On the irreducibility of commuting varieties

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Main questions: Is a commuting variety irreducible? If not, what are the irreducible components?

Pairs of commuting matrices

The ground field is $\mathbb{C}$.

Definition

$\mathcal{C}(2, n) = \{(A, B) \mid AB = BA\} \subset \mathfrak{gl}_n \times \mathfrak{gl}_n$

Theorem

$\mathcal{C}(2, n)$ is irreducible for any n.

- T.S. Motzkin and Olga Taussky (1955)
- M. Gerstenhaber (1961)
- R.W. Richardson (1979)
- Yu. Neretin (1987)
- R. Guralnick (1992)

Neretin's proof of irreducibility

One has to prove that $(A, B) \in \mathcal{C}(2, n)$ can simultaneously be diagonalised after a "small" deformation.

- $W_{\lambda,\mu} = \ker(A - \lambda I)^n \cap \ker(B - \mu I)^n$;
- Replace $\mathbb{C}^n$ with $W_{\lambda,\mu}$ and assume that A, B are nilpotent; (The argument is by induction on $\max\dim W_{\lambda,\mu}$).
- $\exists\, p, q \in \mathbb{C}$ s.t. $R := pA + qB$ is non-regular.
- Then $\exists$ a semisimple C such that $\mathrm{tr}(C) = 0$ and $[C, R] = 0$.
- Deformation in $W_{\lambda,\mu}$: $(A, B) \mapsto (A - \varepsilon qC, B + \varepsilon pC)$, $\varepsilon \in \mathbb{C}$.
- By the induction assumption, $(A - \varepsilon qC, B + \varepsilon pC)$ can be approximated by commuting semisimple matrices.

- H is a connected algebraic group, $\mathfrak{h} = \mathrm{Lie}\,(H)$.
- For $x \in \mathfrak{h}$, $\mathfrak{z}_{\mathfrak{h}}(x)$ is the centraliser of x in $\mathfrak{h}$.

Definition

The commuting variety of a Lie algebra $\mathfrak{h}$ is
$\mathcal{C}(\mathfrak{h}) = \{(x, y) \in \mathfrak{h} \times \mathfrak{h} \mid [x, y] = 0\}$.

Example: $\mathcal{C}(2, n) = \mathcal{C}(\mathfrak{gl}_n)$.

If $\mathfrak{a} \subset \mathfrak{h}$ is commutative, then $\overline{H{\cdot}(\mathfrak{a} \times \mathfrak{a})} \subset \mathcal{C}(\mathfrak{h})$.

The very first step:

Consider the projection $\mathcal{C}(\mathfrak{h}) \to \mathfrak{h}$ and look at the dimension of fibres.

One readily obtains that
$\dim \mathcal{C}(\mathfrak{h}) \geqslant \dim \mathfrak{h} + \min \dim \mathfrak{z}_{\mathfrak{h}}(x) = 2 \dim \mathfrak{h} - \max \dim\{\text{H-orbits in } \mathfrak{h}\}$.

Richardson's theorem–1

Theorem (R.W. Richardson, 1979)

If $\mathfrak{g}$ is reductive, then $\mathcal{C}(\mathfrak{g})$ is irreducible.
More precisely, if $\mathfrak{t}$ is a Cartan subalgebra, then $\mathcal{C}(\mathfrak{g}) = \overline{G{\cdot}(\mathfrak{t}\times\mathfrak{t})}$.

Plan of proof

1. We have to approximate any $(x, y) \in \mathcal{C}(\mathfrak{g})$ by a pair of commuting semisimple elements.
2. The Jordan decomposition and induction on rk $[\mathfrak{g}, \mathfrak{g}]$ allow us to assume that x, y are nilpotent.
3. Next, one can assume that $\mathfrak{z}_{\mathfrak{g}}(x)$ contains no semisimple elements. (Such an x is called distinguished.)
4. For x distinguished, one uses some properties of $\mathfrak{sl}_2$-triples.

