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Inductive formulas for the index of seaweed Lie algebras

Dmitri I. Panyushev

Introduction

Let $\mathfrak a$ be a Lie algebra and ξ a linear form on $\mathfrak a$. Denote by $\mathfrak a_\xi$ the set of all $x \in \mathfrak a$ such that $\xi([\mathfrak a, x]) = 0$. In other words, $\mathfrak a_\xi = \{x \in \mathfrak a \mid (\mathrm{ad}^*x)\cdot\xi = 0\}$, where $\mathrm{ad}^* : \mathfrak a \to \mathfrak{gl}(\mathfrak a^*)$ is the coadjoint representation of $\mathfrak a$. The index $\mathrm{ind}\,\mathfrak a$ of $\mathfrak a$ is defined by

$$\mathrm{ind}\,\mathfrak a = \min_{\xi\in\mathfrak a^*} \dim \mathfrak a_\xi \ .$$

This definition goes back to J. Dixmier, see [4, 11.1.6]. Some general results for the index can be found in [5]. If $\mathfrak a$ is reductive, then $\mathfrak a$ and $\mathfrak a^*$ are isomorphic as $\mathfrak a$-modules and hence $\mathrm{ind}\,\mathfrak a = \mathrm{rk}\,\mathfrak a$. It is therefore interesting to study index for non-reductive Lie algebras. Indices for some classes of semi-direct products of Lie algebras were studied in [9],[10],[11]. In [6], A.G. Elashvili has computed the index for the nilpotent radicals of all parabolic subalgebras of simple Lie algebras. The index for Borel subalgebras of the simple Lie algebras is determined in [12, § 4], see also [1, Prop. 4.2]

In [3], Dergachev and Kirillov introduced the notion of a seaweed subalgebra of $\mathfrak{gl}(V)$ and exhibited a method for computing the index of such algebras. A *seaweed subalgebra* of $\mathfrak{gl}(V)$ (or simply a *seaweed* in $\mathfrak{gl}(V)$) is the subalgebra preserving two "opposite" (partial) flags in V, see section 3 for details. This means, in particular, that any parabolics is a seaweed. The main result of [3] can be stated as follows. To any seaweed $\mathfrak p$ in $\mathfrak{gl}(V)$ one associates a graph with $\dim V$ nodes, a so-called *meander graph*. Write M for this graph. The degree of any node in M is at most 2. Therefore each connected component of M is either a circuit or is homeomorphic to a segment. Then $\mathrm{ind}\,\mathfrak p$ is equal to the number of connected components plus the number of circuits in M. Being rather convenient for explicit computations when $\dim V$ is small, this recipe is not however very helpful in theoretical considerations.

The notion of a seaweed in $\mathfrak{gl}(V)$ admits a natural generalisation to any reductive Lie algebra. Let $\mathfrak g$ be reductive. Two parabolic subalgebras $\mathfrak p$ and $\mathfrak p'$ are called *weakly opposite* whenever $\mathfrak p + \mathfrak p' = \mathfrak g$. Then $\mathfrak p \cap \mathfrak p'$ is said to be a seaweed in $\mathfrak g$. In this paper, we give inductive formulas for the index of seaweeds in classical Lie algebras. It was one of our motivations to verify the conjecture that $\mathrm{ind}\,\mathfrak s \le \mathrm{rk}\,\mathfrak g$ for any seaweed $\mathfrak s \subset \mathfrak g$.

For $\mathfrak{gl}(V) = \mathfrak{gl}_n$, our approach is completely different from that of [3]. Making use of Raïs' theorem for the index of a semi-direct product, we arrange a "shellability" procedure for seaweeds, which results in an inductive formula for the index, see section 4. To explain the answer, some notation is needed. A seaweed subalgebra $\mathfrak p$ of $\mathfrak{gl}_n$ is determined by two strings of positive integers, say $\underline a = (a_1, \dots, a_m)$ and $\underline b = (b_1, \dots, b_t)$, where $\sum a_i = \sum b_j = n$. (See Section 3 for the precise description.) Then we write $\mathfrak p = \mathfrak p(\underline a \mid \underline b)$. A parabolic subalgebra occurs if and only if $m=1$ or $t=1$. Since $\mathfrak p(\underline a \mid \underline b)\simeq \mathfrak p(\underline b\mid \underline a)$, one may assume without loss that $a_1\le b_1$.

