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Short antichains in root systems, semi-Catalan arrangements, and B-stable subspaces

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Let G be a complex simple algebraic group with Lie algebra $\mathfrak{g}$. Fix a Borel subalgebra $\mathfrak{b}$. An ideal of $\mathfrak{b}$ is called ad-*nilpotent*, if it is contained in $[\mathfrak{b}, \mathfrak{b}]$. The goal of this paper is to present a refinement of the enumerative theory of ad-nilpotent ideals in the case, where $\mathfrak{g}$ has roots of different length.

Let $\mathfrak{Ad}$ denote the set of all ad-nilpotent ideals of $\mathfrak{b}$. Any $\mathfrak{c} \in \mathfrak{Ad}$ is completely determined by the corresponding set of roots. The minimal roots in this set are called the *generators* of an ideal. The collection of generators of an ideal forms an *antichain* in the poset of positive roots, and the whole theory can be expressed in the combinatorial language, in terms of antichains. An antichain is called *strictly positive*, if it contains no simple roots. Enumerative results for all and strictly positive antichains were recently obtained in the work of Athanasiadis, Cellini–Papi, Sommers, and this author [1–4, 9, 13].

There are two different theoretical approaches to describing (enumerating) antichains. The first approach consists of constructing a bijection between antichains and the coroot lattice points lying in a certain simplex. An important intermediate step here is a bijection between antichains and the so-called *minimal* elements of the affine Weyl group, $\widehat{W}$. It turns out that the simplex obtained is "equivalent" to a dilation of the fundamental alcove of $\widehat{W}$, so that the problem of counting the coroot lattice points in it can be resolved. For strictly positive antichains, one constructs another bijection and another simplex, and the respective elements of $\widehat{W}$ are called *maximal*; yet, everything is quite similar. The second approach uses the Shi bijection between the ad-nilpotent ideals (or antichains) and the dominant regions of the Catalan arrangement. Under this bijection, the strictly positive antichains correspond to the bounded regions. There is a powerful result of Zaslavsky allowing one to compute the number of all and bounded regions, if the characteristic polynomial of the arrangement is known. Since the characteristic polynomial of the Catalan arrangement was recently computed in [1], the result follows.

If $\mathfrak{g}$ has roots of different length, one can distinguish the length of elements occurring in antichains. We say that an antichain is *short*, if it consists of only short roots. This notion has a natural representation-theoretic incarnation: the short antichains are in

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a one-to-one correspondence with the $\mathfrak{b}$-stable subspaces, without nonzero semisimple elements, in the little adjoint G-module. A short analogue of strictly positive antichains, *strictly s-positive* antichains, is also defined. We are able to carry the above two approaches over to the short antichains. First, we introduce and characterize suitable elements of $\widehat{W}$ (*s-minimal* and *s-maximal* ones), establish bijections between these two sets of elements and the coroot lattice points of certain simplices, and eventually obtain formulae for the number of short and strictly s-positive antichains. Second, we introduce and study the *semi-Catalan arrangement*, which has the same relation to short and strictly s-positive antichains as the usual Catalan arrangement has to all and strictly positive antichains. The difference between the Catalan and semi-Catalan arrangements is that we "deform" only the hyperplanes orthogonal to short roots in the latter. We prove various results connecting the dominant regions of the semi-Catalan arrangement and the elements of $\widehat{W}$ attached to short antichains. Adapting Athanasiadis' argument from [1], we compute the characteristic polynomial for the *extended semi-Catalan arrangements*, or in other words, for m-semi-Catalan arrangements, $\mathrm{Cat}_s^m(\Delta)$, with $m = 0, 1, 2, \ldots$. For $m = 0$, one obtains the Coxeter arrangement of W, and for $m = 1$, the semi-Catalan arrangement.

Here is a part of our results. Let $\alpha_1, \ldots, \alpha_p$ be the simple roots of $\mathfrak{g}$ and θ the highest root. Let $\mathcal{A}$ be the fundamental alcove of $\widehat{W}$ and g the sum of coefficients of the *short* simple roots in the expression of $\theta = \sum c_i\alpha_i$. Then the short (resp. strictly s-positive) antichains are in one-to-one correspondence with the coroot lattice points in $(g+1)\mathcal{A}$ (resp. $(g-1)\mathcal{A}$). If the root system is not of type $\mathbf{G}_2$, this leads to a closed formula for the number of the respective antichains. E.g., the number of short antichains is equal to $\prod_{i=1}^{p} \frac{g+e_i+1}{e_i+1}$, where e_i, $i = 1, 2, \ldots, p$, are the exponents of the Weyl group W. Using this, we found a uniform expression, which covers the $\mathbf{G}_2$-case as well, see Eq. (5.6), but it awaits a conceptual explanation. The characteristic polynomial of $\mathrm{Cat}_s^m(\Delta)$ is $\chi(t) = \prod_{i=1}^{p}(t - mg - e_i)$ (again, if Δ is not of type $\mathbf{G}_2$). For $\mathbf{G}_2$, the formula for $\chi(t)$ depends on the parity of m. We also define a "short" analogue of the extended Shi arrangement, which we call, of course, the *extended semi-Shi arrangement*, and propose a conjectural formula for its characteristic polynomial, see Remarks 6.8.

A rough description of the contents is as follows. In Sections 2 and 3, we give a review of results concerning ideals (antichains) and Catalan arrangements, including the two approaches described above. In particular, we consider minimal and maximal elements of $\widehat{W}$ and their connection with ideals. Some complements to known results are also given. We attempt to present a unified treatment that can be generalized afterwards, without much pains, to the setting of short antichains. Our main results are gathered in Sections 4–7. After a brief description in Section 4 of the relationship between $\mathfrak{b}$-stable subspaces of the little adjoint G-module and short antichains, we turn, in Section 5, to considering s-minimal and s-maximal elements of $\widehat{W}$ and related simplices. In Section 6, we compute the characteristic polynomial for the m-semi-Catalan arrangement with arbitrary $m \in \mathbb{N}$ and study the relationship between the semi-Catalan arrangement (which corresponds to $m = 1$) and short antichains. As a consequence of our theory, we present, in Section 7, several intriguing results whose proof uses case-by-case verification.

To a great extent, this work was inspired by the recent papers of Athanasiadis [1] and Sommers [13].

1. Notation and other preliminaries

1.1. Main notation

Δ is the root system of $(\mathfrak{g}, \mathfrak{t})$ and W is the usual Weyl group. For $\alpha \in \Delta$, $\mathfrak{g}_\alpha$ is the corresponding root space in $\mathfrak{g}$.

Δ^+ is the set of positive roots and $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$.

$\Pi = \{\alpha_1, \ldots, \alpha_p\}$ is the set of simple roots in Δ^+ and θ is the highest root in Δ^+.

We set $V := \mathfrak{t}_{\mathbb{R}} = \oplus_{i=1}^p \mathbb{R}\alpha_i$ and denote by $(\ ,\ )$ a W-invariant inner product on V. As usual, $\mu^\vee = 2\mu/(\mu, \mu)$ is the coroot for $\mu \in \Delta$.

$\mathcal{C} = \{x \in V \mid (x, \alpha) > 0\ \forall\alpha \in \Pi\}$ is the (open) fundamental Weyl chamber.

$\mathcal{A} = \{x \in V \mid (x, \alpha) > 0\ \forall\alpha \in \Pi\ \&\ (x, \theta) < 1\}$ is the fundamental alcove.

$Q^+ = \{\sum_{i=1}^p n_i\alpha_i \mid n_i = 0, 1, 2, \ldots\}$ and $Q^\vee = \oplus_{i=1}^p \mathbb{Z}\alpha_i^\vee \subset V$ is the coroot lattice.

Letting $\widehat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$, we extend the inner product $(\ ,\ )$ on $\widehat{V}$ so that $(\delta, V) = (\lambda, V) = (\delta, \delta) = (\lambda, \lambda) = 0$ and $(\delta, \lambda) = 1$.

$\widehat{\Delta} = \{\Delta + k\delta \mid k \in \mathbb{Z}\}$ is the set of affine real roots and $\widehat{W}$ is the affine Weyl group.

Then $\widehat{\Delta}^+ = \Delta^+ \cup \{\Delta + k\delta \mid k \geq 1\}$ is the set of positive affine roots and $\widehat{\Pi} = \Pi \cup \{\alpha_0\}$ is the corresponding set of affine simple roots, where $\alpha_0 = \delta - \theta$. The inner product $(\ ,\ )$ on $\widehat{V}$ is $\widehat{W}$-invariant. The notation $\beta > 0$ (resp. $\beta < 0$) is shorthand for $\beta \in \widehat{\Delta}^+$ (resp. $\beta \in -\widehat{\Delta}^+$). For α_i $(0 \leq i \leq p)$ we let s_i denote the corresponding simple reflection in $\widehat{W}$. If the index of $\alpha \in \widehat{\Pi}$ is not specified, then we merely write s_α. The length function on $\widehat{W}$ with respect to $s_0, s_1, \ldots, s_p$ is denoted by ℓ. For any $w \in \widehat{W}$, we set

$$N(w) = \{\alpha \in \widehat{\Delta}^+ \mid w(\alpha) \in -\widehat{\Delta}^+\}.$$

It is standard that $\#N(w) = \ell(w)$ and $N(w)$ is *bi-convex*. The latter means that both $N(w)$ and $\widehat{\Delta}^+ \setminus N(w)$ are subsets of $\widehat{\Delta}^+$ that are closed under addition. Furthermore, the assignment $w \mapsto N(w)$ sets up a bijection between the elements of $\widehat{W}$ and the finite bi-convex subsets of $\widehat{\Delta}^+$.

