i \right).$$

Let  $\tilde{\mathcal{O}} \supset \mathcal{O}$  be the open  $L$ -orbit in  $\mathfrak{g}\langle 2d+1 \rangle$ . By Eq. (2.4), its complement is contained in a union of *proper* subspaces. Because  $\mathfrak{g}$  is a simple Lie algebra,  $\mathfrak{g}\langle 2d+1 \rangle$  is a simple  $L$ -module. Therefore this complement must be the origin (for, any non-trivial irreducible component would generate a proper  $L$ -stable subspace). Thus,  $\mathfrak{g}\langle 2d+1 \rangle$  is a two-orbit  $L$ -module.

It is clear that the  $\mathbb{Z}$ -graded Lie subalgebra  $\mathfrak{g}\langle -2d-1 \rangle \oplus \mathfrak{g}\langle 0 \rangle \oplus \mathfrak{g}\langle 2d+1 \rangle$  is reductive. Set  $\mathfrak{q} = [\mathfrak{g}\langle 2d+1 \rangle, \mathfrak{g}\langle -2d-1 \rangle] \subset \mathfrak{g}\langle 0 \rangle$ . It is easily seen that  $\mathfrak{h} := \mathfrak{g}\langle -2d-1 \rangle \oplus \mathfrak{q} \oplus \mathfrak{g}\langle 2d+1 \rangle$  is a (reductive) Lie algebra. Furthermore, since the representation  $\varrho : \mathfrak{g}\langle 0 \rangle \rightarrow \mathfrak{gl}(\mathfrak{g}\langle 2d+1 \rangle)$  is

irreducible,  $\mathfrak{h}$  is simple and  $\mathfrak{g}\langle 0 \rangle = \mathfrak{q} \oplus \text{Ker}(\varrho)$ . Let  $Q \subset L$  be the connected group with Lie algebra  $\mathfrak{q}$ . Our goal is to prove that  $Q \simeq GL(\mathfrak{g}\langle 2d+1 \rangle)$ . Since  $Q$  acts transitively on  $\mathfrak{g}\langle 2d+1 \rangle \setminus \{0\}$ , this quickly follows from the classification of 3-term gradings of simple Lie algebras. An a priori proof is given in the next Lemma.  $\square$

**2.5 Lemma.** *Let  $\mathfrak{h} = \mathfrak{h}\langle -1 \rangle \oplus \mathfrak{h}\langle 0 \rangle \oplus \mathfrak{h}\langle 1 \rangle$  be a simple  $\mathbb{Z}$ -graded Lie algebra. Suppose the group  $H\langle 0 \rangle$  acts transitively on  $\mathfrak{h}\langle 1 \rangle \setminus \{0\}$ . Then  $H\langle 0 \rangle \simeq GL(\mathfrak{h}\langle 1 \rangle)$  and  $H \simeq GL(V')$ , where  $\dim V' = \dim \mathfrak{h}\langle 1 \rangle + 1$ .*

*Proof.* Set  $V = \mathfrak{h}\langle 1 \rangle$ . Then  $\mathfrak{h}\langle -1 \rangle \simeq V^*$ . Let  $\beta$  (resp.  $\theta$ ) be the lowest (resp. highest) weight of the  $H\langle 0 \rangle$ -module  $V$ . (We assume that  $\mathfrak{t}_H \subset \mathfrak{h}\langle 0 \rangle$  and  $\mathfrak{n}_+ \supset \mathfrak{h}\langle 1 \rangle = V$ .) It is clear that  $\theta$  is the highest root and  $\beta$  is a simple root of  $\mathfrak{h}$ . Let  $\Pi'$  be the set of all other simple roots. Write  $\theta = n_\beta \beta + \sum_{\gamma \in \Pi'} n_\gamma \gamma$ . It is well known that  $\beta$  is long and  $n_\beta = 1$ . By assumption,  $[\mathfrak{h}, v] = V$  for any  $v \in V \setminus \{0\}$ . It follows that  $[\xi, v] \neq 0$  for any  $v \in V \setminus \{0\}$  and  $\xi \in V^* \setminus \{0\}$ . In particular,  $[\xi_{-\beta}, v_\theta] \neq 0$ . Hence  $\theta - \beta$  is a (positive) root of  $\mathfrak{h}\langle 0 \rangle$  and  $(\theta, \beta^\vee) = (\theta^\vee, \beta) = 1$ . We have

$$2 = (\theta, \theta^\vee) = 1 + \left( \sum_{\gamma \in \Pi'} n_\gamma \gamma, \theta^\vee \right).$$

It follows that there exists a unique  $\beta' \in \Pi'$  such that  $(\beta', \theta^\vee) = 1$  and  $n_{\beta'} = 1$ . This means that  $\beta'$  is long as well. It is well known that the support of any root on the Dynkin diagram is connected. Therefore  $\beta$  and  $\beta'$  are extreme roots on the Dynkin diagram of  $\mathfrak{h}$ . Hence  $\theta - \beta$  is the highest root in the irreducible root system with basis  $\Pi'$ . Let  $\beta_2$  be the unique simple root adjacent to  $\beta$  on the Dynkin diagram. If  $\beta_2 = \beta'$ , then  $\mathfrak{h} \simeq \mathfrak{sl}_3$ , and we are done. Otherwise, we obtain  $((\theta - \beta)^\vee, \beta_2) = ((\theta - \beta)^\vee, \beta') = 1$ , so that we may argue by induction on the length of the unique chain connecting  $\beta$  and  $\beta'$  on the Dynkin diagram. Finally, we obtain that  $\theta = \sum_{1 \leq i \leq n} \beta_i$ , where  $\beta = \beta_1$ ,  $\beta' = \beta_n$ , and  $\Pi' = \{\beta_2, \dots, \beta_n\}$ . From this, all assertions follows easily.  $\square$

**Remark.** The integer  $s$  appearing in Proposition 2.3 is the number of irreducible constituents of the  $K$ -module  $\mathfrak{g}\langle 2d+1 \rangle$ . Explicit description of the nilpotent elements with odd height shows that one always has  $s = 1$ , i.e.,  $\mathfrak{g}\langle 2d+1 \rangle$  is a simple  $K$ -module. In case  $d = 1$ , this can be proved a priori, see Section 3.

### 3. ON NILPOTENT ORBITS OF HEIGHT 3

Recall that, for any irreducible  $G$ -variety  $X$ , one defines the complexity of  $X$ , denoted  $c_G(X)$ , and the rank of  $X$ , denoted  $r_G(X)$ . Here  $c_G(X)$  is the minimal codimension of orbits in  $X$  of a Borel subgroup of  $G$ . If  $X$  is quasiaffine (which we only need), then  $r_G(X)$  is the dimension of the  $\mathbb{k}$ -vector space in  $\mathfrak{t}_\mathbb{Q}^*$  generated by the highest weights of all simple  $G$ -modules occurring in  $\mathbb{k}[X]$ , see [Pa4, ch. 1] for all this.

Keep the notation introduced in Section 2. In the context of nilpotent orbits, the following

was proved in [Pa1, Theorem 2.3]:

$$(3.1) \quad c_G(G \cdot e) = c_L(L/K) + c_S(\mathfrak{g}\langle \geq 3 \rangle) ,$$

$$(3.2) \quad r_G(G \cdot e) = r_L(L/K) + r_S(\mathfrak{g}\langle \geq 3 \rangle) .$$

Let  $G \cdot e$  be an orbit of height 3. Here  $\mathfrak{k}$  is a symmetric subalgebra of  $\mathfrak{l}$  [Pa1, 3.3]; hence  $L/K$  is an  $L$ -spherical homogeneous space. In this particular case, Eq. (3.1) says that

The orbit  $G \cdot e$  is  $G$ -spherical if and only if  $\mathfrak{g}\langle 3 \rangle$  is a spherical  $S$ -module, i.e., a Borel subgroup of  $S$  has an open orbit in  $\mathfrak{g}\langle 3 \rangle$ .

**3.3 Theorem.** *Suppose  $\text{ht}(e) = 3$ . Then  $G \cdot e$  is spherical.*

*Proof.* By Proposition 2.3, we have  $\mathfrak{g}\langle 3 \rangle = \bigoplus_{i=1}^s V_i$ ,  $K = \prod_{i=1}^s Sp(V_i) \odot K_1$ , and  $L = GL(\mathfrak{g}\langle 3 \rangle) \odot L_1$ . Here  $K_1 \subset L_1$  and the composition

$$\prod_{i=1}^s Sp(V_i) \hookrightarrow L \rightarrow GL(\bigoplus_{i=1}^s V_i)$$

yields the standard embedding. Because  $L/K$  is a spherical homogeneous space, we see that  $GL(\bigoplus_{i=1}^s V_i) / \prod_{i=1}^s Sp(V_i)$  is also a spherical homogeneous space. Since the dimension of the space of  $\prod_{i=1}^s Sp(V_i)$ -fixed vectors in the  $GL(\mathfrak{g}\langle 3 \rangle)$ -module  $\wedge^2 \mathfrak{g}\langle 3 \rangle$  equals  $s$ , one must have  $s = 1$ , i.e.,  $K = Sp(\mathfrak{g}\langle 3 \rangle) \odot K_1$ . Moreover, since  $\mathfrak{k}$  is symmetric in  $\mathfrak{l}$ , the group  $Sp(\mathfrak{g}\langle 3 \rangle)$  cannot be embedded diagonally in  $GL(\mathfrak{g}\langle 3 \rangle) \odot L_1$ . Thus,  $Sp(\mathfrak{g}\langle 3 \rangle) \subset GL(\mathfrak{g}\langle 3 \rangle)$  and  $K_1 \subset L_1$ . In particular,  $\mathfrak{k}_1$  is symmetric in  $\mathfrak{l}_1$ . Then the  $K$ -module  $\mathfrak{g}\langle 2 \rangle$  is isomorphic to  $\text{Lie } L / \text{Lie } K = \mathfrak{g}\langle 0 \rangle / \mathfrak{k} \simeq \wedge^2(\mathfrak{g}\langle 3 \rangle) \oplus (\mathfrak{l}_1 / \mathfrak{k}_1)$ , where  $Sp(\mathfrak{g}\langle 3 \rangle)$  acts only on  $\wedge^2(\mathfrak{g}\langle 3 \rangle)$  and  $K_1$  act only on  $\mathfrak{l}_1 / \mathfrak{k}_1$ . Recall that a generic stabiliser of the  $Sp(V)$ -module  $\wedge^2 V$  is isomorphic to  $(SL_2)^{\dim V/2}$ . Therefore  $S \simeq (SL_2)^d \times S_1$ , where  $d = \dim \mathfrak{g}\langle 3 \rangle / 2$  and  $S_1$  is a generic stabiliser for the  $K_1$ -module  $\mathfrak{l}_1 / \mathfrak{k}_1$ . Since  $S_1$  acts trivially on  $\mathfrak{g}\langle 3 \rangle$ , the explicit structure of this group is unimportant. Thus,  $\mathfrak{g}\langle 3 \rangle$  is isomorphic as  $S$ -module to the sum of simplest representations of different  $SL_2$ . Now it is clear that  $\mathfrak{g}\langle 3 \rangle$  is a spherical  $S$ -module, and we are done.  $\square$

Using the notation of the preceding proof, we can give a formula for the rank of  $G \cdot e$ .

