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Regions in the dominant chamber and nilpotent orbits

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Abstract

Let G be a complex semisimple algebraic group with Lie algebra $\mathfrak{g}$. The goal of this note is to show that combining some ideas of Gunnells and Sommers [Math. Res. Lett. 10 (2–3) (2003) 363–373] and Vinberg and Popov [Invariant Theory, in: Algebraic Geometry IV, in: Encyclopaedia Math. Sci., Vol. 55, Springer, Berlin, 1994, pp.123–284] yields a geometric description of the characteristic of a nilpotent G-orbit in an arbitrary (finite-dimensional) rational G-module.
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Let G be a complex semisimple algebraic group with Lie algebra $\mathfrak{g}$. The goal of this note is to show that combining some ideas of [5] and [1] quickly yields a geometric description of the characteristic of a nilpotent G-orbit in an arbitrary (finite-dimensional) rational G-module.

Fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and a Cartan subalgebra $\mathfrak{t}$ in it. For each $\mathfrak{t}$-weight μ of a G-module $\mathbb{V}$, consider the affine hyperplane $\mathcal{H}_{\mu,2} = \{x \mid (x,\mu) = 2\} \subset \mathfrak{t}_{\mathbb{R}}$. These hyperplanes cut the dominant chamber in finitely many regions, and to any region R one may attach a $\mathfrak{b}$-stable subspace of $\mathbb{V}$ by the following rule:

$$\mathbb{V}_R = \bigoplus_{\mu \in I_R} \mathbb{V}^{\mu},$$

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where I_R is the set of weights of $\mathbb{V}$ such that $(x,\mu) > 2$ for some (equivalently, any) $x \in R$. Given a nilpotent G-orbit $\mathcal{O} \subset \mathbb{V}$, consider the closure of the union of all regions R such that $\mathcal{O} \cap \mathbb{V}_R \neq \emptyset$. Let's call this set $\mathcal{C}_{\mathcal{O}}$. Our first observation is that $\mathcal{C}_{\mathcal{O}}$ contains a unique element of minimal length, and this element is just the dominant characteristic of $\mathcal{O}$ in the sense of [5, 5.5]. Next, we show that if the representation $G \to GL(\mathbb{V})$ is associated with either a periodic or a $\mathbb{Z}$-grading of a reductive algebraic Lie algebra, then the condition "$\mathcal{O} \cap \mathbb{V}_R \neq \emptyset$" can be replaced with "$\mathcal{O} \cap \mathbb{V}_R$ is dense in $\mathbb{V}_R$". This new condition determines a smaller set $\widetilde{\mathcal{C}}_{\mathcal{O}} \subset \mathcal{C}_{\mathcal{O}}$, but these two sets still have the same element of minimal length. This provides another proof and also a generalization of the main result of [1]. It is worth noting that the representations associated with $\mathbb{Z}_m$-gradings are *visible*, i.e., contain finitely many nilpotent orbits, and in this case different orbits have different characteristics.

We also give an example showing that, for an arbitrary visible G-module $\mathbb{V}$, it may happen that different orbits have the same characteristic and that for some orbits $\mathcal{O}$ there are no subspaces of the form $\mathbb{V}_R$ such that $\mathcal{O} \cap \mathbb{V}_R$ is dense in $\mathbb{V}_R$.

Main notation. Δ is the root system of $(\mathfrak{g}, \mathfrak{t})$ and W is the Weyl group of $(\mathfrak{t}, \Delta)$.

Δ^+ is the set of positive roots and $\Pi = \{\alpha_1, \ldots, \alpha_p\}$ is the set of simple roots in Δ^+.

We define $\mathfrak{t}_{\mathbb{R}}$ to be set of all elements of $\mathfrak{t}$ having real eigenvalues in any G-module (a Cartan subalgebra of a split real form of $\mathfrak{g}$). Denote by $(\ ,\ )$ a W-invariant inner product on $\mathfrak{t}_{\mathbb{R}}$. Using $(\ ,\ )$, we identify $\mathfrak{t}_{\mathbb{R}}$ and $\mathfrak{t}^*_{\mathbb{R}}$. So that, one may think that $\mathfrak{t}_{\mathbb{R}} = \bigoplus_{i=1}^p \mathbb{R}\alpha_i$.

$\mathcal{C} = \{x \in V \mid (x,\alpha) > 0\ \forall \alpha \in \Pi\}$ is the (open) fundamental Weyl chamber.

1. The characteristic of a nilpotent orbit

In this section we recall some results published in the survey article [5, §5]. Unfortunately, that simple approach to questions of stability, optimal one-parameter subgroups, and a stratification of the null-cone remained largely unnoticed by the experts.