1.2. Ideals and antichains

Throughout the paper, $\mathfrak{b}$ is the Borel subalgebra of $\mathfrak{g}$ corresponding to Δ^+ and $\mathfrak{u} = [\mathfrak{b}, \mathfrak{b}]$. Let $\mathfrak{c} \subset \mathfrak{b}$ be an ad-nilpotent ideal. Then $\mathfrak{c} = \oplus_{\alpha\in I}\mathfrak{g}_\alpha$ for some $I \subset \Delta^+$. This I is said to be an *ideal* (of Δ^+). More precisely, a set $I \subset \Delta^+$ is an ideal, if whenever $\gamma \in I, \mu \in \Delta^+$, and $\gamma + \mu \in \Delta$, then $\gamma + \mu \in I$. Our exposition will be mostly combinatorial, i.e., in place of ad-nilpotent ideals of $\mathfrak{b}$ we will deal with the respective ideals of Δ^+.

For $\mu, \gamma \in \Delta^+$, write $\mu \preccurlyeq \gamma$, if $\gamma - \mu \in Q^+$. The notation $\mu \prec \gamma$ means that $\mu \preccurlyeq \gamma$ and $\gamma \neq \mu$. We regard Δ^+ as poset under "$\preccurlyeq$". Let $I \subset \Delta^+$ be an ideal. An element $\gamma \in I$ is called a *generator*, if $\gamma - \alpha \notin I$ for any $\alpha \in \Pi$. In other words, γ is a minimal element of I with respect to "$\preccurlyeq$". We write $\Gamma(I)$ for the set of generators of I. It is easily seen that $\Gamma(I)$ is an *antichain* of Δ^+, i.e., $\gamma_i \not\preccurlyeq \gamma_j$ for any pair (γ_i, γ_j) in $\Gamma(I)$. Conversely, if $\Gamma \subset \Delta^+$ is an antichain, then the ideal

$$I\langle\Gamma\rangle := \{\mu \in \Delta^+ \mid \mu \succcurlyeq \gamma_i \text{ for some } \gamma_i \in \Gamma\}$$

has Γ as the set of generators. Let $\mathfrak{An}$ denote the set of all antichains in Δ^+. In view of the above bijection $\mathfrak{Ad} \stackrel{1:1}{\longleftrightarrow} \mathfrak{An}$, we will freely switch between ideals and antichains. An ideal I is called *strictly positive*, if $I \cap \Pi = \varnothing$. The set of strictly positive ideals is denoted by $\mathfrak{Ad}_0$.

2. Ideals, maximal and minimal elements of $\widehat{W}$

In this section we review some recent results by Athanasiadis, Cellini–Papi, Sommers, and this author. A few complements are also given.

The idea of describing ideals of Δ^+ through the use of elements of $\widehat{W}$ goes back to D. Peterson, who exploited minuscule elements for counting Abelian ideals of $\mathfrak{b}$, see [7]. In the general case, given $I \subset \Delta^+$, we want to have $w \in \widehat{W}$ such that $N(w) \subset \cup_{k\ge 1}(k\delta - \Delta^+)$ and $N(w) \cap (\delta - \Delta^+) = \delta - I$. It turns out that, for any ideal I, there is a unique element of minimal length satisfying these properties. In contrast, the element of maximal length exists if and only if I is strictly positive, and in this case such an element is unique, too. Implementation of this program yields also explicit formulae for the number of all and strictly positive ideals.

As is well known, $\widehat{W}$ is isomorphic to a semi-direct product of W and $Q^\vee$. Given $w \in \widehat{W}$, there is a unique decomposition

$$w = v \cdot t_r, \tag{2.1}$$

where $v \in W$ and t_r is the translation corresponding to $r \in Q^\vee$. The word “translation” means the following. The group $\widehat{W}$ has two natural actions:

(a) the linear action on $\widehat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$;

(b) the affine-linear action on V.

We use “$*$” to denote the second action. For $r \in Q^\vee$, the linear action of $t_r \in \widehat{W}$ on $V \oplus \mathbb{R}\delta$ is given by $t_r(x) = x - (x, r)\delta$ (we do not need the formulas for the whole of $\widehat{V}$), while the affine-linear action on V is given by $t_r * y = y + r$. So that t_r is a true translation for the $*$-action on V.

Let us say that $w \in \widehat{W}$ is *dominant*, if $w(\alpha) > 0$ for all $\alpha \in \Pi$. Obviously, w is dominant if and only if $N(w) \subset \cup_{k\ge 1}(k\delta - \Delta^+)$. It also follows from [3, 1.1] that w is dominant if and only if $w^{-1} * \mathcal{A} \subset \mathcal{C}$. Write $\widehat{W}_{\text{dom}}$ for the set of dominant elements.

Proposition 2.2.

(i) *If $w = v \cdot t_r \in \widehat{W}_{\text{dom}}$, then $r \in -\overline{\mathcal{C}}$;*

(ii) *The mapping $\widehat{W}_{\text{dom}} \to Q^\vee$ given by $w = v \cdot t_r \mapsto v(r)$ is a bijection.*

Proof. (i) We have $w^{-1} * x = v^{-1}(x) - r$ for any $x \in V$. In particular, $w^{-1} * 0 = -r$. Since $0 \in \overline{\mathcal{A}}$ and w is dominant, we are done.

(ii) Given $\varkappa \in Q^\vee$, we want to find $w = v \cdot t_r$ such that $w^{-1} * \mathcal{A} = v^{-1}(\mathcal{A}) - r \subset \mathcal{C}$ and $v(r) = \varkappa$. In view of the last equality, the previous containment reads $v^{-1}(\mathcal{A} - \varkappa) \subset \mathcal{C}$.

Therefore v must be the unique element of W taking the alcove $\mathcal{A} - \varkappa$ into the dominant Weyl chamber $\mathcal{C}$. Then $r = v^{-1}(\varkappa)$.

This argument proves both the injectivity and surjectivity of the mapping in question. $\square$

Letting $\delta - I := N(w) \cap (\delta - \Delta^+)$, we easily deduce that I is an ideal, if $w \in \widehat{W}_{\mathrm{dom}}$. We say $\delta - I$ is the *first layer* of $N(w)$ and I is the *first layer ideal* of w. However, an ideal I may well arise from different dominant elements. To obtain a bijection, one has to impose further constraints on dominant elements. One may attempt to consider either maximal or minimal bi-convex subsets with first layer $\delta - I$. This naturally leads to notions of "minimal" and "maximal" elements. This terminology suggested in [13] is also explained by the relationship between these elements and dominant regions of the Shi arrangement; see Section 3. However, the formal definitions do not require invoking arrangements. Furthermore, we want to stress that many results relating the ideals and these two kinds of dominant elements can be obtained without ever mentioning the Shi (or Catalan) arrangement.

Definition 2.3. $w \in \widehat{W}$ is called *minimal*, if

(i) w is dominant;
(ii) if $\alpha \in \widehat{\Pi}$ and $w^{-1}(\alpha) = k\delta + \mu$ for some $\mu \in \Delta$, then $k \geq -1$.

Using (i), condition (ii) can be made more precise. If $k \in \{-1, 0\}$, then $\mu \in \Delta^+$. The set of minimal elements is denoted by $\widehat{W}_{\mathrm{min}}$.

Proposition 2.4 ([3, Proposition 2.12]). *There is a bijection between $\widehat{W}_{\mathrm{min}}$ and $\mathfrak{Ad}$. Namely,*

- *given $w \in \widehat{W}_{\mathrm{min}}$, the corresponding ideal is $\{\mu \in \Delta^+ \mid \delta - \mu \in N(w)\}$;*
- *given $I \in \mathfrak{Ad}$, the corresponding minimal element is determined by the finite bi-convex set*

$$\bigcup_{k\geq 1}(k\delta - I^k) \subset \widehat{\Delta}^+.$$

Here I^k is defined inductively by $I^k = (I^{k-1} + I) \cap \Delta^+$.

If $N \subset \widehat{\Delta}^+$ is a finite convex subset, containing $\delta - I$, then it must also contain $\cup_{k\geq 1}(k\delta - I^k)$. So, the latter is the minimal bi-convex subset containing $\delta - I$.

The first layer ideal of $w \in \widehat{W}_{\mathrm{min}}$ is denoted by I_w.

Proposition 2.5 ([9, Theorem 2.2] [13, 6.3(1)]). *If $w \in \widehat{W}_{\mathrm{min}}$, then $\Gamma(I_w) = \{\gamma \in \Delta^+ \mid w(\delta - \gamma) \in -\widehat{\Pi}\}$.*

Following [4], we give a "geometric" description of the minimal elements. Set

$$D_{\mathrm{min}} = \{x \in V \mid (x, \alpha) \geq -1\ \forall \alpha \in \Pi\ \&\ (x, \theta) \leq 2\}.$$

It is a certain simplex in V.

Proposition 2.6 ([4, Propositions 2 and 3]).