**3.4 Proposition.** *If  $\text{ht}(e) \leq 3$ , then  $r_G(G \cdot e) = r_{L_1}(L_1/K_1) + \dim \mathfrak{g}\langle 3 \rangle$ .*

*Proof.* We apply Eq. (3.2). Since  $\mathfrak{g}\langle 3 \rangle$  as  $S$ -module is a sum of simplest representations of different  $SL_2$ , we have  $r_S(\mathfrak{g}\langle 3 \rangle) = (\dim \mathfrak{g}\langle 3 \rangle) / 2$ . Since  $L/K \simeq GL(\mathfrak{g}\langle 3 \rangle) / Sp(\mathfrak{g}\langle 3 \rangle) \times L_1 / K_1$ , we obtain  $r_L(L/K) = (\dim \mathfrak{g}\langle 3 \rangle) / 2 + r_{L_1}(L_1/K_1)$ .  $\square$

*Remark.* If  $\text{ht}(e) = 2$ , then  $\mathfrak{g}\langle 3 \rangle = 0$ ,  $L_1 = L$ ,  $K_1 = K$ , and the formula applies as well. Since  $L_1/K_1$  is a symmetric space, its rank is a familiar object. E.g., the rank is immediately seen from the Satake diagram.

Let  $Q$  be a connected algebraic group with Lie algebra  $\mathfrak{q}$ . Consider the coadjoint representation of  $Q$  in  $\mathfrak{q}^*$ . The transcendence degree of the field  $\mathbb{k}(\mathfrak{q}^*)^Q$  is called the *index* of  $\mathfrak{q}$ . Recall the following two basic results on the index (see [Pa5] for more details).

- – Rais' formula for the index of semi-direct products: Let  $V$  be a  $Q$ -module. The vector space sum  $\mathfrak{q} \oplus V$  has a natural structure of a Lie algebra such that  $V$  is an Abelian ideal



and  $\mathfrak{q}$  is a subalgebra; the bracket of  $\mathfrak{q}$  and  $V$  is determined by the representation of  $\mathfrak{q}$  on  $V$ . The resulting semi-direct product is denoted by  $\mathfrak{q} \rtimes V$ . Then

$$\text{ind}(\mathfrak{q} \rtimes V) = \text{ind} \mathfrak{q}_\nu + \text{trdeg} \mathbb{k}(V^*)^Q$$

where  $\nu \in V^*$  is a generic point.

- – Vinberg's inequality: if  $\xi \in \mathfrak{q}^*$  is arbitrary, then  $\text{ind} \mathfrak{q}_\xi \geq \text{ind} \mathfrak{q}$ .

**3.5 Theorem.** *Suppose  $\text{ht}(e) = 3$ . Then  $\text{ind} \mathfrak{z}_{\mathfrak{g}}(e) = \text{rk} \mathfrak{g}$ .*

*Proof.* Our plan is as follows. Since  $\mathfrak{g} \simeq \mathfrak{g}^*$ , Vinberg's inequality says that  $\text{ind} \mathfrak{z}_{\mathfrak{g}}(e) \geq \text{rk} \mathfrak{g}$ . We find a special point  $\xi \in \mathfrak{z}_{\mathfrak{g}}(e)^*$  such that  $\mathfrak{z}_{\mathfrak{g}}(e)_\xi$  has a semi-direct product structure. Using Raïs' formula and the above structure results for nilpotent orbits of height 3, we compute that  $\text{ind} \mathfrak{z}_{\mathfrak{g}}(e)_\xi = \text{rk} \mathfrak{g}$ . Then the second application of Vinberg's inequality (with  $\mathfrak{q} = \mathfrak{z}_{\mathfrak{g}}(e)$ ) shows that  $\text{ind} \mathfrak{z}_{\mathfrak{g}}(e) \leq \text{rk} \mathfrak{g}$ .

The centraliser of  $e$  has the following graded structure:

$$\mathfrak{z}_{\mathfrak{g}}(e) = \mathfrak{k} \oplus \mathfrak{z}_{\mathfrak{g}}(e)\langle 1 \rangle \oplus \mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle 3 \rangle .$$

Using  $\Phi$ , consider the opposite element of the  $\mathfrak{sl}_2$ -triple,  $f$ , as a linear form on  $\mathfrak{z}_{\mathfrak{g}}(e)$ . It is an easy exercise based on Proposition 2.1 that  $\mathfrak{z}_{\mathfrak{g}}(e)_f = \mathfrak{k} \oplus \mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle 3 \rangle$ . In other words,  $\mathfrak{z}_{\mathfrak{g}}(e)_f \simeq \mathfrak{k} \rtimes (\mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle 3 \rangle)$ . Write  $V$  for  $\mathfrak{g}\langle 3 \rangle$ . We know that  $\mathfrak{g}\langle 0 \rangle \simeq \mathfrak{gl}(V) \oplus \mathfrak{l}_1$  and  $\mathfrak{k} \simeq \mathfrak{sp}(V) \oplus \mathfrak{k}_1$ , where  $\mathfrak{l}_1$  acts trivially on  $V$  and  $\mathfrak{k}_1 \subset \mathfrak{l}_1$ . Moreover,  $\mathfrak{k}_1$  is a symmetric subalgebra of  $\mathfrak{l}_1$  and  $\mathfrak{g}\langle 2 \rangle \simeq \wedge^2 V \oplus (\mathfrak{l}_1/\mathfrak{k}_1)$  as  $\mathfrak{k}$ -module.

The above description shows that  $\mathfrak{z}_{\mathfrak{g}}(e)_f$  is a direct sum of two semi-direct products:

$$\mathfrak{z}_{\mathfrak{g}}(e)_f = (\mathfrak{sp}(V) \rtimes (\wedge^2 V \oplus V)) \dot{+} (\mathfrak{k}_1 \rtimes (\mathfrak{l}_1/\mathfrak{k}_1)) ,$$

where ' $\dot{+}$ ' stands for the direct sum of Lie algebras. By Raïs' formula, the index of the first summand is  $\dim V$  (a generic stabiliser of  $\mathfrak{sp}(V)$  in  $\wedge^2 V^* \oplus V^*$  is a *commutative* subalgebra of dimension  $(\dim V)/2$ ; hence the transcendence degree of the field  $\mathbb{k}(\wedge^2 V^* \oplus V^*)^{\mathfrak{sp}(V)}$  is also  $(\dim V)/2$ ). The index for the second summand is computed in the proof of Theorem 3.5 in [Pa5]. The only important thing here is that  $\mathfrak{k}_1 \rightarrow \mathfrak{so}(\mathfrak{l}_1/\mathfrak{k}_1)$  is the isotropy representation of the symmetric subalgebra  $\mathfrak{k}_1 \subset \mathfrak{l}_1$ . This implies that the second index is equal to  $\text{rk} \mathfrak{l}_1$ . Finally, the above formula for  $\mathfrak{g}\langle 0 \rangle$  shows that  $\dim V + \text{rk} \mathfrak{l}_1 = \text{rk} \mathfrak{g}\langle 0 \rangle = \text{rk} \mathfrak{g}$ .  $\square$

Together with [Pa5, 3.5], this Theorem shows that  $\text{ind} \mathfrak{z}_{\mathfrak{g}}(e) = \text{rk} \mathfrak{g}$  whenever  $G \cdot e$  is spherical.

#### 4. ON A SPECIFIC NILPOTENT ORBIT OF HEIGHT 3

It was noticed in [Pa1] that orbits of height 3 exist in all simple Lie algebras except  $\mathfrak{sl}(V)$  and  $\mathfrak{sp}(V)$ . Furthermore, the height of any nilpotent orbit in  $\mathfrak{sl}(V)$  or  $\mathfrak{sp}(V)$  is even (see [Pa3, 2.3(1)]). Here we give a partial explanation of this phenomenon by giving an intrinsic construction of a nilpotent orbit of height 3 in each simple Lie algebra whose highest root is fundamental, i.e., for  $\mathfrak{g} \neq \mathfrak{sp}(V)$  or  $\mathfrak{sl}(V)$ . This nilpotent orbit, which will be denoted by  $\mathbb{O}$ , satisfies a number of interesting relations, see below.

For  $\gamma \in \Delta$ , let  $e_\gamma \in \mathfrak{g}$  be a nonzero root vector. We assume that  $\Phi(e_\gamma, e_{-\gamma}) = 1$ . If  $\gamma$  is long,

then the orbit  $G \cdot e_\gamma$  is the minimal nonzero nilpotent orbit, but the word ‘nonzero’ is usually omitted in this case. Consider the  $\mathfrak{sl}_2$ -triple  $\{e_\theta, h_\theta = [e_\theta, e_{-\theta}], e_{-\theta}\}$  and the corresponding  $\mathbb{Z}$ -grading  $\bigoplus_{-2 \leq i \leq 2} \mathfrak{g}\langle i \rangle_{(\theta)}$ . As before, we write  $L$  for the connected group with Lie algebra  $\mathfrak{g}\langle 0 \rangle_{(\theta)}$ , etc. This grading was considered by many authors and for various purposes. Our point of view is invariant theoretic, and we want to deduce our results in a classification-free way. Here  $\mathfrak{g}\langle 2 \rangle_{(\theta)} = \mathbb{k}e_\theta$  and therefore  $\mathfrak{k}$  is a subalgebra of codimension 1 in  $\mathfrak{l}$ . More precisely,  $\mathfrak{l} = \mathfrak{k} \oplus \mathbb{k}h_\theta$ . From now on, we assume that  $\theta$  is a fundamental weight. It is worth mentioning that, assuming this constraint and also that  $\text{rk } \mathfrak{g} \geq 4$ , G. Röhrle [R] describes natural bijections between the set of  $L$ -orbits in  $\mathfrak{g}\langle 1 \rangle_{(\theta)}$  and certain double cosets of  $G$  and its Weyl group.

**4.1 Proposition.** *Suppose  $\theta$  is a fundamental weight. Then*

- (i)  $\mathfrak{g}\langle 1 \rangle_{(\theta)}$  is a simple symplectic  $K$ -module;
- (ii) the quotient space  $\mathfrak{g}\langle 1 \rangle_{(\theta)} // K$  is 1-dimensional;
- (iii) the algebra  $\mathbb{k}[\mathfrak{g}\langle 1 \rangle_{(\theta)}]^K$  is generated by a single polynomial of degree 4, say  $F$ . This polynomial is explicitly given by  $F(x) = \Phi((\text{ad } x)^4 e_{-\theta}, e_{-\theta})$ ,  $x \in \mathfrak{g}\langle 1 \rangle_{(\theta)}$ .