- Suppose $a_1=b_1$. If $t=m=1$, then $\mathfrak p(\underline a\mid \underline b)=\mathfrak{gl}_n$ and the index is equal to n. If $m>1$ (and hence $t>1$), then $\operatorname{ind}\mathfrak p(\underline a\mid\underline b)=a_1+\operatorname{ind}\mathfrak p(a_2,\dots,a_m\mid b_2,\dots,b_t)$.
- Suppose $a_1<b_1$. Then there exists a unique nonnegative integer p such that $\frac{p}{p+1}b_1<a_1\le\frac{p+1}{p+2}b_1$ and in this case

  $$\operatorname{ind}\mathfrak p(\underline a\mid\underline b)=\operatorname{ind}\mathfrak p(a_2,\dots,a_m\mid (p+1)b_1-(p+2)a_1,(p+1)a_1-pb_1,b_2,\dots,b_t)\ .$$

  If $(p+1)b_1=(p+2)a_1$, then the zero coordinate in the right-hand side must be omitted.

In both cases the smaller seaweed subalgebra lies in $\mathfrak{gl}_{n-a_1}$. Therefore the procedure terminates after at most n steps. Another consequence of these formulas is that $\operatorname{ind}\mathfrak p(\underline a\mid\underline b)\le n$ and the equality holds if and only if $\underline a=\underline b$, i.e., for Levi subalgebras. We also give a lower bound for the index. Since $\sum a_i=\sum b_j$, the number of the odd coordinates in $\underline a$ and $\underline b$ is even. Denote this number by $2k$. If $k=0$, then $\operatorname{ind}\mathfrak p(\underline a\mid\underline b)\ge 2$; if $k\ge 1$, then $\operatorname{ind}\mathfrak p(\underline a\mid\underline b)\ge k$.

In section 5, we consider orthogonal and symplectic seaweeds. It turns out that, for $\mathfrak g=\mathfrak{so}(V)$ or $\mathfrak{sp}(V)$, the same shellability procedure applies to any seaweed which is *not* a parabolic subalgebra (Theorem 5.2). It is still based on Raïs' theorem. This procedure terminates when one obtains a parabolic subalgebra of $\mathfrak{sp}(V')$ or $\mathfrak{so}(V')$, respectively, with $\dim V-\dim V'\in 2\mathbb N$. This already enables us to conclude that if $\operatorname{ind}\mathfrak p\le \operatorname{rk}\mathfrak g$ for any parabolic subalgebra $\mathfrak p$ in $\mathfrak{sp}(V)$ (resp. $\mathfrak{so}(V)$), then the same is true for any seaweed in $\mathfrak{sp}(V)$ (resp. $\mathfrak{so}(V)$). Obtaining an inductive formula for the index of parabolic subalgebras requires some stronger tools. To this end, we exploit an extension of Raïs' theorem to $\mathbb Z$-gradings of height 2 associated with *quadratic* nilpotent elements, see 1.5, 1.6. The symplectic case appears to be easier than the orthogonal one. We prove a general closed formula for the index of symplectic parabolic subalgebras, see Theorem 5.6. Together with the previous induction, this shows that $\operatorname{ind}\mathfrak s\le n$ for any seaweed $\mathfrak s\subset\mathfrak{sp}_{2n}$. For orthogonal parabolics, we have a formula for the index only in some special cases. However, all our formulas confirm the above-mentioned conjecture.

This work was strongly influenced by some unpublished results of A.G. Elashvili on the index of parabolic subalgebras. But, unfortunately, his preprint [7] contains no proofs and, moreover, the formulas for the index of parabolics in $\mathfrak{sp}(V)$ and $\mathfrak{so}(V)$ given therein are not quite correct.

Notation. The ground field $\Bbbk$ is algebraically closed and of characteristic zero. All Lie algebras are assumed to be finite-dimensional. The word '(sub)algebra' always means 'Lie (sub)algebra'. The sign '$\oplus$' stands for a direct sum of vector spaces, while '$\dotplus$' denotes a direct sum of Lie algebras.