(1) $w = v \cdot t_r \in \widehat{W}_{\min} \Longleftrightarrow \begin{cases} w \text{ is dominant,} \\ v(r) \in D_{\min} \cap Q^{\vee}. \end{cases}$

(2) *The mapping* $\widehat{W}_{\min} \to D_{\min} \cap Q^{\vee}$, $w = v \cdot t_r \mapsto v(r)$, *is a bijection.*

Proof. (1) "$\Rightarrow$" The first condition is satisfied by the definition.

Next, we have $w^{-1}(x) = v^{-1}(x) + (x, v(r))\delta$ for any $x \in V \oplus \mathbb{R}\delta$. In particular,

$$\begin{aligned} &w^{-1}(\alpha_i) = v^{-1}(\alpha_i) + (\alpha_i, v(r))\delta, \qquad i \geq 1, \\ &w^{-1}(\alpha_0) = -v^{-1}(\theta) + (1 - (\theta, v(r)))\delta. \end{aligned} \tag{2.7}$$

Comparing this with Definition 2.3(ii), one concludes that $v(r) \in D_{\min}$.

"$\Leftarrow$" The previous argument can be reversed.

(2) This follows from part 1 and Proposition 2.2. $\square$

Remark. The above proof applies equally well to Propositions 2.14, 5.3 and 5.10 below. It is a simplified version of the proof of Propositions 2 and 3 in [4].

It follows that $\#(\mathfrak{Ad})$ equals the number of integral points in $D_{\min}$. (Unless otherwise stated, an "integral point" is a point lying in $Q^{\vee}$.) A pleasant feature of this situation is that there is an element of $\widehat{W}$ that takes $D_{\min}$ to a dilated closed fundamental alcove. Namely, $w(D_{\min}) = (h+1)\overline{\mathcal{A}}$ for some $w \in \widehat{W}$, see [4, Theorem 1]. Write θ as a linear combination of simple roots: $\theta = \sum_i c_i\alpha_i$. The integers c_i are said to be the *coordinates of* θ. By a result of Haiman [6, 7.4], the number of integral points in $t\overline{\mathcal{A}}$ is equal to

$$\prod_{i=1}^{p} \frac{t + e_i}{1 + e_i} \tag{2.8}$$

whenever t is relatively prime with all the coordinates of θ. Since this condition is satisfied for $t = h + 1$, one obtains

$$\#(\mathfrak{Ad}) = \prod_{i=1}^{p} \frac{h + e_i + 1}{e_i + 1}. \tag{2.9}$$

It is the main result of [4].

Combining Proposition 2.5 and Eq. (2.7) yields the assertion that $\#\Gamma(I_w) = k$ if and only if $v(r)$ lies on a face of $D_{\min}$ of codimension k [9, Theorem 2.9].

Now, we turn to considering maximal (dominant) elements of $\widehat{W}$ that are introduced and studied by Sommers [13]. Most of the results on these elements are due to him. Because we want to have a uniform treatment for both minimal and maximal elements, some assertions below have no exact counterparts in [13]. For these reason, we also give some proofs.

Definition 2.10. $w \in \widehat{W}$ is called *maximal*, if

(i) w is dominant;

(ii) if $\alpha \in \widehat{\Pi}$ and $w^{-1}(\alpha) = k\delta + \mu$ for some $\mu \in \Delta$, then $k \leq 1$.

Using (i), condition (ii) can be made more precise. If $k = 1$, then $\mu \in -\Delta^+$; if $k = 0$, then $\mu \in \Delta^+$. The set of maximal elements is denoted by $\widehat{W}_{\max}$.

If $I \in \mathfrak{Ad}_0$, then for any $\mu \in \Delta^+$ we define $k(\mu, I)$ as the minimal possible number of summands in the expression $\mu = \sum_i \nu_i$, where $\nu_i \in \Delta^+ \setminus I$. Notice that this definition only makes sense for strictly positive ideals.

Proposition 2.11 ([13, Section 5]). *There is a bijection between $\widehat{W}_{\max}$ and $\mathfrak{Ad}_0$. Namely,*

- *given $w \in \widehat{W}_{\max}$, the corresponding strictly positive ideal is $\{\mu \in \Delta^+ \mid \delta - \mu \in N(w)\}$;*
- *given $I \in \mathfrak{Ad}_0$, the corresponding maximal element is determined by the finite bi-convex set*

$$\{m\delta - \gamma \mid \gamma \in I \ \&\ 1 \le m \le k(\gamma, I) - 1\}. \qquad (\lozenge)$$

Proof. (1) Suppose $w \in \widehat{W}$ is dominant, and let I be the first layer ideal of w. Assuming that $I \cap \Pi \ni \alpha$, we show that w cannot be maximal. For any $\gamma \in I$, let k_γ be the maximal integer such that $k_\gamma\delta - \gamma \in N(w)$, i.e.,

$$N(w) = \{l\delta - \gamma \mid \gamma \in I \ \&\ 1 \le l \le k_\gamma\}.$$

Let $I(\alpha)$ be the ideal generated by α. Clearly, $I(\alpha) \subset I$. Set

$$N(w)^{\langle 2\rangle} = \{l\delta - \gamma \mid \gamma \in I(\alpha) \ \&\ 1 \le l \le 2k_\gamma\} \cup \{l\delta - \gamma \mid \gamma \in I \setminus I(\alpha) \\ \&\ 1 \le l \le k_\gamma\}.$$

Obviously, $N(w)^{\langle 2\rangle}$ is finite and has the same first layer as $N(w)$. It is also easy to verify that $N^{\langle 2\rangle}$ is again bi-convex. Hence $N(w)^{\langle 2\rangle} = N(w')$ for some $w' \in \widehat{W}$. Since $N(w') \supset N(w)$, there is a presentation $w' = uw$, where $\ell(w') = \ell(u) + \ell(w)$. If s_ν ($\nu \in \widehat{\Pi}$) is the rightmost reflection in a reduced decomposition for u, then $w^{-1}(\nu) = k\delta - \mu$ with $k \ge 2$, as the first layers of $N(w')$ and $N(w)$ are the same. Thus, w is not maximal.

(2) Suppose $I \in \mathfrak{Ad}_0$, and let $w \in \widehat{W}$ be any dominant element with first layer ideal I. Since $\widehat{\Delta}^+ \setminus N(w)$ is convex and contains $\delta - (\Delta^+ \setminus I)$, it follows from the very definition of numbers $k(\gamma, I)$ that $l\delta - \gamma \in \widehat{\Delta}^+ \setminus N(w)$ for all $l \ge k(\gamma, I)$. Hence $N(w)$ is contained in the finite set given by Eq. ($\lozenge$) in Proposition 2.11. It only remains to prove that the latter is bi-convex. For this crucial fact, we refer to [13, Lemma 5.2]. $\square$

The strictly positive ideal corresponding to $w \in \widehat{W}_{\max}$ (the first layer ideal of w) is denoted by I^w. For an ideal $I \subset \Delta^+$, we write $\Xi(I)$ for the set of maximal elements of $\Delta^+ \setminus I$. It is immediate that $\Xi(I)$ is an antichain.

Proposition 2.12 ([13, 6.3(2)]). *If $w \in \widehat{W}_{\max}$, then $\Xi(I^w) = \{\gamma \in \Delta^+ \mid w(\delta - \gamma) \in \widehat{\Pi}\}$.*

Remark 2.13. Note that antichains of the form $\Xi(I^w)$ are not arbitrary. From the definition of a strictly positive ideal it readily follows that, given $\Xi \in \mathfrak{An}$, we have $\Xi = \Xi(I)$ for some $I \in \mathfrak{Ad}_0$ if and only if for any $\alpha \in \Pi$ there is a $\gamma \in \Xi$ such that $\gamma \succcurlyeq \alpha$. We shall say that such an antichain *covers* the simple roots.

Now, we proceed to a "geometric" characterization of the maximal elements. Set

$$D_{\max} = \{x \in V \mid (x, \alpha) \le 1 \ \forall \alpha \in \Pi \ \&\ (x, \theta) \ge 0\}.$$

It is a certain simplex in V.

Proposition 2.14 (cf. [13, Proposition 5.6]).

(1) $w = v \cdot t_r \in \widehat{W}_{\max} \Longleftrightarrow \begin{cases} w \text{ is dominant,} \\ v(r) \in D_{\max} \cap Q^\vee. \end{cases}$

(2) *The mapping* $\widehat{W}_{\max} \to D_{\max} \cap Q^\vee$, $w = v \cdot t_r \mapsto v(r)$, *is a bijection.*

Proof. (1) The argument is the same as in Proposition 2.6, taking into account that the constraints for $D_{\max}$ are different.

(2) This follows from part 1 and Proposition 2.2. □

Proposition 2.15 (cf. [13, Proposition 6.2(2)]). *Suppose* $w = v \cdot t_r \in \widehat{W}_{\max}$. *Then* $\#\Xi(I^w) = k$ *if and only* $v(r)$ *lies on a face of codimension* k *of* $D_{\max}$.

Proof. Combine Proposition 2.12 and Eq. (2.7). □

Since Π is the only antichain of cardinality p [9, 2.10(ii)] and it is certainly of the form $\Xi(I^w)$, we see that $D_{\max}$ has a unique integral vertex.