Richardson's theorem–2

- Suppose x is distinguished and $\{x, h, y\}$ is an $\mathfrak{sl}_2$-triple.
- Then $x + \alpha y \sim_G h$ for any $\alpha \neq 0$.
- Since $\dim \mathfrak{z}_{\mathfrak{g}}(x) = \dim \mathfrak{z}_{\mathfrak{g}}(h)$, we have $\mathfrak{z}_{\mathfrak{g}}(x + \alpha y) \underset{\alpha \to 0}{\longrightarrow} \mathfrak{z}_g(x)$.
- $(x + \alpha y, \mathfrak{z}_{\mathfrak{g}}(x + \alpha y)) \subset \overline{G{\cdot}(\mathfrak{t} \times \mathfrak{t})}$ by induction assumption.
- Hence $(x, q) \in \overline{G{\cdot}(\mathfrak{t} \times \mathfrak{t})}$ for any $q \in \mathfrak{z}_{\mathfrak{g}}(x)$.

Corollary

$\dim \mathcal{C}(\mathfrak{g}) = \dim \mathfrak{g} + \dim \mathfrak{t}$.

Example

$\mathfrak{so}_n = \{$skew-symmetric $n \times n$ matrices$\}$.

Therefore $\mathcal{C}^{alt}(2, n)$ is irreducible.

Related problems on $\mathcal{C}(\mathfrak{g})$

- Algebraic-geometric properties
  - prove that $\mathcal{C}(\mathfrak{g})$ is a normal variety
  - prove that $\mathcal{C}(\mathfrak{g})$ has rational singularities
  - construct a resolution of singularities of $\mathcal{C}(\mathfrak{g})$
  - compute the degree of $\mathcal{C}(\mathfrak{g})$
  - compute the Hilbert polynomial of $\mathbb{C}[\mathcal{C}(\mathfrak{g})]$
- Prove that the natural quadratic equations generate the ideal of $\mathcal{C}(\mathfrak{g})$ in $\mathbb{C}[\mathfrak{g} \times \mathfrak{g}]$

What is known:

- the quotient variety $\mathcal{C}(\mathfrak{g})/\!\!/G$ is normal (A. JOSEPH, 1996)
- The degree is computed for $\mathfrak{g} = \mathfrak{gl}(V)$ (A. KNUTSON, P. ZINN-JUSTIN, 2006), see Sequence A029729.
- results for small ranks

d-tuples of commuting matrices

Definition

$\mathcal{C}(d,n) = \{(A_1,\ldots,A_d) \mid [A_i,A_j]=0 \quad \forall\, i,j\} \subset (\mathfrak{gl}_n)^d$

- $\mathcal{C}(d,n)$ is reducible for $n \geqslant 4, d \geqslant 5$ (Gerstenhaber, 1961);
- $\mathcal{C}(d,2)$, $\mathcal{C}(d,3)$ are irreducible for any d, $\mathcal{C}(4,4)$ is reducible (Kirillov–Neretin, 1984);
- $\mathcal{C}(3,n)$ is reducible for $n \geqslant 32$ (Guralnick, 1992), now $n \geqslant 30$;
- $\mathcal{C}(3,n)$ is irreducible for $n \leqslant 8$ (Guralnick-Sethuraman, Holbrook, Omladič, Han, Šivic.)

For more details, attend Šivic's talk tomorrow !

Triangular matrices–1

$\mathfrak{b} = \{\text{upper-triangular } n \times n \text{ matrices}\} \supset \mathfrak{t} = \{\text{diagonal matrices}\}$.

Then $\overline{B{\cdot}\mathfrak{t}} = \mathfrak{b}$.

It can be shown that $\overline{B{\cdot}(\mathfrak{t} \times \mathfrak{t})}$ is always an irreducible component of $\mathcal{C}(\mathfrak{b})$.

Problem

Is it true that $\mathcal{C}(\mathfrak{b}) = \overline{B{\cdot}(\mathfrak{t} \times \mathfrak{t})}$ *?*

The general answer is “no”.