Let $\mathbb{V}$ be a G-module. Write $\mathbb{V}^\mu$ for the μ-weight space of $\mathbb{V}$. Here μ is regarded as element of $\mathfrak{t}_{\mathbb{R}}$. Hence $\mu(x) = (\mu, x)$ for any $x \in \mathfrak{t}_{\mathbb{R}}$.

Suppose $h \in \mathfrak{t}_{\mathbb{R}}$, i.e., h is a rational semisimple element. For a G-module $\mathbb{V}$ and $c \in \mathbb{Q}$, we set

$$\mathbb{V}_h\langle c\rangle = \{v \in \mathbb{V} \mid h\cdot v = cv\}, \quad \mathbb{V}_h\langle \geqslant c\rangle = \bigoplus_{k \geqslant c} \mathbb{V}_h\langle k\rangle, \quad \text{and} \quad \mathbb{V}_h\langle >c\rangle = \bigoplus_{k>c} \mathbb{V}_h\langle k\rangle.$$

For instance, $\mathfrak{g}_h\langle 0\rangle$ is the centralizer of h in $\mathfrak{g}$ (a Levi subalgebra of $\mathfrak{g}$), $\mathfrak{g}_h\langle \geqslant 0\rangle$ is a parabolic subalgebra of $\mathfrak{g}$, and $\mathfrak{g}_h\langle >0\rangle$ is the nilpotent radical of $\mathfrak{g}_h\langle \geqslant 0\rangle$. Clearly,

$$\mathbb{V}_h\langle c\rangle = \bigoplus_{\mu:\ \mu(h)=c} \mathbb{V}^\mu \quad \text{and} \quad \mathfrak{g}_h\langle a\rangle\cdot\mathbb{V}_h\langle c\rangle \subset \mathbb{V}_h\langle a+c\rangle.$$

Recall that an element $v \in \mathbb{V}$, or the orbit $G\cdot v$, is called *nilpotent*, if $\overline{G\cdot v} \ni 0$. It is easy to verify that, for any $h \in \mathfrak{t}_{\mathbb{R}}$, the subspace $\mathbb{V}_h\langle >0\rangle$ consists of nilpotent elements. Conversely, the Hilbert–Mumford criterion asserts that any nilpotent G-orbit in $\mathbb{V}$ meets a subspace of this form for a suitable h.

Definition 1.1. The *characteristic* of a nilpotent orbit $\mathcal{O}$ is a shortest element $h \in \mathfrak{t}_{\mathbb{R}}$ such that $\mathcal{O} \cap \mathbb{V}_h\langle \geqslant 2\rangle \neq \emptyset$.

Remark. In principle, one may choose an arbitrary normalization "$(\geqslant c)$" in the definition. The choice $c = 2$ is explained by the fact that for $\mathbb{V} = \mathfrak{g}$ this leads to the usual (Dynkin) characteristic of a nilpotent element.

It was shown in [5, 5.5] that each nilpotent orbit has a characteristic. Moreover, if $h_1, h_2 \in \mathfrak{t}_{\mathbb{R}}$ are two characteristics of $\mathcal{O}$, then they are W-conjugate. Thus, to any nilpotent orbit $\mathcal{O} \subset \mathbb{V}$ one may attach uniquely the *dominant characteristic*, which is denoted by $h_{\mathcal{O}}$.

If we are given an $h \in \mathfrak{t}_{\mathbb{R}}$ and $u \in \mathbb{V}_h\langle \geqslant 2\rangle$, then it is helpful to have a criterion to decide whether h is a characteristic of $G{\cdot}u$. The following result, attributed in [5, Theorem 5.4] to F. Kirwan and L. Ness, solves the problem. Let $Z_G(h)$ denote the centralizer of h in G and $\widetilde{Z}_G(h) \subset Z_G(h)$ the *reduced centralizer*. That is, the Lie algebra of $\widetilde{Z}_G(h)$ is the orthogonal complement to h in $\mathfrak{g}_h\langle 0\rangle = \operatorname{Lie} Z_G(h)$. Clearly, $\mathbb{V}_h\langle c\rangle$ is a $Z_G(h)$-module for any c.