In order to compute $\#(D_{\max} \cap Q^\vee)$, we replace $D_{\max}$ with another simplex. Let $\{\varpi_i^\vee\}_{i=1}^p$ denote the dual basis of V for $\{\alpha_i\}_{i=1}^p$. Set $\rho^\vee = \sum_{i=1}^p \varpi_i^\vee$. Since the sum of the coordinates of θ equals $h-1$, the translation $x \mapsto t_{-\rho^\vee} * x = x - \sum_{i=1}^p \varpi_i^\vee$ takes $D_{\max}$ to the negative dilated fundamental alcove

$$-(h-1)\overline{\mathcal{A}} = \{x \in V \mid (x,\alpha) \le 0\ \forall \alpha \in \Pi;\ (x,\theta) \ge 1-h\}.$$

It may happen that $\rho^\vee$ does not belong to $Q^\vee$, so that this translation, which is in the extended affine Weyl group, does not belong to $\widehat{W}$, while we wish to have a transformation from $\widehat{W}$. Nevertheless, since $h-1$ is relatively prime with the index of connection of Δ, it follows from [4, Lemma 1] that there is an element of $\widehat{W}$ that takes $D_{\max}$ to $(1-h)\overline{\mathcal{A}}$.

Again, using the above-mentioned result of Haiman, see Eq. (2.8), one obtains the following.

Theorem 2.16 ([1, 9, 13]).

$$\#(\mathfrak{Ad}_0) = \prod_{i=1}^{p} \frac{h+e_i-1}{e_i+1}.$$

Remark. The proofs in [1] and [9] are based on the fact that the strictly positive ideals correspond to the bounded regions of the Catalan arrangement and that the number of bounded regions of any hyperplane arrangement can be computed via the characteristic polynomial of this arrangement, see Section 3.

3. Ideals and dominant regions of the Catalan arrangement

Recall a bijection between the ideals of Δ^+ and the dominant regions of the Catalan arrangement. This bijection is due to Shi [12, Theorem 1.4].

For $\mu \in \Delta^+$ and $k \in \mathbb{Z}$, define the hyperplane $\mathcal{H}_{\mu,k}$ in V as $\{x \in V \mid (x,\mu) = k\}$. The *Catalan arrangement*, $\mathrm{Cat}(\Delta)$, is the collection of hyperplanes $\mathcal{H}_{\mu,k}$, where $\mu \in \Delta^+$ and $k = -1, 0, 1$. The *regions* of an arrangement are the connected components of the complement in V of the union of all its hyperplanes. Obviously, the dominant regions of $\mathrm{Cat}(\Delta)$ are the same as those for the *Shi arrangement* $\mathrm{Shi}(\Delta)$. The latter is the collection of hyperplanes $\mathcal{H}_{\mu,k}$, where $\mu \in \Delta^+$ and $k = 0, 1$. But, it will be more convenient for us to deal with the arrangement $\mathrm{Cat}(\Delta)$, since it is W-invariant.

It is clear that $\mathcal{C}$ is a union of regions of $\mathrm{Cat}(\Delta)$. Any region lying in $\mathcal{C}$ is said to be *dominant*. The Shi bijection takes an ideal $I \subset \Delta^+$ to the dominant region

$$R_I = \{x \in \mathcal{C} \mid (x, \gamma) > 1, \text{ if } \gamma \in I \ \& \ (x, \gamma) < 1, \text{ if } \gamma \notin I\}. \tag{3.1}$$

It should be noted that the proof given by Shi in [12] consists essentially in a reference to his earlier work [11]. It is not, however, easy to extract the actual proof from Shi's papers. The most subtle point is to show that $R_I \neq \varnothing$ for any $I \in \mathfrak{Ad}$. And this fact readily follows from the theory of minimal elements developed by Cellini and Papi in [3, 4]:

If $w \in \widehat{W}$ is the minimal element corresponding to I, then $w^{-1} * \mathcal{A} \subset R_I$.

Indeed, $\mathcal{H}_{\mu,1}$ separates $\mathcal{A}$ and $w^{-1} * \mathcal{A}$ if and only if $w(\delta - \mu) \in -\widehat{\Delta}^+$, see [3, 1.1]. In fact, $w^{-1} * \mathcal{A}$ is the alcove nearest to the origin in R_I.

A region (of an arrangement) is called *bounded*, if it is contained in a sphere about the origin.

Proposition 3.2 ([1, 9]). *$I \in \mathfrak{Ad}(\mathfrak{g})_0$ if and only if the region R_I is bounded.*

If R_I is bounded, then it obviously contains an alcove that is most distant from the origin. It was shown in [13] that if w is the maximal element corresponding to $I \in \mathfrak{Ad}_0$, then $w^{-1} * \mathcal{A}$ is the most distant from the origin alcove in R_I.

The number of regions and bounded regions of any hyperplane arrangement can be counted through the use of a striking result of Zaslavsky. Let $\chi(\mathcal{A}, t)$ denote the characteristic polynomial of a hyperplane arrangement $\mathcal{A}$ in V (see e.g. [1, Section 2] for precise definitions).

Theorem 3.3 (Zaslavsky).

(1) *The number of regions into which $\mathcal{A}$ dissects V equals* $(-1)^p \chi(\mathcal{A}, -1)$.

(2) *The number of bounded regions into which $\mathcal{A}$ dissects V equals* $|\chi(\mathcal{A}, 1)|$.

In [1], Athanasiadis gives a nice case-free proof of the following formula for the characteristic polynomial of the Catalan arrangement:

$$\chi(\mathrm{Cat}(\Delta), t) = \prod_{i=1}^{p} (t - h - e_i). \tag{3.4}$$

Since $\mathrm{Cat}(\Delta)$ is W-invariant, the values $\frac{|\chi(\mathrm{Cat}(\Delta), \pm 1)|}{\#(W)}$ give the number of bounded and all regions in $\mathcal{C}$, respectively. In this way, one obtains explicit formulae for the cardinality of $\mathfrak{Ad}_0$ and $\mathfrak{Ad}$ written already down in Section 2. Thus, the characteristic polynomial of the Catalan arrangement provides an alternative approach to counting ideals and strictly positive ideals.

4. Short antichains and $\mathfrak{b}$-stable subspaces in the little adjoint G-module

For the rest of the paper, we stick to the case in which Δ has roots of different length. Then we naturally have long and short roots, long and short reflections, etc. Our goal is to show that the theory presented in the previous sections can be extended to the setting, where one pays attention to the length of roots involved. A piece of such theory has already appeared in [10], where we studied Abelian ideals of Δ^+ consisting of only long roots. Now, we consider the general case. Our treatment will again be combinatorial. We wish, however, to stress that it has a related representation-theoretic picture. While the ideals (antichains) in Δ^+ correspond bijectively to the $\mathfrak{b}$-stable subspaces in $\mathfrak{g}$ having no nonzero semisimple elements, our short antichains in Δ^+ correspond bijectively to the $\mathfrak{b}$-stable subspaces, without nonzero semisimple elements, in the little adjoint $\mathfrak{g}$-module.

To distinguish various objects associated with long and short roots, we use the subscripts "l" and "s", respectively. For instance, Π_l is the set of long simple roots and Δ_s^+ is the set of short positive roots. Accordingly, each simple reflection s_i is either short or long. Since θ is long, the simple root α_0 and the reflection s_0 are regarded as long. Therefore, $\widehat{\Pi}_l = \Pi_l \cup \{\alpha_0\}$. Write θ_s for the unique short dominant root in Δ^+. A simple $\mathfrak{g}$-module with highest weight θ_s, $\mathbb{V}(\theta_s)$, is said to be *little adjoint*. The set of nonzero weights of $\mathbb{V}(\theta_s)$ is Δ_s, all nonzero weights are simple, and the multiplicity of the zero weight is $\#(\Pi_s)$ [8, 2.8].

Definition 4.1. An antichain $\Gamma \subset \Delta^+$ is called *short*, if it consists of short roots, i.e., $\Gamma \subset \Delta_s^+$. Similarly, one defines a *long* antichain.

If Γ is a short antichain, then $\Gamma^\vee$ is a long antichain in the dual root system $\Delta^\vee$. Therefore, it suffices, in principle, to consider only short antichains. We write $\mathfrak{An}_s$ for the set of all short antichains of Δ^+. The respective set of ideals is denoted by $\mathfrak{Ad}_s$.

Recall that, for any finite-dimensional rational G-module $\mathbb{V}$, there are notions of semisimple and nilpotent elements, generalizing those in $\mathfrak{g}$, see [14]. An element $v \in \mathbb{V}$ is called *semisimple*, if the orbit Gv is closed; it is called *nilpotent*, if the closure of Gv contains the origin. We shall say that a subspace of $\mathbb{V}$ is *nilpotent*, if it consists of nilpotent elements.