*Proof.* (i) We only need to prove that  $\mathfrak{g}\langle 1 \rangle_{(\theta)}$  is simple. For any root vector  $e_\nu \in \mathfrak{g}$ , we have  $[h_\theta, e_\nu] = (\nu, \theta^\vee) e_\nu$ . Let  $\beta$  be the unique simple root such that  $(\theta, \beta) > 0$ . Then  $(\theta^\vee, \beta) = 1$ , i.e.,  $e_\beta \in \mathfrak{g}\langle 1 \rangle_{(\theta)}$ . Since  $[\mathfrak{k}, \mathfrak{k}] = [\mathfrak{l}, \mathfrak{l}]$ , it is clear that  $e_\beta$  is the unique lowest weight vector in  $\mathfrak{g}\langle 1 \rangle_{(\theta)}$  with respect to  $\mathfrak{k} \cap \mathfrak{u}_+$ .

(ii) As  $L$  has finitely many orbits in  $\mathfrak{g}\langle 1 \rangle_{(\theta)}$ , we see that  $\dim(\mathfrak{g}\langle 1 \rangle_{(\theta)} // K) \leq 1$ . Clearly,  $\theta - \beta$  is the highest weight of the  $L$ -module  $\mathfrak{g}\langle 1 \rangle_{(\theta)}$ . For  $\nu \in \mathfrak{t}^*$ , let  $\bar{\nu}$  denote the restriction of  $\nu$  to  $\mathfrak{t} \cap \mathfrak{k}$ . Then  $\overline{\theta - \beta}$  (resp.  $\bar{\beta}$ ) is the highest (resp. lowest) weight of the  $\mathfrak{k}$ -module  $\mathfrak{g}\langle 1 \rangle_{(\theta)}$ . (Notice that  $\bar{\theta} = 0$ .) Because  $\theta$  is the fundamental weight corresponding to  $\beta$ , we have  $(\beta^\vee, \theta) = 1$ . Hence  $(\beta^\vee, \theta) = (\theta^\vee, \beta)$  and therefore  $\beta$  is a long root. Hence  $\theta - 2\beta \notin \Delta$ , i.e.  $\overline{\theta - \beta} - \bar{\beta}$  is not a root of  $\mathfrak{k}$ . It then follows from Corollary 1.3 that  $\mathbb{k}[\mathfrak{g}\langle 1 \rangle_{(\theta)}]^K \neq \mathbb{k}$ .

(iii) It follows from (ii) that  $\mathbb{k}[\mathfrak{g}\langle 1 \rangle_{(\theta)}]^K$  is a polynomial algebra. Since  $\mathfrak{g}\langle 1 \rangle_{(\theta)}$  is simple and symplectic, there are no  $K$ -invariants of degree 1 and 2. On the other hand, Proposition 1.4 says that there is an invariant of degree 4. Therefore, this invariant must generate  $\mathbb{k}[\mathfrak{g}\langle 1 \rangle_{(\theta)}]^K$ . A formula for  $F$  in terms of  $\Phi$  and a quadratic mapping  $\psi$  is given in the proof of (1.4). In our case,  $\psi : \mathfrak{g}\langle 1 \rangle_{(\theta)} \rightarrow \mathfrak{k} \subset \mathfrak{g}\langle 0 \rangle_{(\theta)}$  is given by  $\psi(x) = (\text{ad } x)^2 e_{-\theta}$ . Then the required formula for  $F$  follows by the  $\mathfrak{g}$ -invariance of  $\Phi$ .  $\square$

**Remarks.** 1. If  $\theta$  is fundamental, then  $K = (L, L)$ , and it is connected and semisimple.

2. For  $\mathfrak{g} = \mathfrak{sp}_{2n}$ , we have  $\theta = 2\varphi_1$ . Here  $\mathbb{k}[\mathfrak{g}\langle 1 \rangle_{(\theta)}]^K = \mathbb{k}$ .

3. For  $\mathfrak{g} = \mathfrak{sl}_{n+1}$  ( $n \geq 2$ ), we have  $\theta = \varphi_1 + \varphi_n$ . Here the  $K$ -module  $\mathfrak{g}\langle 1 \rangle_{(\theta)}$  is not simple, and  $\mathbb{k}[\mathfrak{g}\langle 1 \rangle_{(\theta)}]^K$  is generated by a polynomial of degree 2.

**4.2 Theorem.** *Suppose  $\theta$  is fundamental. Then*

- (i) The open  $L$ -orbit in  $\mathfrak{g}\langle 1 \rangle_{(\theta)}$ , say  $\tilde{\mathcal{O}}$ , is affine. We have  $\tilde{\mathcal{O}} = \{x \in \mathfrak{g}\langle 1 \rangle_{(\theta)} \mid \text{ht}(x) = 4\}$ . If  $\tilde{e}$  is any element of  $\tilde{\mathcal{O}}$ , then  $2h_\theta$  is a characteristic of  $\tilde{e}$ ,  $\dim(\text{Im}(\text{ad } \tilde{e})^4) = 1$ , and  $\dim(\text{Im}(\text{ad } \tilde{e})^3) = 2$ .
- (ii) The complement of  $\tilde{\mathcal{O}}$  in  $\mathfrak{g}\langle 1 \rangle_{(\theta)}$  is an irreducible variety. If  $\mathcal{O}$  is the dense  $L$ -orbit in  $\mathfrak{g}\langle 1 \rangle_{(\theta)} \setminus \tilde{\mathcal{O}}$ , then  $\text{ht}(e) = 3$  and  $\dim(\text{Im}(\text{ad } e)^3) = 2$  for any  $e \in \mathcal{O}$ .

*Proof.* (i) Since  $K = (L, L)$  has invariants on  $\mathfrak{g}\langle 1 \rangle_{(\theta)}$ , we can apply [Kac, Prop. 3.3(1)]. That Proposition describes how  $L$ -orbits break into  $K$ -orbits. Namely, the open  $L$ -orbit is affine and it consists of a 1-parameter family of *closed*  $K$ -orbits. All other  $K$ -orbits are unstable and they coincide with  $L$ -orbits. Thus, the complement of the open  $L$ -orbit in  $\mathfrak{g}\langle 1 \rangle_{(\theta)}$  is given by the equation  $F = 0$ . From Proposition 4.1(iii), it follows that  $F(x) \neq 0$  if and only if  $(\text{ad } x)^4 e_{-\theta} \neq 0$ , i.e.,  $\text{ht}(x) = 4$ . For  $\tilde{e} \in \tilde{\mathcal{O}}$ , we have  $\mathfrak{z}_l(\tilde{e}) = \mathfrak{z}_\mathfrak{k}(\tilde{e})$ . It is a reductive Lie algebra, since  $K \cdot \tilde{e}$  is closed. Therefore  $\mathfrak{z}_l(\tilde{e})$  is orthogonal to  $h_\theta$  with respect to  $\Phi$ . Then Proposition 1.2 in [Kac] says that the line  $\mathbb{k}h_\theta$  is in the image of  $\text{ad } \tilde{e}$ . Hence  $2h_\theta$  is a characteristic of  $\tilde{e}$ . The last two equalities easily follow from this and the fact that  $\dim \mathfrak{g}\langle 2 \rangle_{(\theta)} = 1$ .

(ii) Since  $K$  is semisimple,  $F$  is an irreducible polynomial. Hence  $\mathfrak{g}\langle 1 \rangle_{(\theta)} \setminus \tilde{\mathcal{O}}$  is irreducible. If  $F(x) = 0$ , then  $\text{ht}(x) \leq 3$ . However it is not yet clear that there does exist a point with height 3. To see this, consider the differential of  $F$ . It is easy to derive a formula for  $dF_x$ . If  $y \in \mathfrak{g}\langle 1 \rangle_{(\theta)}$ , then

$$(4.3) \quad dF_x(y) = 2\Phi\left((\text{ad } y)(\text{ad } x)^3 e_{-\theta}, e_{-\theta}\right) + 2\Phi\left((\text{ad } x)(\text{ad } y)(\text{ad } x)^2 e_{-\theta}, e_{-\theta}\right).$$

If  $\text{ht}(x) = 2$ , then the first summand is zero. Some standard transformations show that the same is true for the second summand. Indeed,  $(\text{ad } x)(\text{ad } y)(\text{ad } x)^2 e_{-\theta} = (\text{ad } y)(\text{ad } x)^3 e_{-\theta} + (\text{ad } [x, y])(\text{ad } x)^2 e_{-\theta}$ . Notice that  $[x, y] = ce_\theta$ . If  $c = 0$ , then we are done. If not, then up to a scalar factor, the second summand in Eq. (4.3) is equal to

$$\Phi((\text{ad } e_\theta)(\text{ad } x)^2 e_{-\theta}, e_{-\theta}) = -\Phi((\text{ad } x)^2 e_{-\theta}, h_\theta) = -\Phi((\text{ad } x)e_{-\theta}, x) = 0.$$

Thus,  $\text{ht}(e) = 3$  whenever  $dF_e \neq 0$ . The latter holds for  $e$  in the dense  $L$ -orbit in the hypersurface  $\{F = 0\}$ . Since  $G \cdot e$  lies in the closure of  $G \cdot \tilde{e}$ , we have  $0 < \dim(\text{Im}(\text{ad } e)^3) \leq \dim(\text{Im}(\text{ad } \tilde{e})^3) = 2$ . Consider a  $\mathbb{Z}$ -grading associated with  $e$ :  $\mathfrak{g} = \bigoplus_{-3 \leq i \leq 3} \mathfrak{g}\langle i \rangle$ . Since  $\dim \mathfrak{g}\langle 3 \rangle$  is even (see Sect. 2) and  $\mathfrak{g}\langle 3 \rangle = \text{Im}(\text{ad } e)^3$ , the dimension in question is 2.  $\square$

Set  $\tilde{\mathcal{O}} = G \cdot \tilde{\mathcal{O}}$  and  $\mathcal{O} = G \cdot \mathcal{O}$ . We have characterised  $\tilde{\mathcal{O}}$  as the nilpotent orbit whose weighted Dynkin diagram is twice that of the minimal nilpotent orbit. We also described  $\mathcal{O}$  as the unique nilpotent orbit that is dense in  $\mathfrak{g}\langle 1 \rangle_{(\theta)} \setminus (\tilde{\mathcal{O}} \cap \mathfrak{g}\langle 1 \rangle_{(\theta)})$ . Below, we give a more direct characterisation of  $\mathcal{O}$ .

#### 4.4 Proposition.

1.  $G \cdot \mathfrak{g}\langle 1 \rangle_{(\theta)}$  is closed in  $\mathfrak{g}$ ;
2. For any  $x \in \mathfrak{g}\langle 1 \rangle_{(\theta)} \setminus \{0\}$ , we have  $\dim G \cdot x = 2 \dim L \cdot x + 2$ ;
3.  $\mathcal{O}$  is the unique  $G$ -orbit of codimension 2 in the closure of  $\tilde{\mathcal{O}}$ .