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1 Generalities on the index

The index of a Lie algebra $\mathfrak{q}$, as defined in the Introduction, is an integer attached to the adjoint representation of $\mathfrak{q}$. A similar procedure applied to an arbitrary representation of $\mathfrak{q}$ yields the notion of the *index of a representation*. This is quite useful, even though one is mainly interested in the index of Lie algebras. Let $\rho : \mathfrak{q} \to \mathfrak{gl}(V)$ be a finite-dimensional representation of $\mathfrak{q}$, i.e. V is a $\mathfrak{q}$-module. Abusing notation, we write $s{\cdot}v$ in place of $\rho(s)v$, if $s \in \mathfrak{q}$ and $v \in V$. An element $v \in V$ is called *regular* or $\mathfrak{q}$-*regular* whenever its stabiliser $\mathfrak{q}_v = \{s \in \mathfrak{q} \mid s{\cdot}v = 0\}$ has minimal dimension. Because the function $v \mapsto \dim \mathfrak{q}_v$ $(v \in V)$ is upper semicontinuous, the set of all $\mathfrak{q}$-regular elements is open and dense in V. Consider also the dual $\mathfrak{q}$-module V^*.

Definition. The nonnegative integer $\dim V - \max\limits_{\xi \in V^*}(\dim \mathfrak{q}{\cdot}\xi)$ is called the *index* of (the $\mathfrak{q}$-module) V. It will be denoted by $\mathrm{ind}\,(\rho, V)$ or $\mathrm{ind}\,(\mathfrak{q}, V)$.

Notice that in order to define the index of V we use the regular elements in V^*! It is clear that $\mathrm{ind}\,(\mathrm{ad}\,, \mathfrak{q}) = \mathrm{ind}\,\mathfrak{q}$ is the index of $\mathfrak{q}$ in the sense of Introduction. For future reference, we collect some easy properties of the index.

1.1 Lemma.

1. If $\mathfrak{q} = \mathfrak{q}_1 \dot{+} \mathfrak{q}_2$, *then* $\mathrm{ind}\,\mathfrak{q} = \mathrm{ind}\,\mathfrak{q}_1 + \mathrm{ind}\,\mathfrak{q}_2$;
2. $\mathrm{ind}\,\mathfrak{q} \ge \dim \mathfrak{z}(\mathfrak{q})$, *where* $\mathfrak{z}(\mathfrak{q})$ *is the centre of* $\mathfrak{q}$;
3. If $\mathfrak{q}$ *is reductive, then* $\mathrm{ind}\,\mathfrak{q} = \mathrm{rk}\,\mathfrak{q}$. □

The bracket in $\mathfrak{q}$ can be regarded as a skew-symmetric bilinear form on $\mathfrak{q}$ with values in $\mathfrak{q}$: $K = [\ ,\ ] : \mathfrak{q} \times \mathfrak{q} \to \mathfrak{q}$. Evaluating this form at an arbitrary point $\xi \in \mathfrak{q}^*$ yields a usual skew-symmetric form:

$$K_\xi : \mathfrak{q} \times \mathfrak{q} \to \mathfrak{q} \xrightarrow{\xi(\cdot)} \Bbbk \ .$$

This K_ξ is nothing but the *Kirillov form* associated with ξ. The following facts are standard.

1.2 Proposition.

1. $\mathrm{Ker}\,K_\xi = \mathfrak{q}_\xi$;
2. $\mathrm{rank}\,K_\xi = \dim \mathfrak{q}{\cdot}\xi$. □

Regarding K as a skew-symmetric form with coefficients in the symmetric algebra $\mathcal{S}^\bullet(\mathfrak{q})$, we see that $\mathrm{rank}\,K = \max_\xi \mathrm{rank}\,K_\xi = \dim \mathfrak{q} - \mathrm{ind}\,\mathfrak{q}$. Having chosen a basis $(s_1, \dots, s_n)$ for $\mathfrak{q}$, we may identify the skew-symmetric form K with the $n \times n$ matrix $([s_i, s_j])_{i,j=1}^n \in Mat_n(\mathcal{S}^\bullet(\mathfrak{q}))$. Since $\mathrm{rank}\,K_\xi$ is always even, we conclude that $\dim \mathfrak{q} - \mathrm{ind}\,\mathfrak{q}$ is even for any Lie algebra. A Lie algebra $\mathfrak{q}$ is called *Frobenius*, if $\mathrm{ind}\,\mathfrak{q} = 0$, i.e., K_ξ is non-degenerate for some $\xi \in \mathfrak{q}^*$.