Theorem 1.2. *Under the previous notation, h is a characteristic of $G{\cdot}u$ if and only if the projection of u to $\mathbb{V}_h\langle 2\rangle$ is not a nilpotent element with respect to the action of $\widetilde{Z}_G(h)$.*

2. Regions and characteristics

Let $\mathbb{V}$ be a G-module. Write $\mathcal{P}^*(\mathbb{V})$ for the set of nonzero weights of $\mathbb{V}$ with respect to $\mathfrak{t}$. For any $\mu \in \mathcal{P}^*(\mathbb{V})$, consider the affine hyperplane $\mathcal{H}_{\mu,2} = \{x \in \mathfrak{t}_{\mathbb{R}} \mid (x, \mu) = 2\}$. The number "2" is determined by the normalization in Definition 1.1. We will be interested in the hyperplanes meeting the dominant Weyl chamber. It is easily seen that the following is true.

Lemma 2.1. *We have*

$$\mathcal{H}_{\mu,2} \cap \mathcal{C} \neq \emptyset \quad \Leftrightarrow \quad \mu \text{ has a positive coefficient in the expression } \mu = \sum_{i=1}^{p} a_i\alpha_i \quad (a_i \in \mathbb{Q}).$$

The set of all such hyperplanes cuts $\mathcal{C}$ in regions. That is, a *region* (associated with $\mathbb{V}$) is a connected component of $\mathcal{C} \setminus \bigcup_{\mu} \mathcal{H}_{\mu,2}$. The set of all regions is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{V})$.

Clearly, the closure of each region is a convex polytope. Given a region R, consider all hyperplanes separating R from the origin, and the corresponding weights in $\mathcal{P}^*(\mathbb{V})$. This set of weights is denoted by I_R. More precisely, if $x \in R$, then

$$I_R = \bigl\{\mu \in \mathcal{P}^*(\mathbb{V}) \mid (x, \mu) > 2\bigr\}.$$

Lemma 2.2. *Let $R \in \mathcal{R}$.*

(i) *if $\mu \in I_R$, $\gamma \in \Delta^+$, and $\mu + \gamma \in \Delta^+$, then $\mu + \gamma \in I_R$.*
(ii) *The subspace $\mathbb{V}_R := \bigoplus_{\mu \in I_R} \mathbb{V}^{\mu} \subset \mathbb{V}$ is $\mathfrak{b}$-stable.*

(iii) *Each G-orbit meeting V_R is nilpotent.*

Proof. (i) – obvious; (ii) follows from (i); (iii) we have $\lim_{t\to-\infty} \exp(tx)\cdot u = 0$ for any $x \in R$ and $u \in V_R$. □

Suppose $\mathcal{O} \subset \mathbb{V}$ is a nilpotent G-orbit. One may attach to $\mathcal{O}$ a collection of regions, as follows. Set

$$M_{\mathcal{O}} = \{R \in \mathcal{R} \mid \mathcal{O} \subset G\cdot\mathbb{V}_R\} = \{R \in \mathcal{R} \mid \mathcal{O} \cap \mathbb{V}_R \neq \emptyset\}, \tag{2.1}$$

and

$$\mathcal{C}_{\mathcal{O}} = \bigcup_{R\in M_{\mathcal{O}}} \overline{R} \subset \overline{\mathcal{C}}.$$

Thus, $\mathcal{C}_{\mathcal{O}}$ is a closed subset of $\overline{\mathcal{C}}$ determined by $\mathcal{O}$. Let $h' \in \mathcal{C}_{\mathcal{O}}$ be an element of minimal length.

Proposition 2.3. *h' is a unique element of minimal length in $\mathcal{C}_{\mathcal{O}}$, and $h' = h_{\mathcal{O}}$.*

Proof. By the very construction, h' has the property that $\mathcal{O} \cap \mathbb{V}_{h'}\langle \geqslant 2\rangle \neq \emptyset$ and it is a shortest dominant element with this property. It then follows from results described in Section 1 that $h' = h_{\mathcal{O}}$ and $\mathcal{C}_{\mathcal{O}}$ contains a unique element of minimal length. □

The above construction is inspired by [1], where the case $\mathbb{V} = \mathfrak{g}$ is considered. However, condition (2.1) was slightly different there. Namely, the set of regions attached to $\mathcal{O}$ was determined by the condition that $\mathcal{O} \cap \mathfrak{g}_R$ be dense in $\mathfrak{g}_R$. But this stronger condition cannot lead in general to satisfactory results, unless $\mathbb{V}$ is a *visible G*-module. For, the number of subspaces of the form $\mathbb{V}_R$ is finite and therefore the set of such regions would be empty for infinitely many nilpotent orbits, if $\mathbb{V}$ is not visible. Moreover, even if $\mathbb{V}$ is visible, it may happen that, for a given nilpotent orbit, there is no subspace $\mathbb{V}_R$ ($R \in \mathcal{R}$) such that $\mathcal{O} \cap \mathbb{V}_R$ is dense in $\mathbb{V}_R$ (see example below).