*Proof.* 1. Since  $\mathfrak{g}\langle \geq 1 \rangle_{(\theta)}$  is the nilpotent radical of a parabolic,  $G \cdot (\mathfrak{g}\langle \geq 1 \rangle_{(\theta)})$  is closed. Recall that  $\mathfrak{g}\langle \geq 1 \rangle_{(\theta)}$  is a Heisenberg Lie algebra (see [Jos]) and  $\mathfrak{g}\langle 2 \rangle_{(\theta)} = \mathbb{k}e_\theta$ . Therefore if  $x = x_1 + x_2 \in \mathfrak{g}\langle \geq 1 \rangle$  and  $x_1 \neq 0$ , then  $G \cdot x$  has a representative in  $\mathfrak{g}\langle 1 \rangle_{(\theta)}$ . Finally,  $G \cdot e_\theta$  has a representative in  $\mathfrak{g}\langle 1 \rangle_{(\theta)}$ , since  $\mathfrak{g}\langle 1 \rangle_{(\theta)}$  contains long root spaces.

2. Let us look at the graded structure of  $\mathfrak{z}_{\mathfrak{g}}(x)$ . Using the Kostant-Kirillov form associated with  $x$ , one sees that  $\dim \mathfrak{g}\langle i \rangle_{(\theta)} - \dim \mathfrak{z}_{\mathfrak{g}}(x)\langle i \rangle = \dim \mathfrak{g}\langle -i-1 \rangle_{(\theta)} - \dim \mathfrak{z}_{\mathfrak{g}}(x)\langle -i-1 \rangle$  for all  $i$ . Clearly,  $\mathfrak{z}_{\mathfrak{g}}(x)\langle 2 \rangle = \mathfrak{g}\langle 2 \rangle_{(\theta)}$ ,  $\dim \mathfrak{z}_{\mathfrak{g}}(x)\langle 1 \rangle = \dim \mathfrak{g}\langle 1 \rangle_{(\theta)} - 1$ , and  $\dim L \cdot x = \dim \mathfrak{g}\langle 0 \rangle_{(\theta)} - \dim \mathfrak{z}_{\mathfrak{g}}(x)\langle 0 \rangle$ . Now, the conclusion is obtained by a simple counting.

3. By part (1),  $\overline{\mathbb{O}} = G \cdot \mathfrak{g}\langle 1 \rangle_{(\theta)}$ , and we have proved in Theorem 4.2(ii) that  $\mathbb{O}$  is the unique  $L$ -orbit of codimension 1 in  $\mathfrak{g}\langle 1 \rangle_{(\theta)}$ .  $\square$

*Remark.* Since  $\mathbb{O}$  is  $G$ -spherical, Proposition 4.4(3) implies that  $c_G(\tilde{\mathbb{O}}) \leq 2$ . It can conceptually be proved that the complexity in question equals 2. Moreover,  $\mathbb{O}$  is *the* unique minimal non-spherical orbit, if  $\theta$  is fundamental and  $\mathfrak{g} \neq F_4$ , see [Pa3, 4.4].

Now, we are able to describe the weighted Dynkin diagram and other properties of  $\mathbb{O}$ .

**4.5 Theorem.** Let  $e \in \mathbb{O}$  and let  $\bigoplus_{i=-3}^3 \mathfrak{g}\langle i \rangle$  be the  $\mathbb{Z}$ -grading associated with an  $\mathfrak{sl}_2$ -triple  $\{e, h, f\}$ . Then

- (i)  $\dim \mathfrak{g}\langle 3 \rangle = 2$  and  $\dim \mathfrak{g}\langle 1 \rangle = 2 \dim \mathfrak{g}\langle 2 \rangle$ ;
- (ii) Let  $\beta \in \Pi$  be the unique root such that  $(\theta, \beta) > 0$ , and let  $\{\alpha_i\}$  ( $i \in I$ ) be all simple roots adjacent to  $\beta$  on the Dynkin diagram. Then weighted Dynkin diagram of  $\mathbb{O}$  is obtained by attaching ‘1’ to the all  $\alpha_i$ ’s ( $i \in I$ ) and ‘0’ to all other simple roots.

*Proof.* (i) The first equality is already proved in Theorem 4.2(ii). To prove the second, we construct a bi-grading of  $\mathfrak{g}$ . Take a nonzero element  $x \in \mathfrak{g}\langle 3 \rangle$ . By a modification of the Morozov-Jacobson theorem, one can find an  $\mathfrak{sl}_2$ -triple  $\{x, \tilde{h}, y\}$  such that  $\tilde{h} \in \mathfrak{g}\langle 0 \rangle$  and  $y \in \mathfrak{g}\langle -3 \rangle$ . Recall that  $\mathfrak{g}\langle 3 \rangle \setminus \{0\}$  is a single  $L$ -orbit, see Proposition 2.3(2). Therefore  $x$  lies in the minimal nilpotent orbit and  $\text{ht}(x) = 2$ . The pair of commuting elements  $h, \tilde{h}$  determines a refinement of the original  $\mathbb{Z}$ -grading:

$$\mathfrak{g} = \bigoplus_{i,j} \mathfrak{g}\langle i, j \rangle,$$

where  $\mathfrak{g}\langle i, j \rangle = \{z \in \mathfrak{g} \mid [h, z] = iz, [\tilde{h}, z] = jz\}$ ,  $-3 \leq i \leq 3$ , and  $-2 \leq j \leq 2$ . We write  $\mathfrak{g}\langle i, * \rangle$  for  $i$ -th eigenspace of  $h$  and  $\mathfrak{g}\langle *, j \rangle$  for  $j$ -eigenspace of  $\tilde{h}$ . Notice that  $x \in \mathfrak{g}\langle 3, 2 \rangle$  and therefore  $\text{ad } x$  takes  $\mathfrak{g}\langle i, j \rangle$  to  $\mathfrak{g}\langle i+3, j+2 \rangle$ . We are going to show that many subspaces of this bi-grading are equal to zero.

By Proposition 2.1(iv), the linear mappings  $(\text{ad } x)^2 : \mathfrak{g}\langle *, -2 \rangle \rightarrow \mathfrak{g}\langle *, 2 \rangle$  and  $\text{ad } x : \mathfrak{g}\langle *, -1 \rangle \rightarrow \mathfrak{g}\langle *, 1 \rangle$  are one-to-one; therefore  $\mathfrak{g}\langle i, -2 \rangle = 0$  for  $i \geq -2$ ,  $\mathfrak{g}\langle i, -1 \rangle = 0$  for  $i \geq 1$ , and  $\dim \mathfrak{g}\langle i, -1 \rangle = \dim \mathfrak{g}\langle i+3, 1 \rangle$  for  $i \leq 0$ . We also have  $[\mathfrak{g}\langle 0, * \rangle, x] = \mathfrak{g}\langle 3, * \rangle$ , hence  $\mathfrak{g}\langle 3, 0 \rangle = \{0\}$ .

Using the central symmetry for dimensions and the fact that  $\dim \mathfrak{g}\langle 3, * \rangle = 2$ , we obtain

$$\dim \mathfrak{g}\langle 3, 2 \rangle = \dim \mathfrak{g}\langle 3, 1 \rangle = \dim \mathfrak{g}\langle 0, -1 \rangle = \dim \mathfrak{g}\langle 0, 1 \rangle = 1$$

and

$$\dim \mathfrak{g}\langle 2, 1 \rangle = \dim \mathfrak{g}\langle -1, -1 \rangle = \dim \mathfrak{g}\langle 1, 1 \rangle = \dim \mathfrak{g}\langle -2, -1 \rangle =: a$$

Thus, we obtain the following “matrix” of numbers  $\dim \mathfrak{g}\langle i, j \rangle$ , where only possible non-zero entries are indicated.

$$\begin{array}{c|ccccccc} & & & & & & & 1 \\ & & & & & & & \\ 2 & & & & & & & \\ 1 & & & & & 1 & a & a & 1 \\ 0 & & & c & b & d & b & c & \\ -1 & & 1 & a & a & 1 & & & \\ -2 & & 1 & & & & & & \\ \hline j/i & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \end{array}$$

Since  $e$  lies in  $\mathfrak{g}\langle 2, 0 \rangle \oplus \mathfrak{g}\langle 2, 1 \rangle$  and  $\text{ad } e : \mathfrak{g}\langle -1, * \rangle \rightarrow \mathfrak{g}\langle 1, * \rangle$  is one-to-one, we see that  $\text{ad } e$  maps  $\mathfrak{g}\langle -1, -1 \rangle$  injectively to  $\mathfrak{g}\langle 1, 0 \rangle$ . Hence  $a \leq b$ . Now we compute  $\dim \mathfrak{z}_{\mathfrak{g}}(e)$  in two different ways. Using  $i$ -grading and Proposition 2.1(v), we obtain  $\dim \mathfrak{z}_{\mathfrak{g}}(e) = \dim \mathfrak{g}\langle 0, * \rangle + \dim \mathfrak{g}\langle 1, * \rangle = 2 + a + b + d$ . On the other hand, the  $j$ -grading arises as grading connected with the minimal nilpotent orbit. Therefore  $2\tilde{h}$  is a characteristic for  $\tilde{\mathcal{O}}$  (Theorem 4.2(i)) and hence  $\dim \mathfrak{z}_{\mathfrak{g}}(\tilde{e}) = \dim \mathfrak{g}\langle *, 0 \rangle = 2b + 2c + d$  for any  $\tilde{e} \in \tilde{\mathcal{O}}$ . Because  $\dim \mathcal{O} = \dim \tilde{\mathcal{O}} - 2$  (Proposition 4.4(3)), we obtain  $2b + 2c + d = a + b + d$ . Hence  $c = 0$  and  $a = b$ .

(ii) Let  $y$  be a nonzero element of the 1-dimensional space  $\mathfrak{g}\langle 0, 1 \rangle$ . Obviously, the “highest” subspace in each column lies in  $\mathfrak{z}_{\mathfrak{g}}(y)$ ; next, a subspace of codimension 1 in  $\mathfrak{g}\langle 0, 0 \rangle$  also lies there. Hence  $\dim \mathfrak{z}_{\mathfrak{g}}(y) \geq 4a + d + 2$ . On the other hand, we deduce from considering the  $j$ -grading that  $\dim \mathfrak{z}_{\mathfrak{g}}(x) = 4a + d + 2$ . Therefore  $y$  lies in the minimal nilpotent orbit as well, and one has equality for  $\dim \mathfrak{z}_{\mathfrak{g}}(y)$ . This implies that  $\text{ad } y : \mathfrak{g}\langle 1, 0 \rangle \rightarrow \mathfrak{g}\langle 1, 1 \rangle$  is one-to-one.