A powerful tool for computing index is a theorem of M. Raïs. Let $\rho : \mathfrak{q} \to \mathfrak{gl}(V)$ be a linear representation of $\mathfrak{q}$. The linear space $V \times \mathfrak{q}$ has a natural structure of Lie algebra, with bracket $[\ ,\ ]^\sim$ defined by the equality

$$[(v_1, s_1), (v_2, s_2)]^\sim = (\rho(s_1)v_2 - \rho(s_2)v_1, [s_1, s_2]) \ .$$

Following Raïs, the resulting Lie algebra is denoted by $V \times_\rho \mathfrak{q}$. It is a semi-direct product of $\mathfrak{q}$ and an Abelian ideal V. We identify the dual space $(V \times_\rho \mathfrak{q})^*$ with $V^* \oplus \mathfrak{q}^*$.

1.3 Theorem (Raïs [9]). *Let $\xi \in V^*$ be a $\mathfrak{q}$-regular element such that $\xi + \eta$ is a regular element in $(V \times_\rho \mathfrak{q})^*$ for some $\eta \in \mathfrak{q}^*$. Then* $\mathrm{ind}\,(V \times_\rho \mathfrak{q}) = \mathrm{ind}\,(\mathfrak{q}, V) + \mathrm{ind}\,\mathfrak{q}_\xi$.

It is not assumed that the stabilisers of all (or almost all) regular elements are conjugated. We are going to apply Theorem 1.3 in the following specific situation.

1.4 Corollary. *Suppose $\mathfrak{q}$ is an algebraic Lie algebra and* $\mathrm{ind}\,(\mathfrak{q}, V) = 0$. *Then* $\mathrm{ind}\,(V \times_\rho \mathfrak{q}) = \mathrm{ind}\,\mathfrak{q}_\xi$, *where $\xi \in V^*$ is any $\mathfrak{q}$-regular element.*

Proof. Let Q be a connected algebraic group with Lie algebra $\mathfrak{q}$. The condition on $\mathrm{ind}\,(\mathfrak{q}, V)$ is equivalent to the fact that Q has an open orbit in V^*. Therefore the set of $\mathfrak{q}$-regular elements in V^* is just this orbit. □

The semi-direct product $V \times_\rho \mathfrak{q} =: \mathfrak{h}$ can be thought of as a $\mathbb{Z}$-graded Lie algebra with the nonzero summands of grade 0 and 1; namely, $\mathfrak{h}(0) \simeq \mathfrak{q}$ and $\mathfrak{h}(1) \simeq V$. Whereas Raïs' theorem appears to be sufficient for applications to seaweeds in $\mathfrak{gl}(V)$ (see Section 4), parabolic subalgebras in other classical series require understanding a more general situation. In Section 5, we exploit an extension of Raïs' theorem to some $\mathbb{Z}$-graded Lie algebras with three nonzero summands.

Let $\mathfrak{h} = \mathfrak{h}(0) \oplus \mathfrak{h}(1) \oplus \cdots \oplus \mathfrak{h}(d)$ be a $\mathbb{Z}_{\geq 0}$-graded Lie algebra. Then $\mathfrak{h}^*$ is a graded $\mathfrak{h}$-module with nonzero summands of grade $-d, \dots, -1, 0$. More precisely, $\mathfrak{h}^*(-j) = \mathrm{Ann}\,(\oplus_{i \neq j} \mathfrak{h}(i))$. Hence $\mathfrak{h}^*(-j) \simeq \mathfrak{h}(j)^*$. We have $\mathrm{ad}^*(\mathfrak{h}(i)) \cdot \mathfrak{h}^*(-j) \subset \mathfrak{h}^*(i-j)$; in particular, each $\mathfrak{h}^*(-j)$ is an $\mathfrak{h}(0)$-module.

We consider such algebras with $d = 2$. In this case, the index of the subalgebra $\mathfrak{h}^{ev} := \mathfrak{h}(0) \oplus \mathfrak{h}(2)$ can be computed using Theorem 1.3.

1.5 Proposition. *Suppose that there exists an $\mathfrak{h}(0)$-regular element $\xi \in \mathfrak{h}^*(-2)$ such that $\mathfrak{h}_\xi \cap \mathfrak{h}(1) = \{0\}$. Then*

$$\mathrm{ind}\,\mathfrak{h} = \mathrm{ind}\,\mathfrak{h}^{ev} = \mathrm{ind}\,\mathfrak{h}(0)_{\xi'} + \mathrm{ind}\,(\mathfrak{h}(0), \mathfrak{h}(2)),$$

where ξ' is a suitable regular element in $\mathfrak{h}^*(-2)$.