However, one may formally set

$$\begin{aligned}\widetilde{M}_{\mathcal{O}} &= \{R \in \mathcal{R} \mid \mathcal{O} \text{ is dense in } G\cdot\mathbb{V}_R\}\\ &= \{R \in \mathcal{R} \mid \mathcal{O} \cap \mathbb{V}_R \text{ is dense in } \mathbb{V}_R\},\end{aligned} \tag{2.2}$$

and

$$\widetilde{\mathcal{C}}_{\mathcal{O}} = \bigcup_{R\in \widetilde{M}_{\mathcal{O}}} \overline{R} \subset \overline{\mathcal{C}}.$$

Clearly, $\widetilde{M}_{\mathcal{O}} \subset M_{\mathcal{O}}$ and $\widetilde{\mathcal{C}}_{\mathcal{O}} \subset \mathcal{C}_{\mathcal{O}}$. We also define $\tilde{h}_{\mathcal{O}}$ to be an element of $\widetilde{\mathcal{C}}_{\mathcal{O}}$ of minimal length (if $\widetilde{\mathcal{C}}_{\mathcal{O}} \neq \emptyset$!).

Example 2.4. Here we give an example of a visible module such that (i) $\widetilde{M}_{\mathcal{O}} = \emptyset$ for some $\mathcal{O}$, and (ii) $h_{\mathcal{O}_1} = h_{\mathcal{O}_2}$ for different nilpotent orbits.

Let $G = SL(V_1) \times SL(V_2)$, $\dim V_1 = \dim V_2 = 2$, and $\mathbb{V} = (V_1 \otimes V_2) \oplus V_1$. Identifying $\mathbb{V}$ with the space of 2 by 3 matrices, we write

$$v = \begin{pmatrix} m & n & x \\ p & q & y \end{pmatrix}$$

for a generic element in $\mathbb{V}$. Here

$$V_1 \otimes V_2 = \left\{ \begin{pmatrix} m & n & 0 \\ p & q & 0 \end{pmatrix} \right\}.$$

There are five nilpotent orbits in $\mathbb{V}$, and representatives of the non-trivial orbits are:

$$\mathcal{O}_2\colon v_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{O}_3\colon v_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\mathcal{O}_4\colon v_4 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{O}_5\colon v_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One has $\dim \mathcal{O}_i = i$ for $2 \leqslant i \leqslant 5$. Since $\operatorname{rk} G = 2$, we have $\mathfrak{t}_{\mathbb{R}}$ is $\mathbb{R}^2$ and it is not hard to depict all the regions and determine the characteristics. The dominant chamber is the positive quadrant. There are four lines of the form $\mathcal{H}_{\mu,2}$ meeting the positive quadrant, which correspond to the roots $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 - \alpha_2, -\alpha_1 + \alpha_2$. Here α_i is the simple root of $SL(V_i)$. Hence one gets six regions marked by Roman numbers. We have $\widetilde{M}_{\mathcal{O}_2} = \emptyset$, $\widetilde{M}_{\mathcal{O}_3} = \{\mathrm{II}, \mathrm{III}\}$, $\widetilde{M}_{\mathcal{O}_4} = \{\mathrm{IV}, \mathrm{V}\}$, $\widetilde{M}_{\mathcal{O}_5} = \{\mathrm{VI}\}$. Since $\overline{\mathcal{O}_4} \supset \mathcal{O}_2$ and $\overline{\mathcal{O}_3} \not\supset \mathcal{O}_2$, we conclude that $h_3 = (1, 1)$, $h_2 = h_4 = (2, 0)$, and $h_5 = (2, 4)$. The elements h_i are circled in Fig. 1.

Therefore one should not expect that $\tilde{h}_{\mathcal{O}}$ is always defined and that different orbits have different characteristics. At the rest of the section, we give a sufficient condition for this to happen.

Let $\mathfrak{l}$ be a reductive algebraic Lie algebra. Consider a $\mathbb{Z}_m$-grading of $\mathfrak{l}$, where $m \in \mathbb{N}$ or $m = \infty$. That is, we have $\mathfrak{l} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{l}_i$ if m is finite, and $\mathfrak{l} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{l}_i$ is a $\mathbb{Z}$-grading in the second case. Here $\mathfrak{l}_0$ is reductive and each $\mathfrak{l}_i$ is an $\mathfrak{l}_0$-module. Let G be a connected (reductive) group with Lie algebra $\mathfrak{l}_0$, and set $\mathbb{V} = \mathfrak{l}_1$. Then we shall say that the representation $G \to GL(\mathbb{V})$ is associated with a $\mathbb{Z}_m$-grading (of $\mathfrak{l}$). By a famous result of Vinberg [3, § 2], $\mathbb{V}$ is a visible G-module in this situation.