Without loss of generality, we may assume that the triangular decomposition of  $\mathfrak{g}$  is chosen so that  $h, \tilde{h} \in \mathfrak{t} \subset \mathfrak{g}\langle 0, 0 \rangle$  and  $\mathfrak{u}_+ \supset \mathfrak{g}\langle i, j \rangle$  for all  $(i, j)$  such that  $i + j > 0$ . Then  $x$  becomes a highest weight vector with respect to this choice of  $\Pi$ , i.e.,  $x = e_{\theta}$ . Since  $y$  is a lowest weight vector in the *simple*  $\mathfrak{g}\langle *, 0 \rangle$ -module  $\mathfrak{g}\langle *, 1 \rangle$ , neither of the root spaces in  $\mathfrak{g}\langle i, 1 \rangle$ ,  $i \geq 1$ , corresponds to simple roots. Thus, the simple root spaces lie only in  $\mathfrak{g}\langle 0, 0 \rangle$ ,  $\mathfrak{g}\langle 0, 1 \rangle$  and  $\mathfrak{g}\langle 1, 0 \rangle$ . In particular,  $y = e_{\beta}$  for some  $\beta \in \Pi$ . Let  $\{\alpha_i\}$  be all simple roots such that the corresponding root spaces belong to  $\mathfrak{g}\langle 1, 0 \rangle$ . The argument in the previous paragraph and bi-graded structure of  $\mathfrak{g}$  show that  $\beta$  is the only simple root such that  $(\beta, \theta) \neq 0$  and the  $\alpha_i$ 's are exactly the simple roots that are not orthogonal to  $\beta$ . Thus,  $\alpha(h) \leq 1$  for all  $\alpha \in \Pi$ , and  $\alpha(h) = 1$  if and only if  $(\alpha, \beta) \neq 0$ .  $\square$

**4.6 Remarks.** 1. Once the equalities  $c = 0$ ,  $b = a$  are proved, one may observe that the above “dimension matrix” gains the apparent  $G_2$ -symmetry. For, after making an affine transformation, the matrix can be depicted as follows:

$j$				
2	1			
1	1	$a$	$a$	1
0		$a$	$d$	$a$
-1	1	$a$	$a$	1
-2	1			

Each one-dimensional space represents the highest root space with respect to a suitable choice of  $\Delta^+$ . It is not hard to prove that 6 one-dimensional spaces generate a Lie subalgebra of type  $A_2$ . Next, our nilpotent element  $e$  of height 3 lies in one of  $a$ -dimensional spaces. It can be shown that 6 one-dimensional spaces together with  $e$  generate a Lie algebra of type  $G_2$ . Therefore the  $\mathbb{Z} \times \mathbb{Z}$ -grading constructed in the previous proof yields also an instance of a Lie algebra graded by root system  $G_2$ . (See [Ne] and references therein for the general notion of a Lie algebra graded by a root system.) In this way, one obtains an intrinsic construction of a  $G_2$ -grading for any simple Lie algebra whose highest root is fundamental.

2. Explicit calculations show that one always has  $r_G(\mathbb{O}) = r_G(\tilde{\mathbb{O}}) = \min\{\text{rk } \mathfrak{g}, 4\}$ .

## 5. REGULAR FUNCTIONS ON A NILPOTENT ORBIT

In this section, we propose an approach to describing the algebra of covariants on nilpotent orbits.

Maintain notation of Section 2. Let  $\{e, h, f\}$  be an arbitrary  $\mathfrak{sl}_2$ -triple. Without loss, we may assume that  $\alpha(h) \geq 0$  for all  $\alpha \in \Pi$  (i.e.,  $h = h_+$ ). Then  $\mathfrak{u}_+ = \mathfrak{g}\langle 0 \rangle_+ \oplus \mathfrak{g}\langle \geq 1 \rangle$  and  $\mathfrak{u}_- = \mathfrak{g}\langle 0 \rangle_- \oplus \mathfrak{g}\langle \leq -1 \rangle$ , where  $\mathfrak{g}\langle 0 \rangle_{\pm} = \mathfrak{u}_{\pm} \cap \mathfrak{g}\langle 0 \rangle$ . Let  $\Delta\langle i \rangle$  denote the subset of  $\Delta$  corresponding to  $\mathfrak{g}\langle i \rangle$ . Set

$$U_- = \exp(\mathfrak{u}_-), \quad U = \exp(\mathfrak{u}_+), \quad U(L)_- = \exp(\mathfrak{g}\langle 0 \rangle_-), \quad U(L) = \exp(\mathfrak{g}\langle 0 \rangle_+).$$

These are maximal unipotent subgroups of  $G$  and  $L$ , respectively. Then  $U(L)_- = U_- \cap P = U_- \cap L$  and  $U_- \simeq N \times U(L)_-$ , where  $N = \exp(\mathfrak{g}\langle \leq -1 \rangle)$ . Since  $\mathfrak{g}\langle \geq 2 \rangle$  is a  $P$ -module, one can form the homogeneous vector bundle  $G *_P \mathfrak{g}\langle \geq 2 \rangle$  over  $G/P$ . The following is fairly well known:

$$(5.1) \quad \mathbb{k}[G *_P \mathfrak{g}\langle \geq 2 \rangle] = \mathbb{k}[G \cdot e] = \mathbb{k}[(\overline{G \cdot e})_n],$$

where  $(\ )_n$  denotes the normalisation, and

$$(5.2) \quad \text{the collapsing } G *_P \mathfrak{g}\langle \geq 2 \rangle \rightarrow \overline{G \cdot e} \text{ is a projective birational morphism.}$$

It follows from Eq. (5.1) and (5.2) that  $\mathfrak{g}\langle \geq 2 \rangle$  can be regarded as closed subvariety of  $(\overline{G \cdot e})_n$ . Therefore there is the restriction homomorphism  $\mathbb{k}[G \cdot e] \rightarrow \mathbb{k}[\mathfrak{g}\langle \geq 2 \rangle]$ , which is onto. Clearly, it takes  $U_-$ -invariant functions on  $G \cdot e$  to  $U(L)_-$ -invariant functions on  $\mathfrak{g}\langle \geq 2 \rangle$ . Thus, we obtain the restriction homomorphism

$$\tau^o : \mathbb{k}[G \cdot e]^{U_-} \rightarrow \mathbb{k}[\mathfrak{g}\langle \geq 2 \rangle]^{U(L)_-}.$$

Notice that  $T = \exp(\mathfrak{t})$  is a maximal torus in both  $L$  and  $G$ , and that both subalgebras under consideration are  $T$ -stable.

**5.3 Theorem.**  $\tau^\circ$  is  $T$ -equivariant, injective and birational.

*Proof.* The first claim is obvious. Since  $U_- \cdot P$  is dense in  $G$  and hence  $U_- \cdot \mathfrak{g}\langle \geq 2 \rangle$  contains a dense open subset of  $G \cdot e$ , we see that  $\tau^\circ$  is injective. To prove birationality, we dualize the picture and consider the morphism of the corresponding spectra:

$$\tau : \mathfrak{g}\langle \geq 2 \rangle // U(L)_- \rightarrow G \cdot e // U_- .$$

The birationality of  $\tau$  is equivalent to the fact that there exists a dense open subset  $\mathcal{D}$  of  $\mathfrak{g}\langle \geq 2 \rangle$  such that

$$(*) \quad U_- \cdot v \cap \mathfrak{g}\langle \geq 2 \rangle = U(L)_- \cdot v \text{ for all } v \in \mathcal{D}.$$

We take  $\mathcal{D} = P \cdot e$ . To prove  $(*)$ , we first notice that  $(**)$   $G \cdot e \cap \mathfrak{g}\langle \geq 2 \rangle = P \cdot e$ . Indeed,  $P \cdot e$  is dense in  $\mathfrak{g}\langle \geq 2 \rangle$  in view of Proposition 2.1(iii), and if  $v \in \mathfrak{g}\langle \geq 2 \rangle \setminus P \cdot e$ , then  $\dim Z_P(v) > \dim Z_P(e) = \dim Z_G(e)$ . Second, we notice that if  $v \in P \cdot e$ ,  $g \in N$ , and  $g \cdot v \in \mathfrak{g}\langle \geq 2 \rangle$ , then  $g = 1$ . This follows from  $(**)$  and the containment  $Z_G(e) \subset P$ . Now,  $(*)$  easily follows.  $\square$

The following observation will allow us to switch between  $\mathbb{k}[G \cdot e]^U$  and  $\mathbb{k}[G \cdot e]^{U_-}$ .

**5.4 Lemma.** Given  $e \in \mathcal{N}$ , let  $\Gamma = \Gamma(G \cdot e)$  be the monoid of all highest weights occurring in  $\mathbb{k}[G \cdot e]$ , i.e. the monoid of  $T$ -weights in  $\mathbb{k}[G \cdot e]^U$ . Then the monoid of all lowest weights in  $\mathbb{k}[G \cdot e]$  is  $-\Gamma$ .

*Proof.* By Frobenius reciprocity, this is equivalent to the fact that if  $V$  is a simple  $G$ -module and  $V^{Z_G(e)} \neq 0$ , then  $(V^*)^{Z_G(e)} \neq 0$ . The last assertion follows from the following two properties of a Weyl involution  $\vartheta \in \text{Aut } G$ :

(a) the  $G$ -module  $V$  equipped with the twisted action  $(g, v) \mapsto \vartheta(g) \cdot v$ ,  $g \in G$ ,  $v \in V$ , is isomorphic to  $V^*$ ;

(b)  $\vartheta$  acts trivially on the set of nilpotent orbits [LS, 2.10]. In particular,  $\vartheta(Z_G(e))$  is  $G$ -conjugate to  $Z_G(e)$ .  $\square$

Given a (locally-finite)  $T$ -module  $\mathcal{A}$ , let  $\mathcal{A}_\mu$  denote the  $\mu$ -weight space of  $\mathcal{A}$ . We have  $\mathbb{k}[G \cdot e]^{U_-} = \bigoplus_{\mu \in -\Gamma} \mathbb{k}[G \cdot e]_\mu^{U_-}$  and  $\tau^\circ : \mathbb{k}[G \cdot e]_\mu^{U_-} \hookrightarrow \mathbb{k}[\mathfrak{g}\langle \geq 2 \rangle]_\mu^{U(L)_-}$ .

Let  $\mathcal{X}_+(G)$  (resp.  $\mathcal{X}_-(G)$ ) denote the cone of dominant (resp. antidominant) weights of  $G$ . The mapping  $\tau^\circ$  is not always onto. There are two related difficulties:

(1) since  $L$  has fewer simple roots than  $G$ , the cone of (anti)dominant weights for  $L$  is wider than that for  $G$ . So that  $L$ -lowest weight vectors corresponding to “extra-weights” (if any) cannot lie in the image of  $\tau^\circ$ ;

(2) even for  $\mu \in \mathcal{X}_-(G)$ , the dimensions of  $\mathbb{k}[G \cdot e]_\mu^{U_-}$  and  $\mathbb{k}[\mathfrak{g}\langle \geq 2 \rangle]_\mu^{U(L)_-}$  can be different. To overcome the first difficulty, one may consider the subalgebra of  $\mathbb{k}[\mathfrak{g}\langle \geq 2 \rangle]^{U(L)_-}$  corresponding to  $\mathcal{X}_-(G)$ . Still, this does not yield an isomorphism in general (e.g., in case of the principal nilpotent orbit in  $\mathfrak{sl}_3$ ), because of the presence of the second problem. If  $G \cdot e$  is

spherical, then  $\mathfrak{g}\langle \geq 2 \rangle$  is a spherical  $L$ -module [Pa3, 4.2(2)]. Hence all *nonzero* weight spaces in  $\mathbb{k}[G \cdot e]^{U_-}$  and  $\mathbb{k}[\mathfrak{g}\langle \geq 2 \rangle]^{U(L)_-}$  are 1-dimensional. This partly resolves the second problem and gives some hope that a more precise statement holds in the spherical case. To begin with, consider the orbits of height 2. Then  $\overline{G \cdot e}$  is normal by a result of W. Hesselink [He]. Therefore there is no difference between  $\mathbb{k}[G \cdot e]$  and  $\mathbb{k}[\overline{G \cdot e}]$ . Hesselink used Kempf's theorem on the collapsing of homogeneous bundles with completely reducible fibres. Since this result applies not only to nilpotent orbits in  $\mathfrak{g}$ , we will work for a while in a more general setting. Let  $P$  be a parabolic subgroup of  $G$  and let  $U_-$  be a maximal unipotent subgroup of  $G$  such that  $U_- \cdot P$  is dense in  $G$ . Then  $P \cap U_- =: U(L)_-$  is a maximal unipotent subgroup for some Levi subgroup  $L \subset P$ .