Triangular matrices–2

Example

$n = 3m, \quad \dim \overline{B{\cdot}(\mathfrak{t} \times \mathfrak{t})} = \dim \mathfrak{b} + \dim \mathfrak{t} = \frac{9}{2}(m^2 + m).$

$$\text{Take } \mathfrak{a} = \left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\} \subset \mathfrak{b}.$$

Then $\dim \mathfrak{a} = 3m^2$ and $\dim[\mathfrak{a}, \mathfrak{a}] = m^2$.

Therefore $\dim\big((\mathfrak{a} \times \mathfrak{a}) \cap \mathcal{C}(\mathfrak{b})\big) = \dim \mathcal{C}(\mathfrak{a}) \geqslant 5m^2$.

Hence $\mathcal{C}(\mathfrak{a}) \not\subset \overline{B{\cdot}(\mathfrak{t} \times \mathfrak{t})}$ for $m \geqslant 10$ and $\mathcal{C}(\mathfrak{b})$ is reducible.

Related problem

Determine parabolic subalgebras $\mathfrak{p} \subset \mathfrak{gl}_n$ having the property that $\mathcal{C}(\mathfrak{p})$ is irreducible.

Centralisers of nilpotent elements of $\mathfrak{gl}_n$ — I

The nilpotent orbits are parametrised by partitions of n.
Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_s)$ be a partition of n. (Notation: $\underline{\lambda} \dashv n$.)
Then $\mathcal{O}(\underline{\lambda})$ and $\mathfrak{z}(\underline{\lambda})$ denote the corresponding orbit and centraliser, respectively.

Examples

- if $\underline{\lambda} = (n)$, then $\mathcal{O}(\underline{\lambda})$ is regular and $\mathfrak{z}(\underline{\lambda})$ is abelian,
- if $\underline{\lambda} = (2, 1, \ldots, 1)$, then $\mathcal{O}(\underline{\lambda})$ is the minimal (nonzero) orbit.

Theorem (M.G. Neubauer and B. A. Sethuraman, 1999)

If $\underline{\lambda}$ has two nonzero parts, then $\mathcal{C}(\mathfrak{z}(\underline{\lambda}))$ is irreducible.

If a matrix is 2-regular and nilpotent, then it has at most two Jordan blocks.

Centralisers of nilpotent elements of $\mathfrak{gl}_n$ — II

- There is a connection between $\mathcal{C}(\mathfrak{z}(\underline{\lambda}))$ and commuting triples:

Theorem (O. Yakimova, 2006)

If $\mathcal{C}(\mathfrak{z}(\underline{\lambda}))$ is irreducible for any $\underline{\lambda} \dashv m$ with $m \leqslant n$, then $\mathcal{C}(3, n)$ is irreducible as well.

Corollary

For $n \geqslant 30$, there is a $\underline{\lambda} \dashv n$ such that $\mathcal{C}(\mathfrak{z}(\underline{\lambda}))$ is reducible.

However, no explicit examples is known.

Nilpotent commuting varieties – I

$\mathcal{N} \subset \mathfrak{g}$ – the cone of nilpotent elements.

Definition

The nilpotent commuting variety is $\mathcal{C}(\mathcal{N}) := \mathcal{C}(\mathfrak{g}) \cap (\mathcal{N} \times \mathcal{N})$.

Theorem (A. Premet, 2003)

(i) *The irreducible components of $\mathcal{C}(\mathcal{N})$ are parametrised by the distinguished nilpotent G-orbits in $\mathfrak{g}$.*

(ii) *The variety $\mathcal{C}(\mathcal{N})$ is of pure dimension* $\dim \mathfrak{g}$.

- Consider the projection $p : \mathcal{C}(\mathcal{N}) \to \mathcal{N}$
- If $\mathcal{O} \subset \mathcal{N}$ is an orbit, then $\dim p^{-1}(\mathcal{O}) \leqslant \dim \mathfrak{g}$ and the equality exactly means that $\mathcal{O}$ is distinguished.