Proposition 4.2. *There is a one-to-one correspondence between $\mathfrak{An}_s$ and the nilpotent $\mathfrak{b}$-stable subspaces of $\mathbb{V}(\theta_s)$.*

Proof. If Γ is a short antichain, then the corresponding subspace $\mathbb{U}_\Gamma \subset \mathbb{V}(\theta_s)$ is defined as the sum of weight spaces $\mathbb{V}(\theta_s)_\mu$, $\mu \in \Delta_s^+$, such that $\gamma_i \preccurlyeq \mu$ for some $\gamma_i \in \Gamma$. Being a subset of Δ_s^+, the weights of $\mathbb{U}_\Gamma$ lie in an open halfspace of V. Hence all elements of $\mathbb{U}_\Gamma$ are nilpotent, see e.g. [14, 5.4]. Conversely, if $\mathbb{U}$ is a $\mathfrak{b}$-stable subspace of $\mathbb{V}(\theta_s)$, then it is a sum of weight spaces. Assume $\mathbb{U}$ contains a weight space $\mathbb{V}(\theta_s)_\mu$ with $\mu \in -\Delta_s^+$. It then follows from the $\mathfrak{b}$-invariance that $\mathbb{U}$ has non-empty intersection with $\mathbb{V}(\theta_s)_0$. Hence $\mathbb{U}$ is not nilpotent, because the orbit Gx is closed for any $x \in \mathbb{V}(\theta_s)_0$. Thus, the weights of $\mathbb{U}$ form a subset of Δ_s^+. The minimal elements of this set of weights give us the required short antichain. $\square$

If Γ is a short antichain, then the set of weights of the corresponding nilpotent $\mathfrak{b}$-stable subspace of $\mathbb{V}(\theta_s)$ is $I\langle\Gamma\rangle \cap \Delta_s^+$.

5. Short antichains, s-minimal and s-maximal elements of $\widehat{W}$

Our goal is to show that the theory described in Section 2 extends well to short antichains.

Definition 5.1. $w \in \widehat{W}$ is called *s-minimal*, if

(i) w is dominant;
(ii) if $\alpha \in \Pi_s$ and $w^{-1}(\alpha) = k\delta + \mu$ with $\mu \in \Delta$, then $k \geq -1$;
(iii) if $\alpha \in \widehat{\Pi}_l$ and $w^{-1}(\alpha) = k\delta + \mu$ with $\mu \in \Delta$, then $k \geq 0$.

Using (i), conditions (ii), (iii) can be made more precise. If $k = 0$ or $k = -1$ in (ii), then $\mu \in \Delta^+$.

We write $\widehat{W}_{\min}^{(s)}$ for the set of all s-minimal elements. Notice that $\widehat{W}_{\min}^{(s)} \subset \widehat{W}_{\min}$.

Proposition 5.2. *The bijection between $\mathfrak{Ad}$ and $\widehat{W}_{\min}$ described in Proposition 2.4 gives rise to a bijection between $\mathfrak{An}_s$ (or $\mathfrak{Ad}_s$) and $\widehat{W}_{\min}^{(s)}$.*

Proof. (1) Suppose $w \in \widehat{W}_{\min}^{(s)}$, and let I_w be the corresponding ideal. It follows from Definition 5.1(ii), (iii) and Proposition 2.5 that $\Gamma(I_w) \subset \Delta_s^+$. Thus, we obtain a short antichain.

(2) The use of Proposition 2.5 gives also the converse. □

Now, we give a geometric description of s-minimal elements in the spirit of Section 2. Set

$$D_{\min}^{(s)} = \{x \in V \mid (x, \alpha) \geq -1 (\alpha \in \Pi_s);\ (x, \alpha) \geq 0 (\alpha \in \Pi_l); (x, \theta) \leq 1\},$$

and recall that $w = v \cdot t_r$, where $v \in W$ and $r \in Q^\vee$.

Proposition 5.3.

(1) $w = v \cdot t_r \in \widehat{W}_{\min}^{(s)} \Longleftrightarrow \begin{cases} w \text{ is dominant,} \\ v(r) \in D_{\min}^{(s)} \cap Q^\vee. \end{cases}$

(2) *The mapping $\widehat{W}_{\min}^{(s)} \to D_{\min}^{(s)} \cap Q^\vee$, $w = v \cdot t_r \mapsto v(r)$, is a bijection.*

Proof. The argument is the same as in Proposition 2.6, taking into account that the constraints for $D_{\min}^{(s)}$ are different. □

In order to compute the number $\#(D_{\min}^{(s)} \cap Q^\vee)$, we perform the following transformation.

Set $\rho_s^\vee = \sum_{\alpha_i \in \Pi_s} \varpi_i^\vee$. It is easily seen that the translation $t_{\rho_s^\vee}$ takes $D_{\min}^{(s)}$ to the dilated closed fundamental alcove

$$(g+1)\overline{\mathcal{A}} = \{x \in V \mid (x, \alpha) \geq 0\ \forall\, \alpha \in \Pi; (x, \theta) \leq g+1\}.$$

Here $g = (\theta, \sum_{\alpha_i \in \Pi_s} \varpi_i^\vee)$, i.e., it is the sum of the *short* coordinates of θ (i.e., those corresponding to the short simple roots). It is easy to obtain other formulae for g.

E.g., $g = (\#\Delta_s)/p = (\rho_s, \theta^\vee)$, where ρ_s is the half-sum of all positive short roots. If we want to explicitly indicate that g depends on Δ, we write g_Δ.

Although the above translation may not belong to $\widehat{W}$, the very existence of such a transformation and Lemma 1 in [4] show that the following is true

Lemma 5.4. *If $g+1$ and the index of connection of Δ are relatively prime, then there is an element of $\widehat{W}$ that takes $D^{(s)}_{\min}$ to $(g+1)\overline{\mathcal{A}}$.*

The numbers g for all root systems with roots of different lengths are as follows:

Δ	$\mathbf{C}_p$	$\mathbf{B}_p$	$\mathbf{F}_4$	$\mathbf{G}_2$
g	$2p-2$	2	6	3

It follows that Lemma 5.4 always applies and hence $\#(\mathfrak{A}\mathfrak{n}_s) = \#(\widehat{W}^{(s)}_{\min}) = \#((g+1)\overline{\mathcal{A}} \cap Q^\vee)$. In turn, if $g+1$ is relatively prime with the coordinates of θ, then this number is computed by Eq. (2.8). One sees that the condition of relative primeness does not hold only for $\mathbf{G}_2$. (However, this case can be studied by hand.) Thus, we obtain

Theorem 5.5. *Suppose $|\theta|^2/|\theta_s|^2 = 2$. Then*

$$\#(\mathfrak{A}\mathfrak{n}_s) = \prod_{i=1}^{p} \frac{g+e_i+1}{e_i+1}.$$

It is easily seen that $\#(\mathfrak{A}\mathfrak{n}_s) = 4$ for $\mathbf{G}_2$.

Looking at the factors occurring in the formula of Theorem 5.5, one may notice that there is a nice formula for $\mathfrak{A}\mathfrak{n}_s$, which resembles Eq. (2.9) and also covers the case of $\mathbf{G}_2$. Here it is.

Suppose $e_1 < e_2 < \cdots < e_p$ and set $n = \#(\Pi_s)$. Then for any Δ we have

$$\#(\mathfrak{A}\mathfrak{n}_s) = \prod_{i=1}^{n} \frac{h+e_i+1}{e_i+1}. \tag{5.6}$$

But it is not clear how to prove this a priori.

Changing the role of long and short roots in Definition 5.1, one may define l-*minimal* elements, which are in a one-to-one correspondence with the *long* antichains. Since the proofs here are similar, we state only results. The simplex associated with the l-minimal elements is

$$D^{(l)}_{\min} = \{x \in V \mid (x,\alpha) \geq 0 (\alpha \in \Pi_s);\ (x,\alpha) \geq 1 (\alpha \in \Pi_l); (x,\theta) \leq 2\},$$

and the l-minimal elements bijectively correspond to the integral points of $D^{(l)}_{\min}$. The shift in the direction of $\rho_l^\vee = \sum_{\alpha_i \in \Pi_l} \varpi_i^\vee$ takes $D^{(l)}_{\min}$ to $(h+1-g)\mathcal{A}$. Therefore, if Δ is not of type $\mathbf{G}_2$, then

$$\#(\mathfrak{A}\mathfrak{n}_l) = \prod_{i=1}^{p} \frac{h-g+e_i+1}{e_i+1}. \tag{5.7}$$

Since this number can also be computed as $\#(\mathfrak{An}_s)$ for the dual root system $\Delta^\vee$, a relation between g for Δ and $\Delta^\vee$ emerges. Namely, $g_\Delta + g_{\Delta^\vee} = h$.

Now we turn to considering a "short" analogue of maximal elements.

Definition 5.8. $w \in \widehat{W}$ is called *s-maximal*, if

(i) w is dominant;
(ii) if $\alpha \in \Pi_s$ and $w^{-1}(\alpha) = k\delta + \mu$ with $\mu \in \Delta$, then $k \le 1$;
(iii) if $\alpha \in \widehat{\Pi}_l$ and $w^{-1}(\alpha) = k\delta + \mu$ with $\mu \in \Delta$, then $k \le 0$.

Using (i), conditions (ii), (iii) can be made precise. If $k = 0$, then $\mu \in \Delta^+$; if $k = 1$ in (ii), then $\mu \in -\Delta^+$.

We write $\widehat{W}^{(s)}_{\max}$ for the set of all s-maximal elements. Notice that $\widehat{W}^{(s)}_{\max} \subset \widehat{W}_{\max}$.