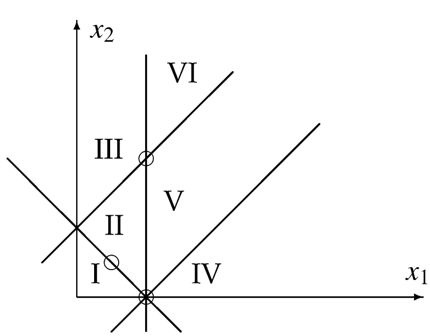

Fig. 1.

Theorem 2.5. *Suppose the representation $G \to GL(\mathbb{V})$ is associated with a $\mathbb{Z}_m$-grading. Then different nilpotent G-orbits in $\mathbb{V}$ have different characteristics and for any nilpotent G-orbit $\mathcal{O} \subset \mathbb{V}$ we have $\tilde{h}_{\mathcal{O}} = h_{\mathcal{O}}$.*

Proof. (1) Let $e \in \mathbb{V} = \mathfrak{l}_1$ be a nilpotent element, and $\mathcal{O} = G{\cdot}e$. By a generalization of the Morozov–Jacobson theorem [4, Theorem 1(1)], there is an $\mathfrak{sl}_2$-triple (e, h, f) such that $h \in \mathfrak{l}_0 = \mathfrak{g}$ and $f \in \mathfrak{l}_{-1}$. The rational semisimple element h determines a $\mathbb{Z}$-grading of $\mathfrak{l}$, and we have $e \in \mathbb{V}_h\langle 2\rangle \subset \mathfrak{l}_h\langle 2\rangle$. It is well known that, in the Lie algebra $\mathfrak{l}$, we have $\widetilde{Z}_L(h){\cdot}e$ is closed in $\mathfrak{l}_h\langle 2\rangle$. Hence, by the Richardson–Vinberg lemma [3, § 2], $\widetilde{Z}_G(h){\cdot}e$ is closed in $\mathbb{V}_h\langle 2\rangle$. Without loss of generality, one may assume that h is a dominant element in $\mathfrak{t}_{\mathbb{R}}$. Then, in view of Theorem 1.2, $h = h_{\mathcal{O}}$ is the dominant characteristic of $\mathcal{O}$. Next, $L{\cdot}e \cap \mathfrak{l}_h\langle \geqslant 2\rangle$ is dense in $\mathfrak{l}_h\langle \geqslant 2\rangle$ and hence $\mathcal{O} \cap \mathbb{V}_h\langle \geqslant 2\rangle$ is dense in $\mathbb{V}_h\langle \geqslant 2\rangle$. Thus, $h = \tilde{h}_{\mathcal{O}}$.

(2) That different nilpotent orbits have different characteristics stems from [4, Theorem 1(4)]. □

This result applies, in particular, to the adjoint representations ($m = 1$), where one obtains another proof for the main result in [1]. Another interesting case is that of the little adjoint G-module, if $\mathfrak{g}$ is a simple Lie algebra having roots of different lengths. Let θ_s be the short dominant root. Then the simple G-module with highest weight θ_s is called the *little adjoint*. It is denoted by $\mathfrak{g}_{la}$. The set $\mathcal{P}^*(\mathfrak{g}_{la})$ is Δ_s, the set of short roots. This module is always associated with a $\mathbb{Z}_m$-grading ($m = 2$ for $\mathbf{B}_p$, $\mathbf{C}_p$, $\mathbf{F}_4$; $m = 3$ for $\mathbf{G}_2$). Therefore the set of regions $\mathcal{R}(\mathfrak{g}_{la})$ allows us to determine the characteristics of the nilpotent G-orbits in $\mathfrak{g}_{la}$. The arrangement of hyperplanes $\mathcal{H}_{\mu,2}$ ($\mu \in \Delta_s^+$) inside of $\mathcal{C}$ was studied in [2], where it was shown that there is a bijection between the set of regions $\mathcal{R}(\mathfrak{g}_{la})$ and the set of all $\mathfrak{b}$-stable subspaces of $\mathfrak{g}_{la}$ without semisimple elements. We also give in [2] an explicit formula for the number $\#\mathcal{R}(\mathfrak{g}_{la})$.

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