**5.5 Lemma.** *Let  $V$  be a  $G$ -module and  $M \subset V$  a  $P$ -submodule. Suppose  $M$  is completely reducible and the collapsing  $\pi : G *_P M \rightarrow G \cdot M \subset V$  is birational. Then*

$$\mathbb{k}[G \cdot M]^{U_-} \simeq \mathbb{k}[G *_P M]^{U_-} \simeq \mathbb{k}[M]^{U(L)_-} .$$

*Proof.* Since  $\pi$  is proper and birational, we have  $\mathbb{k}[G \cdot M] \simeq \mathbb{k}[G *_P M]$ , and hence  $\mathbb{k}[G \cdot M]^{U_-} \simeq \mathbb{k}[G *_P M]^{U_-}$ .

The second map is given by the restriction to the fibre over  $\{P\}$  of the projection  $G *_P M \rightarrow G/P$ . Obviously, this restriction is injective (even if  $M$  is not completely reducible). Onto-ness follows by an application of the Borel-Weil-Bott theorem that is due to G. Kempf (see Hesselink's version in [He, sect. 3]). Namely,

$$\mathbb{k}[G *_P M] \simeq \bigoplus_{n \geq 0} H^0(G/P, \mathcal{L}(S^n M^*)) .$$

As  $S^n M^*$  is a completely reducible  $P$ -module and  $S^n V^* \rightarrow S^n M^*$  is onto, each simple  $P$ -module (=  $L$ -module)  $E$  in  $S^n M^*$  gives rise to a nontrivial simple  $G$ -module  $H^0(G/P, \mathcal{L}(E))$  in  $\mathbb{k}[G *_P M]$ .  $\square$

Now, we come back to nilpotent orbits.

**5.6 Theorem.** *Suppose  $\text{ht}(e) = 2$  and hence  $\mathfrak{g}\langle \geq 2 \rangle = \mathfrak{g}\langle 2 \rangle$ . Then*

- (i)  $\tau^o : \mathbb{k}[\overline{G \cdot e}]^{U_-} \xrightarrow{\sim} \mathbb{k}[\mathfrak{g}\langle 2 \rangle]^{U(L)_-}$ , and it is a polynomial algebra;
- (ii)  $\mathbb{k}[\mathfrak{g}\langle 2 \rangle]$  is a free  $\mathbb{k}[\mathfrak{g}\langle 2 \rangle]^{U(L)_-}$ -module;
- (iii)  $\mathbb{k}[\overline{G \cdot e}]$  is a free  $\mathbb{k}[\overline{G \cdot e}]^{U_-}$ -module.

*Proof.* (i) Since  $G \cdot e$  is spherical,  $\mathfrak{g}\langle 2 \rangle$  is a spherical  $L$ -module. That  $\mathbb{k}[\mathfrak{g}\langle 2 \rangle]^{U(L)_-}$  is a polynomial algebra is a standard property of spherical representations. Since  $\mathfrak{g}\langle 2 \rangle$  is a completely reducible  $P$ -module, onto-ness of  $\tau^o$  follows from Lemma 5.5.

(ii) This is essentially proved in [Pa2, 4.6]. To get in that situation, one has to consider the ‘‘even’’ reductive subalgebra  $\mathfrak{g}^{ev} = \mathfrak{g}\langle -2 \rangle \oplus \mathfrak{g}\langle 0 \rangle \oplus \mathfrak{g}\langle 2 \rangle \subset \mathfrak{g}$ . Then  $\mathfrak{g}\langle 2 \rangle$  is the Abelian nilpotent radical of  $\mathfrak{p}^{ev} = \mathfrak{g}\langle 0 \rangle \oplus \mathfrak{g}\langle 2 \rangle$ .

(iii) In [Pa2, 5.5], we proved a general sufficient condition for  $\mathbb{k}[X]$  to be a flat  $\mathbb{k}[X]^{U_-}$ -module, where  $G$  is reductive and  $X$  is an affine  $G$ -variety. This reads as follows:



Suppose  $\mathbb{k}[X]^U$  is polynomial and let  $\lambda_1, \dots, \lambda_r$  be the  $T$ -weights of free generators of  $\mathbb{k}[X]^U$ . Suppose the  $\lambda_i$ 's are linearly independent and any fundamental weight (♠) of  $G$  occurs in at most one  $\lambda_i$  (i.e., the  $\lambda_i$ 's depend on pairwise disjoint subsets of fundamental weights). Then  $\mathbb{k}[X]$  is a flat  $\mathbb{k}[X]^U$ -module. (If  $G$  is semisimple, then the linear independence of the  $\lambda_i$ 's follows from the second assumption.)

In the graded situation, “free” is equivalent to “flat”. Hence our aim is to verify this property for  $X = \overline{G \cdot e}$ . Using the isomorphism in part (i) and some results of [Pa2], one can give the explicit description of the  $\lambda_i$ 's. Let  $\gamma_1, \dots, \gamma_r$  be the *upper canonical string* (= u.c.s.) of roots in  $\Delta\langle 2 \rangle$ . By definition, this means that  $\gamma_1 = \theta$ ;  $\gamma_2$  is the maximal root in  $\Delta^{(1)}\langle 2 \rangle = \{\alpha \in \Delta\langle 2 \rangle \mid (\alpha, \gamma_1) = 0\}$ , and so on... The procedure terminates when  $\Delta^{(i)}\langle 2 \rangle$  becomes empty. See [Pa2, Sect. 1] for more details. (Our  $\Delta\langle 2 \rangle$  here is  $\Delta(1)$  in [Pa2]. In course of constructing the u.c.s., one may ignore the subset  $\Delta\langle 1 \rangle$ .) This string consists of pairwise orthogonal long positive roots.

It follows from [Pa2, Corollary 4.2] that the weights of free generators of  $\mathbb{k}[\mathfrak{g}\langle 2 \rangle]^{U(L)-}$  are  $-\gamma_1, -\gamma_1 - \gamma_2, \dots, -\gamma_1 - \dots - \gamma_r$ . More precisely, that Corollary describes the weights for  $\mathbb{k}[\mathfrak{g}\langle 2 \rangle]^{U(L)}$  in terms of the *lower canonical string* (= l.c.s.). Using the longest element in the Weyl group of  $L$ , it is then easy to realise that the description of  $\mathbb{k}[\mathfrak{g}\langle 2 \rangle]^{U(L)-}$  can be given in terms of the u.c.s., as above. Then using (i) and Lemma 5.4, we conclude that the weights of free generators of  $\mathbb{k}[\overline{G \cdot e}]^U$ , i.e., the generators of  $\Gamma$ , are

$$(5.7) \quad \lambda_1 = \gamma_1, \lambda_2 = \gamma_1 + \gamma_2, \dots, \lambda_r = \gamma_1 + \dots + \gamma_r.$$

Thus, we are to verify that, for any  $\alpha \in \Pi$ , there is at most one  $i \in \{1, \dots, r\}$  such that  $(\alpha, \lambda_i) > 0$ . The proof of this will be given in the next Lemma.  $\square$

**5.8 Lemma.** *Suppose  $\text{ht}(e) = 2$  and let  $\gamma_1, \dots, \gamma_r$  be the u.c.s. in  $\Delta\langle 2 \rangle$ . Then for any  $\alpha \in \Pi$ , there is at most one  $i \in \{1, \dots, r\}$  such that  $(\alpha, \gamma_1 + \dots + \gamma_i) > 0$ .*

*Proof.* Let  $\Pi\langle i \rangle$  denote the subset of  $\Pi$  which lies in  $\Delta\langle i \rangle$ . Then  $\Pi\langle 0 \rangle$  is the set of simple roots for  $L$ . For  $\alpha \in \Pi\langle 0 \rangle$ , the desired property has been proved in [Pa2, 4.6]. And this is the key property that yields the freeness in [Pa2] and hence in Theorem 5.6(ii). Dealing with the other simple roots, one has to distinguish two possibilities.

(a)  $\mathfrak{g}\langle 1 \rangle = 0$ , i.e.  $e$  is an even nilpotent element.

Then there is a unique root  $\beta \in \Pi\langle 2 \rangle$ . This  $\beta$  is the lowest weight of the  $L$ -module  $\mathfrak{g}\langle 2 \rangle$  with respect to  $U(L)$ . Since the  $\mathbb{Z}$ -grading of  $\mathfrak{g}$  is associated with of a nilpotent orbit, one can show that the u.c.s. and l.c.s. in  $\Delta\langle 2 \rangle$  coincide as sets (cf. also Prop. 1.5 in [Pa2]). In particular, this means that  $\beta = \gamma_r$ . Hence  $(\beta, \lambda_i) = 0$  if  $i < r$  and  $(\beta, \lambda_r) = (\beta, \beta) > 0$ .

(b)  $\mathfrak{g}\langle 1 \rangle \neq 0$ .

Then there are one or two roots in  $\Pi\langle 1 \rangle$ . Let  $\alpha \in \Pi\langle 1 \rangle$ . It is clear that  $(\alpha, \gamma_i) \geq 0$  for all  $i$ , since  $\Delta\langle j \rangle = \emptyset$  for  $j \geq 3$ .

(b<sub>1</sub>) There is at most one index  $i$  such that  $(\alpha, \gamma_i) > 0$ .

Indeed, if  $(\alpha, \gamma_i) > 0$  and  $(\alpha, \gamma_j) > 0$ , then  $\gamma_i - \alpha \in \Delta\langle 1 \rangle$  and  $(\gamma_i - \alpha, \gamma_j) < 0$ . Whence  $\gamma_i + \gamma_j - \alpha \in \Delta\langle 3 \rangle$ . A contradiction!