As in case of maximal elements, we wish to set up a one-to-one correspondence between the s-maximal elements and a certain subset of $\mathfrak{An}_s$. In order to distinguish the right subset we need some preparations. Recall that, although Δ_s is not a sub-root system of Δ, it is a root system in its own right. Clearly, Δ_s^+ is the set of positive roots for Δ_s. Let us write $\Pi(\Delta_s^+)$ for the corresponding set of simple roots. Since Δ_s spans the whole space V, we have $\#\Pi(\Delta_s) = \dim V = \#\Pi$. Obviously, $\Pi_s \subset \Pi(\Delta_s^+)$. Other roots in $\Pi(\Delta_s^+)$ are in a natural bijection with Π_l. Namely, each $\beta \in \Pi_l$ is replaced by a short root as follows. Let α be the closest to β (in the sense of the Dynkin diagram) short simple root. The sum of all simple roots in the string connecting α and β is a short root, which is a simple root for Δ_s^+. That one really obtains a basis for Δ_s^+ is easily verified case-by-case. A conceptual proof can be given using the fact that both Π_l and Π_s form connected subsets of the Dynkin diagram.

Warning. Although we often consider antichains lying in (certain subsets of) Δ_s^+, it is always meant that the ordering "$\preccurlyeq$" is inherited from the whole of Δ^+.

Proposition 5.9. *The bijection between $\mathfrak{Ad}_0$ and $\widehat{W}_{\max}$ described in Proposition 2.11 gives rise to a bijection between $\widehat{W}^{(s)}_{\max}$ and the short antichains lying in $\Delta_s^+ \setminus \Pi(\Delta_s^+)$.*

Proof. The correspondence described in Proposition 2.11 attaches to a maximal element w its first layer ideal, I^w. But even if w is s-maximal, the generators of I^w may not be short roots. So, we do not immediately obtain a required short antichain. To correct this, we take $I^w \cap \Delta_s^+$. (It is also the set of weights of a nilpotent $\mathfrak{b}$-stable subspace of $\mathbb{V}(\theta_s)$.) The set of generators (minimal elements) of $I^w \cap \Delta_s^+$ is a short antichain of Δ^+, which we attach to $w \in \widehat{W}^{(s)}_{\max}$.

Now, we prove that the resulting antichain lies in $\Delta_s^+ \setminus \Pi(\Delta_s^+)$ and that this correspondence is really a bijection.

Recall from Section 2 that $\Xi(I)$ is the set of maximal elements of $\Delta^+ \setminus I$ and that in case of maximal elements $\Xi(I^w)$ is described in Proposition 2.12. That description implies that, for $w \in \widehat{W}^{(s)}_{\max}$, $\Xi(I^w)$ consists of short roots. Since $\Xi(I^w)$ covers all simple roots (see Remark 2.13) and consists of short roots, it also covers all roots from $\Pi(\Delta_s^+)$. (Use the explicit description of $\Pi(\Delta_s^+)$ given above.) This means that the short antichain $\Gamma(I^w \cap \Delta_s^+)$ does not contain roots from $\Pi(\Delta_s^+)$.

Injectivity. If $w, w' \in \widehat{W}^{(s)}_{\max}$ are different, then $\Xi(I^w) \ne \Xi(I^{w'})$. Since these two sets consist of short roots, we obviously have $I^w \cap \Delta_s^+ \ne I^{w'} \cap \Delta_s^+$.

Surjectivity. If Γ is an antichain of Δ^+ lying in $\Delta_s^+ \setminus \Pi(\Delta_s^+)$, then take all maximal short roots in $\Delta^+ \setminus I\langle\Gamma\rangle$. More precisely, let Ξ be the set of short roots μ such that if $\nu \in \Delta_s^+$ and $\nu \succ \mu$ then there is $\gamma \in \Gamma$ such that $\nu \succcurlyeq \gamma$. Then Ξ is a short antichain that covers all roots in $\Pi(\Delta_s^+)$ and hence the whole of Π. In view of Remark 2.13, Ξ is of the form $\Xi(I^w)$ for some $w \in \widehat{W}_{\max}$. Finally, since Ξ consists of short roots, this w is s-maximal. □

The antichains of Δ^+ lying in $\Delta_s^+ \setminus \Pi(\Delta_s^+)$ are said to be *strictly s-positive*. The corresponding subset of $\mathfrak{An}_s$ is denoted by $\mathfrak{An}_{s,0}$.

Once again, the next part of our program is a geometric description. Set

$$D_{\max}^{(s)} = \{x \in V \mid (x, \alpha) \le 1 (\alpha \in \Pi_s);\ (x, \alpha) \le 0 (\alpha \in \Pi_l);\ (x, \theta) \ge 1\}.$$

Proposition 5.10.

(1) $w = v \cdot t_r \in \widehat{W}_{\max}^{(s)} \Longleftrightarrow \begin{cases} w \text{ is dominant,} \\ v(r) \in D_{\max}^{(s)} \cap Q^\vee. \end{cases}$

(2) *The mapping* $\widehat{W}_{\max}^{(s)} \to D_{\max}^{(s)} \cap Q^\vee$, $w = v \cdot t_r \mapsto v(r)$, *is a bijection.*

Proof. The argument is the same as in Proposition 2.6, taking into account that the constraints for $D_{\max}^{(s)}$ are different. □

The translation in the direction of $-\rho_s^\vee$, which belongs to the extended affine Weyl group, takes $D_{\max}^{(s)}$ to $(1 - g)\mathcal{A}$. Since $g - 1$ is always relatively prime with the index of connection, there is also an element of $\widehat{W}$ that does the same, cf. Lemma 5.4. As in the case of s-minimal elements, we have $g - 1$ is relatively prime with the coordinates of θ, if Δ is not of type $\mathbf{G}_2$. Therefore, if $\Delta \in \{\mathbf{B}_p, \mathbf{C}_p, \mathbf{F}_4\}$, then

$$\#(\mathfrak{An}_{s,0}) = \#(\widehat{W}_{\max}^{(s)}) = \prod_{i=1}^{p} \frac{g + e_i - 1}{e_i + 1}. \tag{5.11}$$

For $\mathbf{G}_2$, this set consists of two elements.

6. Short antichains and the semi-Catalan arrangement

In this section, we study a hyperplane arrangement in V that has the same connection with short antichains in Δ^+ as the Catalan arrangement has with all antichains. This provides yet another approach to counting the short and strictly s-positive antichains.

Definition 6.1. (1) The *semi-Catalan arrangement* in V, $\mathrm{Cat}_s(\Delta)$, consists of the hyperplanes $\mathcal{H}_{\mu,k}$ ($\mu \in \Delta_s^+$, $k = -1, 0, 1$) and $\mathcal{H}_{\nu,0}$ ($\nu \in \Delta_l^+$).

(2) The *m-semi-Catalan arrangement* in V, $\mathrm{Cat}_s^m(\Delta)$, consists of the hyperplanes $\mathcal{H}_{\mu,k}$ ($\mu \in \Delta_s^+$, $k = -m, \dots, -1, 0, 1, \dots, m$) and $\mathcal{H}_{\nu,0}$ ($\nu \in \Delta_l^+$).

All these arrangements are deformations of the Coxeter arrangement. Notice also that $\mathrm{Cat}_s^0(\Delta)$ is the usual Coxeter arrangement, and $\mathrm{Cat}_s(\Delta) = \mathrm{Cat}_s^1(\Delta)$.

First, we are interested in the dominant regions of $\mathrm{Cat}_s(\Delta)$ and their relation to short antichains. Define a mapping

$$\psi : \mathfrak{An}_s \to \{\text{the dominant regions of } \mathrm{Cat}_s(\Delta)\}$$

as follows. For $\Gamma \in \mathfrak{An}_s$, let

$$\Gamma \overset{\psi}{\mapsto} R_\Gamma^{(s)} := \{x \in \mathcal{C} \mid (x, \mu) > 1, \text{ if } \mu \in I\langle\Gamma\rangle \cap \Delta_s^+ \ \&\ (x, \mu) < 1,\\ \text{if } \mu \in \Delta_s^+ \setminus I\langle\Gamma\rangle\}.$$

Theorem 6.2.

(i) *The mapping ψ is well-defined, and it is a bijection;*

(ii) *$R_\Gamma^{(s)}$ is bounded if and only if $\Gamma \in \mathfrak{An}_{s,0}$.*

Proof. (i) 1. Regarding Γ as a "usual" antichain, we can construct a region $R_{I\langle\Gamma\rangle}$, as prescribed by Eq. (3.1). Obviously, $R_{I\langle\Gamma\rangle} \subset R_\Gamma^{(s)}$. Hence the latter is non-empty.

2. Since the definition of the set $R_\Gamma^{(s)}$ includes a constraint for any hyperplane in $\mathrm{Cat}_s(\Delta)$ meeting $\mathcal{C}$, $R_\Gamma^{(s)}$ cannot contain more than one region. It is also clear that $R_\Gamma^{(s)} \neq R_{\Gamma'}^{(s)}$, if $\Gamma \neq \Gamma'$. For, if $\gamma \in \Gamma \setminus I\langle\Gamma'\rangle$, then $\mathcal{H}_{\gamma,1}$ separates $R_\Gamma^{(s)}$ and $R_{\Gamma'}^{(s)}$. Hence ψ is injective.

3. The surjectivity of ψ follows from the existence of the inverse map. Given a region R, take the set of walls of R separating R from the origin. Then the corresponding set of roots form a short antichain.

(ii) If $\Pi(\Delta_s^+) \cap I\langle\Gamma\rangle = \varnothing$, then $R_\Gamma^{(s)}$ belong to the bounded domain $\{x \in \mathcal{C} \mid (x, \mu) < 1, \mu \in \Pi(\Delta_s^+)\}$.