Nilpotent commuting varieties – II

Corollary (Baranovsky, 2001; Basili, 2003)

For $\mathfrak{g} = \mathfrak{sl}_n$, the variety $\mathcal{C}(\mathcal{N})$ is irreducible.

(The only other cases are $\mathfrak{so}_5$ and $\mathfrak{so}_7$.)

- It is an interesting problem to describe nilpotent matrices commuting with a given nilpotent matrix ($\rightarrow$ talk of P. Oblak tomorrow).

The diagonal commutator scheme

Suppose $\mathfrak{g}$ is semisimple

- $\psi = [\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ – the usual bracket and $\mathcal{C}(\mathfrak{g}) = \psi^{-1}(0)$.
- ψ is onto and the generic fibre of ψ is of dimension $\dim \mathfrak{g}$.
- The expected dimension of $\psi^{-1}(\mathfrak{t})$ is $\dim \mathfrak{g} + \dim \mathfrak{t}$.

Theorem (Knutson, 2005)

If $\mathfrak{g} = \mathfrak{sl}_n$, then $\psi^{-1}(\mathfrak{t})$ is a reduced complete intersection. It has two irreducible components of dimension $\dim \mathfrak{g} + \dim \mathfrak{t}$.

Knutson constructs a degeneration of $\psi^{-1}(\mathfrak{t})$ into the scheme

$$\{(A, B) \mid AB \text{ is upper triangular, } BA \text{ is lower triangular}\}.$$

The latter has $n!$ irreducible components, which are parametrised by permutations. Knutson also studies the degree of irreducible components.

Involutions and commuting varieties

For an involution $\vartheta \in \mathrm{Aut}(\mathfrak{g})$, let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the corresponding $\mathbb{Z}_2$-grading. Then $(\mathfrak{g}, \mathfrak{g}_0)$ is called a symmetric pair.

Definition

The commuting variety associated with ϑ is $\mathcal{C}(\mathfrak{g}_1) = \mathcal{C}(\mathfrak{g}) \cap (\mathfrak{g}_1 \times \mathfrak{g}_1)$.

- $\mathcal{C}(\mathfrak{g}_1)$ is the zero fibre of $\psi_1 : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$;
- The group G_0 acts on $\mathfrak{g}_1$ and $\mathcal{C}(\mathfrak{g}_1)$;
- If $\mathfrak{c}$ is a Cartan subspace of $\mathfrak{g}_1$, then $\overline{G_0{\cdot}\mathfrak{c}} = \mathfrak{g}_1$.
- $\overline{G_0{\cdot}(\mathfrak{c} \times \mathfrak{c})}$ is an irreducible component of $\mathcal{C}(\mathfrak{g}_1)$.
- $\dim \overline{G_0{\cdot}(\mathfrak{c} \times \mathfrak{c})} = \dim \mathfrak{g}_1 + \dim \mathfrak{c}$.

The induction scheme of Richardson basically applies here. The problem reduces to study of ϑ-distinguished (nilpotent) G_0-orbits in $\mathfrak{g}_1$.

Involutions of maximal rank

We say that ϑ is of maximal rank if $\dim \mathfrak{c} = \operatorname{rk} \mathfrak{g}$. In this case $\mathfrak{c}$ is Cartan and $\dim \mathfrak{g}_1 - \dim \mathfrak{g}_0 = \operatorname{rk} \mathfrak{g}$.

Theorem (Panyushev, 1994)

If ϑ is of maximal rank, then $\mathcal{C}(\mathfrak{g}_1)$ is an irreducible normal complete intersection. The ideal of $\mathcal{C}(\mathfrak{g}_1)$ in $\mathbb{C}[\mathfrak{g}_1 \times \mathfrak{g}_1]$ is generated by quadrics.