( $b_2$ ) Assume that  $(\alpha, \gamma_i) > 0$  for  $i < r$ . Then we also have  $\gamma_{i+1}$ . Since  $\gamma_{i+1} \neq \theta$ , there is  $\nu \in \Pi\langle 0 \rangle$  such that  $\gamma_{i+1} + \nu$  is a root (in  $\Delta\langle 2 \rangle$ ). Notice that  $\gamma_{i+1} + \nu \neq \gamma_i$ , since  $\gamma_i$  and  $\gamma_{i+1}$  are orthogonal long roots. By the definition of the u.c.s., we have  $(\gamma_i, \gamma_{i+1} + \nu) \neq 0$  and hence it is positive. Therefore  $(\gamma_i, \nu) > 0$  and

$$(\diamond) \quad \gamma_i - \gamma_{i+1} - \nu \in \Delta\langle 0 \rangle^+.$$

(This root is positive, since  $\text{ht}(\gamma_i) > \text{ht}(\gamma_{i+1})$ .) We have  $\gamma_i - \alpha \in \Delta\langle 1 \rangle$  and  $(\gamma_i - \alpha, \gamma_{i+1} + \nu) = (\gamma_i, \nu) - (\alpha, \nu) > 0$ . Hence

$$(\diamond\diamond) \quad \gamma_{i+1} + \nu - \gamma_i + \alpha \in \Delta\langle 1 \rangle.$$

Taking the sum of  $(\diamond)$  and  $(\diamond\diamond)$ , we conclude that  $\alpha$  is not simple. A contradiction! Thus,  $(\alpha, \gamma_i) = 0$  if  $i < r$  and  $(\alpha, \gamma_r) \geq 0$ , which implies the assertion.  $\square$

**5.9 Complement** (to Theorem 5.6(i)). *Let  $f_i \in \mathbb{k}[\overline{G \cdot e}]^U$  be the generator with weight  $\lambda_i$ . Then  $\deg f_i = i$ .*

*Proof.* The isomorphism  $\tau^o$  preserve the degree of functions, and the degrees of generators of  $\mathbb{k}[\mathfrak{g}\langle 2 \rangle]^{U(L)}$  were described in [Pa2, 4.1].  $\square$

**Remark.** For  $G = SL_n$ , the fact that  $\mathbb{k}[G \cdot e]^U$  is polynomial was previously proved in [Shm, Sect. 8]. Shmelkin also explicitly describes the generators as certain minors of  $n$  by  $n$  matrices.

Making use of Theorem 5.6, Eq. (5.7) and Complement 5.9, it is quite easy to determine the structure of  $\mathbb{k}[\overline{G \cdot e}]$  as graded  $G$ -module, if  $\text{ht}(e) = 2$ . The relevant information will be presented in Table 1 below.

$$\text{Set } \mathbb{k}[\mathfrak{g}\langle \geq 2 \rangle]_{(-)}^{U(L)-} = \bigoplus_{\mu \in \mathcal{X}_-(G)} \mathbb{k}[\mathfrak{g}\langle \geq 2 \rangle]_{\mu}^{U(L)-}. \text{ Clearly, } \tau^o(\mathbb{k}[G \cdot e]^{U-}) \subset \mathbb{k}[\mathfrak{g}\langle \geq 2 \rangle]_{(-)}^{U(L)-}.$$

**5.10 Conjecture.** *Suppose  $G \cdot e$  is spherical. Then  $\tau^o : \mathbb{k}[G \cdot e]^{U-} \xrightarrow{\sim} \mathbb{k}[\mathfrak{g}\langle \geq 2 \rangle]_{(-)}^{U(L)-}$ .*

The preceding exposition shows that Conjecture 5.10 is true if  $\text{ht}(e) = 2$ , and one even has  $\mathbb{k}[\mathfrak{g}\langle 2 \rangle]_{(-)}^{U(L)-} = \mathbb{k}[\mathfrak{g}\langle 2 \rangle]^{U(L)-}$ . But for non-spherical orbits the above map can fail to be an isomorphism. Thus, the only interesting open case is that of orbits of height 3. It is more convenient to work with  $U$ -invariants, because then the weights of generators are dominant. To this end, Conjecture 5.10 can be restated in the ‘‘dominant’’ form:

**5.11 Conjecture’.** *Suppose  $G \cdot e$  is spherical. Then  $\hat{\tau}^o : \mathbb{k}[G \cdot e]^U \xrightarrow{\sim} \mathbb{k}[\mathfrak{g}\langle \leq -2 \rangle]_{(+)}^{U(L)} \simeq \mathcal{S}(\mathfrak{g}\langle \geq 2 \rangle)_{(+)}^{U(L)}$ .*

Here  $\mathcal{S}(\cdot)$  stands for the symmetric algebra, and the subscript  $(+)$  means that we take only weight spaces from  $\mathcal{X}_+(G)$ .

We say that a monoid  $\Gamma \subset \mathcal{X}_+(G)$  is *saturated*, if  $\mathbb{Z}\Gamma \cap \mathcal{X}_+(G) = \Gamma$ . For instance, it follows from Eq. (5.7) and Lemma 5.8 that  $\Gamma(G \cdot e)$  is saturated if  $\text{ht}(e) = 2$ .

**5.12 Conjecture.** *If  $G \cdot e \subset \mathcal{N}$  is spherical, then  $\Gamma(G \cdot e)$  is saturated.*

*Remarks.* 1. If  $G \cdot e$  is the regular nilpotent orbit, then  $\mathbb{k}[G \cdot e] = \mathbb{k}[\mathcal{N}]$ . Here  $\Gamma(\mathcal{N})$  is intersection of the root lattice with  $\mathcal{X}_+(G)$ ; hence it is saturated. More generally, for  $e$  even, it follows from [Mc] that  $\Gamma(G \cdot e) = \Gamma(G/L)$ . This means it is interesting to study saturatedness property for  $G/L$  with an arbitrary Levi subgroup  $L$ . It is likely that Littelmann's path method can be helpful for the last problem. It was pointed out by the referee that  $\Gamma(G \cdot e)$  is not always saturated; e.g. in case of 10-dimensional orbit of  $G_2$ , one has  $2\varphi_1, 3\varphi_1 \in \Gamma$ , whereas  $\varphi_1 \notin \Gamma$ .

2. The approach of [AHV] successfully computes the monoid  $\Gamma$  in case of 128-dimensional nilpotent orbit in type  $E_8$ . This can also be applied to some other cases, e.g. 40-dimensional nilpotent orbit in type  $E_6$ .

Using methods of [Pa4], it can be shown that Conjecture 5.12 implies Conjecture 5.10. This would provide an easy way for determining the structure of the algebra of covariants and, in particular, the monoid  $\Gamma(G \cdot e)$  for *all* spherical orbits, see Section 6.

## 6. TABLES AND EXAMPLES

In Tables 1 and 2, we collect some information on spherical nilpotent orbits. Numbering of the simple roots and fundamental weights of  $\mathfrak{g}$  is the same as in [VO, Tables]. In the classical case, we follow the standard notation for roots via  $\varepsilon_i$ 's. Roots of exceptional Lie algebras are presented via the Dynkin diagrams. For instance, the array  $\begin{pmatrix} n_1 & n_2 & n_3 & n_4 & n_5 & n_6 & n_7 \\ & & & & & & n_8 \end{pmatrix}$  represents the root  $\sum n_i \alpha_i$  for  $E_8$ . This picture also demonstrates our numbering of fundamental weights for  $E_8$ . In both Tables, nilpotent orbits are described by partitions (the classical case) and weighted Dynkin diagrams (the exceptional case).

In Table 1, we present the information on the monoid  $\Gamma(G \cdot e)$  for the nilpotent orbits of height 2. The relations between the  $\gamma_i$ 's and the  $\lambda_i$ 's are given in Eq. (5.7). As well-known, a very even partition in the orthogonal case gives rise to two nilpotent orbits. This happens if  $l = 0$  for the second item in the  $\mathfrak{so}_n$ -case. However, we include in the Table only one possibility. The second possibility is (only for  $l = 0$ ):  $\gamma_r = \varepsilon_{2r-1} - \varepsilon_{2r}$  and  $\lambda_r = 2\varphi_{2r-1}$ . We also did not include in Table 1 the minimal nilpotent orbits for the exceptional Lie algebras, because we only have  $\gamma_1 = \lambda_1 = \theta$  in these cases.

In Table 2, we present our computations of the homogeneous generators of  $\mathcal{S}(\mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle 3 \rangle)_{(+)}^{U(L)}$  for all nilpotent orbits of height 3. By Conjecture 5.11, these should also be the generators of the algebra  $\mathbb{k}[G \cdot e]^U = \mathbb{k}[(\overline{G \cdot e})_n]^U$ . Each generator is represented by its  $T$ -weight and degree, the latter being the usual degree in the symmetric algebra. The corresponding monoid of dominant weights is called  $\tilde{\Gamma}$ , and the third column gives the dimension of the subspace in  $\mathfrak{t}_{\mathbb{Q}}^*$  generated by  $\tilde{\Gamma}$  or, equivalently,  $r_G(G \cdot e)$  or the Krull dimension of  $\mathbb{k}[G \cdot e]^U$ . Then using Theorem 4.5(ii) or Remark 4.6(2), the interested reader may realise which orbit is  $\mathbb{O}$  for each  $\mathfrak{g}$ . The rightmost column gives known information about the normality of  $\overline{G \cdot e}$ . (The classical cases are due to Kraft and Procesi; the case of  $G_2$  is due to Levasseur and Smith; the other exceptional cases are due to Broer, see [Bro, p.959].) To fill in the