Conversely, assume $\beta \in I \cap \Pi(\Delta_s^+)$. Recall from Section 5 that $\Pi(\Delta_s^+)$ is in bijection with Π (β either belong to Π_s or is obtained via a simple procedure from a long simple root). Let β' be the simple root in Π corresponding to β and $\varphi_{\beta'}$ the respective fundamental weight of $\mathfrak{g}$. Then we claim that if $x \in R_\Gamma^{(s)}$, then $x + t\varphi_{\beta'} \in R_\Gamma^{(s)}$ for any $t \in \mathbb{R}_{\geq 0}$. Indeed, β is the minimal short root having nonzero β'-coordinate. Therefore all short roots having nonzero β'-coordinate are in $I\langle\Gamma\rangle$. This means that $R_\Gamma^{(s)}$ has no upper bound in the direction of $\varphi_{\beta'}$. Thus, $R_\Gamma^{(s)}$ is unbounded. $\square$

Let us look at the relationship between s-minimal and s-maximal elements on one hand, and dominant regions of $\mathrm{Cat}_s(\Delta)$ on the other hand.

Proposition 6.3.

(i) *Suppose $w \in \widehat{W}_{\min}^{(s)}$, and let $\Gamma \in \mathfrak{An}_s$ be the corresponding antichain. Then $w^{-1} * \mathcal{A} \subset R_\Gamma^{(s)}$, and it is the alcove nearest to the origin in $R_\Gamma^{(s)}$.*

(ii) *Suppose $w \in \widehat{W}_{\max}^{(s)}$, and let $\Gamma \in \mathfrak{An}_{s,0}$ be the corresponding antichain. Then $w^{-1} * \mathcal{A} \subset R_\Gamma^{(s)}$, and it is the alcove most distant from the origin in $R_\Gamma^{(s)}$.*

Proof. (i) It was already observed before that $w^{-1} * \mathcal{A} \subset R_{I\langle\Gamma\rangle} \subset R_\Gamma^{(s)}$. Suppose we are inside $w^{-1} * \mathcal{A}$. To get in an alcove that is closer to the origin, we must cross a wall separating $w^{-1} * \mathcal{A}$ from the origin. These walls correspond to the roots $\alpha \in \widehat{\Pi}$ such that $w^{-1}(\alpha) < 0$. But then $w^{-1}(\alpha) = -\delta + \mu$, where $\mu \in \Delta_s^+$. So that having crossed this wall, we get in another dominant region of $\mathrm{Cat}_s(\Delta)$.

(ii) Suppose $w \in \widehat{W}_{\max}^{(s)}$ and we are inside $w^{-1} * \mathcal{A}$. To get in an alcove that is more distant from the origin, we must cross a wall of $w^{-1} * \mathcal{A}$ that does not separate $w^{-1} * \mathcal{A}$ from the origin. These walls correspond to the roots $\alpha \in \widehat{\Pi}$ such that $w^{-1}(\alpha) > 0$. In view of the definition of s-maximal elements, there are two possibilities: (a) if $w^{-1}(\alpha) = \mu \in \Delta^+$, then crossing such a wall we leave $\mathcal{C}$; (b) if $w^{-1}(\alpha) = \delta - \mu$, $\mu \in \Delta_s^+$, then crossing such a wall we get in another dominant region of $\mathrm{Cat}_s(\Delta)$. Hence $w^{-1} * \mathcal{A}$ is the most distant from the origin alcove in a certain region.

The hyperplanes of $\mathrm{Cat}_s(\Delta)$ separating $w^{-1} * \mathcal{A}$ from the origin (not necessarily walls of $w^{-1} * \mathcal{A}$) correspond to the short roots μ such that $w(\delta - \mu) < 0$, i.e., these roots are exactly the short roots in the first layer ideal of w. According to the correspondence described in the proof of Proposition 5.9, the minimal elements of this set form the short antichain Γ attached to w. Thus, $w^{-1} * \mathcal{A}$ lies in the required alcove. □

Theorem 6.2 implies that the number of short or strictly s-positive antichains can be found through the use of the characteristic polynomial of $\mathrm{Cat}_s(\Delta)$. In fact, we are able to compute the characteristic polynomial for $\mathrm{Cat}_s^m(\Delta)$ with any m. One should just follow the scheme of Athanasiadis' proof for the usual m-Catalan arrangement, see [1, Theorem 3.1]. Let $P^\vee$ be the coweight lattice and $f = [P^\vee : Q^\vee]$. (Hence f is the index of connection of Δ.)

Theorem 6.4. *Suppose $t \in \mathbb{N}$, $t > mg$, and both t, $t - mg$ are relatively prime with all the coordinates of θ. Then*

$$\chi(\mathrm{Cat}_s^m(\Delta), t) = \frac{\#W}{f} \#((t - mg)\mathcal{A} \cap P^\vee).$$

Proof. We give only a sketch of the proof, where we indicate essential distinction from Athanasiadis' proof for an m-Catalan arrangement, referring to [1] for all details.

Let $\mathcal{P}$ denote the fundamental parallelepiped $\{\sum_{i=1}^p y_i \varpi_i^\vee \mid 0 \le y_i \le 1\}$. Set $\mathcal{P}_t = \mathcal{P} \cap \frac{1}{t} P^\vee$. Also, let $V_{\Delta,t}^m$ be the set of hyperplanes

$$\mathcal{H}_{\mu, k + \frac{n}{t}} (n, k \in \mathbb{Z}, |n| \le m, \mu \in \Delta_s) \qquad \text{and} \qquad \mathcal{H}_{\gamma,k} (k \in \mathbb{Z}, \gamma \in \Delta_l).$$

Note that fractional indices are allowed only for hyperplanes orthogonal to short roots, so that our $V_{\Delta,t}^m$ is different from that of Athanasiadis.

Given an arrangement $\mathcal{A}$ in V, according to a general result (Athanasiadis, Björner-Ekedahl), the value $\chi(\mathcal{A}, t)$ is equal to the number of points in the complement of all hyperplanes, counted after reduction modulo t, i.e., in $(\mathbb{Z}_t)^p$. More precisely, this equality holds for infinitely many t (this can be made even more precise, see [1, Section 2]). In our situation, as well as in [1], this means that t must be relatively prime with all the coefficients of θ.

Then the above general result leads to the equality $\chi(\mathrm{Cat}_s^m(\Delta), t) = \#\{\mathcal{P}_t \setminus V_{\Delta,t}^m\}$. Using the standard fact that $\mathcal{P}$ contains $(\#W)/f$ alcoves, this can be transformed to

$$\chi(\mathrm{Cat}_s^m(\Delta), t) = \frac{\#W}{f} \cdot \# \left(\left(\mathcal{A} \cap \frac{1}{t} P^\vee \right) \Big\backslash V_{\Delta,t}^m \right) = \frac{\#W}{f} \cdot \#((t\mathcal{A} \cap P^\vee) \backslash t V_{\Delta,t}^m).$$

It easily follows from the definition of $V_{\Delta,t}^m$ that $(t\mathcal{A} \cap P^\vee) \backslash t V_{\Delta,t}^m$ is obtained from $t\,\mathcal{A} \cap P^\vee$ by deleting the coweights lying on the hyperplanes $\mathcal{H}_{\alpha,i}$ with $\alpha \in \Pi_s$ and

$1 \le i \le m$. That is, the set in question is equal to

$$\{x \in P^\vee \mid (x,\alpha) > m(\alpha \in \Pi_s);\ (x,\alpha) > 0(\alpha \in \Pi_l);\ (x,\theta) < t\}.$$

Finally, the translation by the negative of $m\rho_s^\vee$ (which lies in $P^\vee$) takes this set to the points of $P^\vee$ lying in the open simplex $(t - gm)\mathcal{A}$. $\square$

Let us discuss consequences of this result. We use values of g given in Section 5. If $\Delta \in \{\mathbf{B}_p, \mathbf{C}_p, \mathbf{F}_4\}$ and t is relatively prime with the coordinates of θ, then the same holds for $t - mg$ with any m. It follows that

$$\chi(\mathrm{Cat}_s^m(\Delta), t) = \chi(\mathrm{Cat}_s^0(\Delta), t - mg) = \prod_{i=1}^{p}(t - mg - e_i).$$

(The first equality holds for infinitely many values of t; hence it holds always, as both parts are polynomials in t. The second equality is a statement about Coxeter arrangements.) In particular,

$$\chi(\mathrm{Cat}_s(\Delta), t) = \prod_{i=1}^{p}(t - g - e_i). \tag{6.5}$$

Combining Theorems 3.3 and 6.2, we conclude that

$$\#(\mathfrak{An}_s) = |\chi(\mathrm{Cat}_s(\Delta), -1)|/\#W \qquad \text{and} \qquad \#(\mathfrak{An}_{s,0}) = |\chi(\mathrm{Cat}_s(\Delta), 1)|/\#W,$$

which coincides, of course, with the formula in Theorem 5.5 and Eq. (5.11).

For $\mathbf{G}_2$, we have $g = 3$ and the assumption of Theorem 6.4 is satisfied only if m is even. Therefore

$$\chi(\mathrm{Cat}_s^{2m}(\mathbf{G}_2), t) = \chi(\mathrm{Cat}_s^0(\mathbf{G}_2), t - 6m) = (t - 6m - 1)(t - 6m - 5).$$

Using ad hoc arguments, one may derive the “odd” formula

$$\chi(\mathrm{Cat}_s^{2m+1}(\mathbf{G}_2), t) = (t - 6m - 5)(t - 6m - 7). \tag{6.6}$$

It is also easy to compute $\chi(\mathrm{Cat}_s^1(\mathbf{G}_2), t)$ directly from the definition of a characteristic polynomial.