- ψ_1 is onto, hence $\dim \mathcal{C}(\mathfrak{g}_1) \geqslant 2 \dim \mathfrak{g}_1 - \dim \mathfrak{g}_0 = \dim \mathfrak{g}_1 + \operatorname{rk} \mathfrak{g}$;
- $\dim \overline{G_0{\cdot}(\mathfrak{c} \times \mathfrak{c})} = \dim \mathfrak{g}_1 + \operatorname{rk} \mathfrak{g}$;
- The complement of $\overline{G_0{\cdot}(\mathfrak{c} \times \mathfrak{c})}$ forms a subvariety of dimension less than $\dim \mathfrak{g}_1 + \operatorname{rk} \mathfrak{g}$;

Here $\psi_1 : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ is flat and all the fibres are irreducible, normal, etc.

Some examples

Example

For $\mathfrak{g} = \mathfrak{gl}_n$, the involution of maximal rank is given by $\vartheta(A) = -A^t$. It follows that $\mathfrak{g}_0 = \mathfrak{so}_n$ and $\mathfrak{g}_1 = \{\text{symmetric } n \times n \text{ matrices}\}$.

Therefore $\mathcal{C}^{sym}(2, n)$ is irreducible.

Unpleasant fact: $\mathcal{C}(\mathfrak{g}_1)$ is not always irreducible.

Example (Panyushev and Yakimova, 2007)

$s = \text{diag}(\underbrace{1, \dots, 1}_{m}, \underbrace{-1, \dots, -1}_{n}) \in GL_{n+m}$ and $\vartheta = \text{Int}(s)$. Then

$\mathfrak{g}_0 = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \simeq \mathfrak{gl}_m \oplus \mathfrak{gl}_n$, $\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\}$, and $\dim \mathfrak{c} = \min\{n, m\}$.

Here $\mathcal{C}(\mathfrak{g}_1)$ is reducible unless $n = m$.

The rank one case

Theorem (Panyushev, 2004)

If $\dim \mathfrak{c} = 1$, *then* $\#Irr(\mathcal{C}(\mathfrak{g}_1)) = \#\{$*nonzero* G_0*-orbits in* $\mathcal{N} \cap \mathfrak{g}_1\}$.

Number of irreducible components

$(\mathfrak{so}_n, \mathfrak{so}_{n-1})$ $n \geqslant 3$	$(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2} \times \mathfrak{sp}_2)$ $n \geqslant 2$	$(\mathbf{F}_4, \mathfrak{so}_9)$	$(\mathfrak{sl}_n, \mathfrak{sl}_{n-1} \times T_1)$ $n \geqslant 3$
1	2	2	3

Fact: The standard component $\overline{G_0{\cdot}(\mathfrak{c} \times \mathfrak{c})}$ is always a unique irreducible component of maximal dimension.

If $\dim \mathfrak{c} = 1$, then all other components are of dimension $\dim \mathfrak{g}_1$.

Open problems

- For what partitions λ is the variety $\mathcal{C}(\mathfrak{z}(\lambda))$ irreducible?
- Study the triples of commuting *nilpotent* matrices.
- Study the triples of commuting *symmetric* matrices.
- Is the variety $\{(A_1, A_2, A_3) \in \mathcal{C}(3, n) \mid A_1 A_2 = A_3^2\}$ irreducible ?
- Describe $\mathrm{Irr}\,\mathcal{C}(\mathfrak{g}_1)$ for the symmetric pair $(\mathfrak{gl}_{n+m}, \mathfrak{gl}_n \times \mathfrak{gl}_m)$, $n \neq m$.
- The irreducibility of $\mathcal{C}(\mathfrak{g}_1)$ is not known for 3 cases. Two serial cases concern *classical* symmetric pairs: $(\mathfrak{so}_{4n}, \mathfrak{gl}_{2n})$ and $(\mathfrak{sp}_{2n+2m}, \mathfrak{sp}_{2n} \times \mathfrak{sp}_{2m})$, $\min(n, m) \geqslant 3$.
- There is no general principle for the irreducibility of $\mathcal{C}(\mathfrak{g}_1)$!