TABLE 1. Nilpotent orbits of height 2

$\mathfrak{g}$	nilpotent orbit	u.c.s.	Free generators of $\Gamma$
$\mathfrak{sl}_n$	$(2^r, 1^{n-2r})$ $r \leq n/2$	$\gamma_i = \varepsilon_i - \varepsilon_{n-i+1}$ $i = 1, \dots, r$	$\lambda_i = \varphi_i + \varphi_{n-i}$
$\mathfrak{sp}_{2n}$	$(2^r, 1^{2n-2r})$ $r \leq n$	$\gamma_i = 2\varepsilon_i$ $i = 1, \dots, r$	$\lambda_i = 2\varphi_i$
$\mathfrak{so}_n$	$(3, 1^l)$ $l \geq 4, n = 3+l$	$\gamma_1 = \varepsilon_1 + \varepsilon_2$ $\gamma_2 = \varepsilon_1 - \varepsilon_2$	$\lambda_1 = \varphi_2$ $\lambda_2 = 2\varphi_1$
	$(2^{2r}, 1^l)$ $n = 4r+l$	$\gamma_i = \varepsilon_{2i-1} + \varepsilon_{2i}$ $i = 1, \dots, r$	$\lambda_i = \varphi_{2i} \ (i < r)$ $\lambda_r = \begin{cases} 2\varphi_{2r}, & l \leq 1 \\ \varphi_{2r} + \varphi_{2r+1}, & l = 2 \\ \varphi_{2r}, & l \geq 3 \end{cases}$
$E_6$	$1-0-0-0-1$ $\quad \quad \quad \downarrow$ $\quad \quad \quad 0$	$\gamma_1 = \begin{pmatrix} 1^2 3^2 2^1 \\ 2 \end{pmatrix}$ $\gamma_2 = \begin{pmatrix} 1^1 1^1 1^1 \\ 0 \end{pmatrix}$	$\lambda_1 = \varphi_6$ $\lambda_2 = \varphi_1 + \varphi_5$
$E_7$	$0-1-0-0-0-0$ $\quad \quad \quad \downarrow$ $\quad \quad \quad 0$	$\gamma_1 = \begin{pmatrix} 1^2 3^4 3^2 \\ 2 \end{pmatrix}$ $\gamma_2 = \begin{pmatrix} 1^2 2^2 1^0 \\ 1 \end{pmatrix}$	$\lambda_1 = \varphi_6$ $\lambda_2 = \varphi_2$
	$2-0-0-0-0-0$ $\quad \quad \quad \downarrow$ $\quad \quad \quad 0$	$\gamma_1 = \begin{pmatrix} 1^2 3^4 3^2 \\ 2 \end{pmatrix}$ $\gamma_2 = \begin{pmatrix} 1^2 2^2 1^0 \\ 1 \end{pmatrix}$ $\gamma_3 = \begin{pmatrix} 1^0 0^0 0^0 \\ 0 \end{pmatrix}$	$\lambda_1 = \varphi_6$ $\lambda_2 = \varphi_2$ $\lambda_3 = 2\varphi_1$
$E_8$	$0-0-0-0-0-0-1$ $\quad \quad \quad \downarrow$ $\quad \quad \quad 0$	$\gamma_1 = \begin{pmatrix} 2^3 4^5 6^4 2 \\ 3 \end{pmatrix}$ $\gamma_2 = \begin{pmatrix} 0^1 2^3 4^3 2 \\ 2 \end{pmatrix}$	$\lambda_1 = \varphi_1$ $\lambda_2 = \varphi_7$
$F_4$	$1-0 \Leftarrow 0-0$	$\gamma_1 = (2 \ 4 \ 3 \ 2)$ $\gamma_2 = (2 \ 2 \ 1 \ 0)$	$\lambda_1 = \varphi_4$ $\lambda_2 = 2\varphi_1$

fourth column, we first compute the weights and degrees of the homogeneous generators of  $\mathcal{S}(\mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle 3 \rangle)^{U(L)}$ . In doing so, one has to take into account not only the fundamental weights of  $L$ , but also the weights of the central torus in  $L$ . The main technical problem is then to express the weights obtained in terms of fundamental weights of  $G$ . Once this is done, it is rather easy to determine the monomials in generators whose weights are  $G$ -dominant.

**Example.** Take the second nilpotent orbit for  $E_7$  in Table 2. Its dimension is 70 and the Dynkin-Bala-Carter label is  $4A_1$ . Using the weighted Dynkin diagram, it is easily seen that  $L = SL_6 \odot T_2$ . Write  $\varphi_i$  (resp.  $\tilde{\varphi}_i$ ) for fundamental weights of  $E_7$  (resp.  $SL_6$ ). Then  $\mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle 3 \rangle = V_{\tilde{\varphi}_2} + V_0 + V_{\tilde{\varphi}_5}$  as  $SL_6$ -module. It follows that  $\mathcal{S}(\mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle 3 \rangle)^{U(L)}$  is polynomial. Working inside of  $E_7$  and taking into account weights of  $T_2$ , one finds the weights of free

TABLE 2. The nilpotent orbits of height 3

$\mathfrak{g}$	nilpotent orbit	$\dim \tilde{\Gamma}$	Weights and degrees of the generators	Norm.
$\mathfrak{so}_{4t+3}$ ( $B_{2t+1}$ )	$(3, 2^{2t})$ $t \geq 1$	$2t+1$	$(\varphi_{2i-1}, i), \quad i = 1, \dots, t$ $(\varphi_{2i}, i), \quad i = 1, \dots, t$ $(2\varphi_{2t+1}, t+1)$	–
$\mathfrak{so}_{4t+4}$ ( $D_{2t+2}$ )	$(3, 2^{2t}, 1)$ $t \geq 1$	$2t+2$	$(\varphi_{2i-1} + \varphi_{2j-1}, i+j), \quad 1 \leq i \leq j \leq t$ $(\varphi_{2i}, i), \quad i = 1, \dots, t$ $(\varphi_{2j-1} + \varphi_{2t+1} + \varphi_{2t+2}, t+j+1), \quad j = 1, \dots, t$ $(2\varphi_{2t+1}, t+1), (2\varphi_{2t+2}, t+1)$	+
$\mathfrak{so}_{4t+l+3}$	$(3, 2^{2t}, 1^l)$ $t \geq 1, l \geq 2$	$2t+2$	$(\varphi_{2i}, i), \quad i = 1, \dots, t$ $(\varphi_{2i-1} + \varphi_{2j-1}, i+j), \quad 1 \leq i \leq j \leq t+1$ $\left\{ \begin{array}{l} (2\varphi_{2t+2}, t+1), \quad l = 2 \\ (\varphi_{2t+2} + \varphi_{2t+3}, t+1), \quad l = 3 \\ (\varphi_{2t+2}, t+1), \quad l \geq 4 \end{array} \right.$	+
$E_6$	$0-0-1-0-0$ $\quad \quad \quad \downarrow$ $\quad \quad \quad 0$	4	$(\varphi_6, 1), (\varphi_1 + \varphi_5, 2), (\varphi_3, 3), (\varphi_2 + \varphi_4, 4)$	+
$E_7$	$0-0-0-0-1-0$ $\quad \quad \quad \downarrow$ $\quad \quad \quad 0$	4	$(\varphi_6, 1), (\varphi_2, 2), (\varphi_5, 3), (\varphi_4, 4)$	+
	$1-0-0-0-0-0$ $\quad \quad \quad \downarrow$ $\quad \quad \quad 1$	7	$(\varphi_6, 1), (\varphi_2, 2), (\varphi_5, 3), (2\varphi_1, 3),$ $(\varphi_4, 4), (\varphi_1 + \varphi_7, 4), (\varphi_1 + \varphi_3, 5),$ $(2\varphi_7, 5), (\varphi_3 + \varphi_7, 6), (2\varphi_3, 7)$	+
$E_8$	$0-1-0-0-0-0-0$ $\quad \quad \quad \downarrow$ $\quad \quad \quad 0$	4	$(\varphi_1, 1), (\varphi_7, 2), (\varphi_2, 3), (\varphi_3, 4)$	+
	$0-0-0-0-0-0-0$ $\quad \quad \quad \downarrow$ $\quad \quad \quad 1$	8	$(\varphi_1, 1), (\varphi_7, 2), (\varphi_2, 3), (\varphi_3, 4)$ $(\varphi_8, 4), (\varphi_6, 5), (\varphi_4, 6), (\varphi_5, 7)$	+
$F_4$	$0-0-0-1-0$	4	$(\varphi_4, 1), (2\varphi_1, 2), (\varphi_3, 3), (2\varphi_2, 4)$	+
$G_2$	$1 \Leftarrow 0$	2	$(\varphi_1, 1), (\varphi_2, 1)$	–

generators in terms of  $\varphi_i$ 's. Namely, the generators of  $\mathcal{S}(\mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle 3 \rangle)^{U(L)}$  are:

$f_1 = (\varphi_6, 1)$ ,  $f_2 = (\varphi_2, 2)$ ,  $f_3 = (\varphi_5 + \varphi_1 - \varphi_7, 2)$ ,  $f_4 = (\varphi_4 + \varphi_1 - \varphi_7, 3)$ ,  $f_5 = (2\varphi_1, 3)$ ,  $f_6 = (\varphi_7 - \varphi_1, 1)$ ,  $f_7 = (\varphi_3 - \varphi_7, 1)$ . Therefore  $\mathcal{S}(\mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle 3 \rangle)_{(+)}^{U(L)}$  is generated by

$$f_1, f_2, f_3 f_6, f_5, f_4 f_6, f_5 f_6, f_5 f_6 f_7, f_5 f_6^2, f_5 f_6^2 f_7, f_5 f_6^2 f_7^2.$$

This ordering of generators corresponds to their ordering in Table 2.

Comparing our computations with known structure of  $\mathbb{k}[G \cdot e]$  as  $G$ -module when possible, we see that Conjectures 5.11 and 5.12 are valid for several items in Table 2; that is, the generators of  $\mathcal{S}(\mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle 3 \rangle)_{(+)}^{U(L)}$  given in the fourth column do belong to  $\mathbb{k}[G \cdot e]^U$ . As far as I know, this refers to the orbits in  $E_6, F_4, G_2$ , and the second orbit for  $E_8$ . The last case is especially interesting. It was proved in [AHV] that the algebra of regular functions on

this 128-dimensional orbit is a model algebra for  $E_8$ , i.e., each simple finite-dimensional  $E_8$ -module occurs in  $\mathbb{k}[G \cdot e]$  exactly once. On the other hand,  $\mathcal{S}(\mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle 3 \rangle)_{(+)}^{U(L)}$  is a polynomial algebra and the weights of free generators are exactly the fundamental weights of  $E_8$ . As  $\hat{\tau}^\circ$  is an embedding of  $\mathbb{k}[G \cdot e]^U$  into  $\mathcal{S}(\mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle 3 \rangle)_{(+)}^{U(L)}$ , we see that  $\hat{\tau}^\circ$  is an isomorphism in this case. Furthermore, our computation provides the degree in which each fundamental  $E_8$ -module appears in the graded algebra  $\mathbb{k}[\overline{G \cdot e}]^U$ .

For all spherical orbits, it is easy to determine the structure of  $\mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle 3 \rangle$  as  $(L, L)$ -module. Then using results of M. Brion [Bri], we conclude that  $\mathcal{S}(\mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle 3 \rangle)^{U(L)}$  is always polynomial. However, this is not always the case for the subalgebra  $\mathcal{S}(\mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle 3 \rangle)_{(+)}^{U(L)}$ . For instance, the latter has  $(t+1)(t+4)/2$  generators in the case of  $\mathfrak{so}_{4t+4}$  or  $\mathfrak{so}_{4t+l+3}$ , whereas its Krull dimension equals  $2t+2$ . Modulo Conjecture 5.11, this means that the homological dimension of the algebra of covariants for the closure of a spherical nilpotent orbit may be arbitrarily large.

**Remark.** Our computations for the orbits of height 3 yield the following assertions:

- If  $\text{ht}(e) = 3$ , then  $\mathcal{S}(\mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle 3 \rangle)$  is a free  $\mathcal{S}(\mathfrak{g}\langle 2 \rangle \oplus \mathfrak{g}\langle 3 \rangle)^{U(L)}$ -module. (Here one has to apply condition ( $\spadesuit$ ) given in the proof of Theorem 5.6(iii) to the fundamental weights of  $L$ , not  $G$ .)
- the validity of Conjecture 5.11 would imply that if  $\mathbb{k}[G \cdot e]^U$  is polynomial, then  $\mathbb{k}[G \cdot e]$  is a free  $\mathbb{k}[G \cdot e]^U$ -module.

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