Again, it is noteworthy that formulae (6.5) and (6.6) for $\chi(\mathrm{Cat}_s(\Delta), t)$ admit a uniform presentation for all non-simply laced cases, cf. Eq. (5.6).

Theorem 6.7. *If $n = \#(\Pi_s)$ and the exponents of Δ are increasingly ordered, then*

$$\chi(\mathrm{Cat}_s(\Delta), t) = \prod_{i \le n}(t - h - e_i) \prod_{i \ge n+1}(t - e_i).$$

Of course, it would be interesting to have a uniform proof (explanation) for this.

Remarks 6.8. (1) One may consider “short” analogues for other arrangements associated with root systems. For instance, the *extended semi-Shi arrangement*, $\mathrm{Shi}_s^m(\Delta)$, is the collection of hyperplanes $\mathcal{H}_{\mu,k}$ ($\mu \in \Delta_s^+, k = -m+1, \ldots, -1, 0, \ldots, m$) and $\mathcal{H}_{\nu,0}$ ($\nu \in \Delta_l^+$). It is not hard to compute that, for $\mathbf{C}_2$, the characteristic polynomial is equal to $(t-2m-1)^2$. For $\mathbf{G}_2$, it is equal to $(t-3m-1)(t-3m-2)$, at least if $m \le 3$. I conjecture that the following formula holds in general:

$$\chi(\mathrm{Shi}_s^m(\Delta), t) = \prod_{i=1}^{p}(t - mg_{\Delta^\vee} - e_i(\Delta_l)),$$

where $\{e_i(\Delta_l)\}$ are the exponents for the root system Δ_l. For instance, in the case of $\mathbf{F}_4$ we have $g_\Delta = g_{\Delta^\vee} = 6$ and Δ_l is of type $\mathbf{D}_4$. Therefore the conjectural expression is $(t-6m-1)(t-6m-3)^2(t-6m-5)$.

(2) The dominant regions of $\mathrm{Cat}_s(\Delta)$ provide a connection, in the spirit of [5], with nilpotent G-orbits in $\mathbb{V}(\theta_s)$. I hope to discuss this topic in a forthcoming publication.

7. Some numerical complements

In this section, we collect several results that can be proved in a case-by-case fashion.

7.1.

We know the number of all and short antichains for all irreducible reduced root systems. Using this, one may observe that $\#(\mathfrak{An}_s)$ divides $\#(\mathfrak{An})$ in all cases. Furthermore, the ratio has, a posteriori, an interesting description. Namely, let $\Delta(\Pi_l)$ be the root system whose set of simple roots is Π_l. Notice that $\Delta(\Pi_l)$ is smaller than Δ_l, and that the former is irreducible, since Π_l is a connected subset of the Dynkin diagram. Write $\mathfrak{An}(\Pi_l)$ for the set of all antichains in $\Delta(\Pi_l)^+$.

Theorem 7.1. $\#(\mathfrak{An}) = \#(\mathfrak{An}_s) \cdot \#(\mathfrak{An}(\Pi_l))$.

Proof. Case-by-case verification. For instance, in case of $\mathbf{F}_4$ we have $\#(\mathfrak{An}) = 105$, $\#(\mathfrak{An}_s) = 21$, and $\Delta(\Pi_l)$ is of type $\mathbf{A}_2$, where one has five antichains. $\square$

Of course, this proof is not illuminating. One may consider a natural mapping $\mathfrak{An} \to \mathfrak{An}_s$ that takes Γ to the set of minimal elements of $I\langle\Gamma\rangle \cap \Delta_s^+$. For $\mathbf{C}_p$, all fibres of this mapping have the same cardinality, which is 2. To some extent, this is an explanation in this case. Unfortunately, the "equicardinality" property does not hold for $\mathbf{F}_4$ and $\mathbf{G}_2$. The statement of Theorem 7.1 can be compared with another equality, which is easy to prove. The reflection $s_\gamma \in W$ is called short, if $\gamma \in \Delta_s^+$. Let W_s be the (normal) subgroup of W generated by all short reflections, and let $W(\Pi_l)$ be the Weyl group of $\Delta(\Pi_l)$. Then $W \simeq W_s \rtimes W(\Pi_l)$ (a semidirect product).

7.2.

We have shown that the short antichains of Δ^+ lying in $\Delta_s^+ \setminus \Pi(\Delta_s^+)$ are in a one-to-one correspondence with s-maximal elements, and then computed their number. However, it is also natural to enumerate the short antichains lying in $\Delta_s^+ \setminus \Pi_s$. (Recall that Π_s is a proper subset of $\Pi(\Delta_s^+)$.) Set $\mathfrak{An}_{ss} = \{\Gamma \in \mathfrak{An}_s \mid \Gamma \cap \Pi_s = \varnothing\}$. I did not find a suitable bijection for $\mathfrak{An}_{ss}$, but the following formula for the cardinality is true:

$$\#(\mathfrak{An}_{ss}) = \prod_{i=1}^{n} \frac{h + e_i - 1}{e_i + 1}, \tag{7.2}$$

where the notation is the same as in Theorem 6.7. Again, this formula bears a striking resemblance with Theorem 2.16. Direct calculations show that this gives us the correct number for $\mathbf{B}_p$ (this is easy, because there is only a few short roots), $\mathbf{F}_4, \mathbf{G}_2$.

The argument for $\mathbf{C}_p$ goes as follows. The set of positive roots $\Delta^+(\mathbf{C}_p)$ is naturally represented by the shifted Ferrers diagram of shape $(2p-1, 2p-3, \dots, 1)$, and the ideals are represented by suitable subdiagrams of it, see slightly different versions in [12, Section 2], [3, Section 3], [9, Section 5]. In these interpretations, the long roots are represented by the boxes in an extreme diagonal of this shifted Ferrers diagram, and the simple roots correspond to the boxes of another ("opposite") diagonal. These two diagonals have a unique common box, corresponding to the long simple root. If we want to obtain an ideal whose generators are short and contain no short simple roots, then we just erase both these diagonals and consider a subdiagram of the smaller shifted diagram. But this smaller shifted Ferrers diagram, which is of shape $(2p-3, 2p-5, \dots, 1)$, can be thought of as the set of positive roots for $\mathbf{C}_{p-1}$. Thus, the number $\#(\mathfrak{An}_{ss})$ for $\mathbf{C}_p$ equals the number $\#(\mathfrak{An})$ for $\mathbf{C}_{p-1}$. The latter is known to equal $\binom{2p-2}{p-1}$, which is consistent with Eq. (7.2). Actually, we obtain more. Our bijection between $\mathfrak{An}_{ss}(\mathbf{C}_p)$ and $\mathfrak{An}(\mathbf{C}_{p-1})$ preserves the number of elements. Therefore, we conclude that the number of k-element antichains in $\mathfrak{An}_{ss}(\mathbf{C}_p)$ is equal to $\binom{p-1}{k}^2$, $k = 0, 1, \dots, p-1$.

7.3.

Counting antichains with respect to the number of generators yields an interesting q-analogue of $\#(\mathfrak{An})$, see [2], [9]. In case of two root lengths, one may consider, of course, a 2-parameter refinement. Set

$$\mathfrak{An}\langle k, m\rangle = \{\Gamma \in \mathfrak{An} \mid \#(\Gamma \cap \Delta_s^+) = k \ \&\ \#(\Gamma \cap \Delta_l^+) = m\}, \quad a_{k,m} = \#\mathfrak{An}\langle k, m\rangle,$$

and consider the generating function $\mathcal{F}(t,u) = \sum_{k,m\geq 0} a_{k,m} t^k u^m$. We have

$\mathbf{G}_2$: $\mathcal{F}(t,u) = 1 + 3t + 3u + tu$.
$\mathbf{F}_4$: $\mathcal{F}(t,u) = 1 + 12t + 12u + 8t^2 + 39tu + 8u^2 + 12t^2u + 12tu^2 + t^2u^2$.

The symmetry of these polynomials stems from the fact the corresponding root systems are self-dual. Since the root systems of type **B** and **C** are dual to each other, the corresponding matrices $(a_{k,m})$ are mutually transposed. So, it suffices to handle the case of $\mathbf{C}_p$. Each pair of long roots in $\Delta^+(\mathbf{C}_p)$ is comparable, hence any antichain contains at most one long root. So that we are to determine the coefficients $a_{k,0}, a_{k,1}$, $(k = 0, 1, \dots, p-1)$. In [9, Section 5], we constructed an involution on the set $\mathfrak{An}(\mathbf{C}_p)$, which maps $\mathfrak{An}\langle k, 0\rangle$ onto $\mathfrak{An}\langle p-1-k, 1\rangle$. Hence $a_{p-k-1,1} = a_{k,0}$ and we have to only count the number of short antichains with k elements. Using shifted Ferrers diagrams, it can be shown that $a_{k,0} = \binom{p}{k}\binom{p-1}{k}$. (In this situation, short simple roots are allowed, so that one has to erase only one diagonal and work with the shifted Ferrers diagram of shape $(2p-2, 2p-4, \dots, 2)$.)

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