Proof. Let $\eta = \eta_{-2} + \eta_{-1} + \eta_0$ be a $\mathfrak{h}$-regular element in $\mathfrak{h}^*$, where $\eta_i \in \mathfrak{h}^*(i)$. Without loss of generality, one may assume that $\eta_{-2} = \xi'$ is an $\mathfrak{h}(0)$-regular element in $\mathfrak{h}^*(-2)$, satisfying the assumption of the Proposition. By dimension reason, we then have $\mathrm{ad}^*(\mathfrak{h}(1)) \cdot \xi' = \mathfrak{h}^*(-1)$. Consider the unipotent group $N := exp\,(\mathfrak{h}(1) \oplus \mathfrak{h}(2))$ and its coadjoint action in $\mathfrak{h}^*$. The previous relation implies that the orbit $N \cdot \eta$ contains a representative of the form $\eta' = \xi' + \eta'_0$. Since η' is a regular element in $\mathfrak{h}^*$, $\eta' \in (\mathfrak{h}^{ev})^*$, and $\mathfrak{h}_{\eta'} \subset \mathfrak{h}^{ev}$, we see that $\mathrm{ind}\,\mathfrak{h} = \mathrm{ind}\,\mathfrak{h}^{ev}$. The second equality is nothing but Theorem 1.3. □

1.6 Example. A vast reserve of algebras satisfying the hypothesis of the previous proposition occurs in connection with *quadratic* nilpotent elements. We refer to [2] for standard facts about nilpotent orbits. Let $\mathfrak{g}$ be semisimple and let $x \in \mathfrak{g}$ be a nilpotent element such that $(\mathrm{ad}\, x)^3 = 0$. Take an $\mathfrak{sl}_2$-triple containing x, say $\{x, h, y\}$. Here h is semisimple, $[h, x] = 2x$, and $[h, y] = -2y$. Let $\mathfrak{g}(i)$ denote the eigenspace of $\mathrm{ad}\, h$ belonging to $i \in \mathbb{Z}$. It follows from the assumption on x that the eigenvalues of $\mathrm{ad}\, h$ in $\mathfrak{g}$ lie in $\{-2, -1, 0, 1, 2\}$ and therefore $\mathfrak{g} = \oplus_{i=-2}^{2} \mathfrak{g}(i)$. It is a $\mathbb{Z}$-grading of *height* 2. We have $y \in \mathfrak{g}(-2)$ and $\mathfrak{g}_y \cap (\mathfrak{g}(1) \oplus \mathfrak{g}(2)) = \{0\}$. Consider the parabolic subalgebra $\tilde{\mathfrak{p}} := \oplus_{i=0}^{2} \mathfrak{g}(i)$. Then the dual space $\tilde{\mathfrak{p}}^*$ is isomorphic to $\mathfrak{g}/(\mathfrak{g}(1) \oplus \mathfrak{g}(2))$ as $\tilde{\mathfrak{p}}$-module, and the image of y in $\mathfrak{p}^*$ can be taken as ξ.

More generally, if $\mathfrak{p}(0)$ is any parabolic subalgebra of $\mathfrak{g}(0)$, then $\mathfrak{p} := \mathfrak{p}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ still satisfies the hypothesis of Proposition 1.5. This is for trivial reason that the first and second graded pieces have remained intact. This specific situation possesses a nice property that $\mathfrak{g}(2)$ and $\mathfrak{g}(-2)$ are spherical $G(0)$-modules, where $G(0)$ is a connected (reductive) algebraic group with Lie algebra $\mathfrak{g}(0)$ (see e.g. [8, Prop. 3.2]). Therefore $\mathrm{ind}\,(\mathfrak{p}(0), \mathfrak{g}(2)) = 0$ for any parabolic subalgebra $\mathfrak{p}(0) \subset \mathfrak{g}(0)$. Thus Proposition 1.5 and Corollary 1.4 imply that $\mathrm{ind}\, \mathfrak{p} = \mathrm{ind}\, \mathfrak{p}(0)_\xi$, where $\xi \in \mathfrak{g}(2)^* \simeq \mathfrak{g}(-2)$ is any $\mathfrak{p}(0)$-regular element.

2 Seaweed subalgebras of reductive Lie algebras

Seaweed subalgebras of $\mathfrak{gl}(V)$ were recently introduced in [3]. Here we discuss a generalisation of this notion to arbitrary reductive Lie algebras. Let $\mathfrak{g}$ be a reductive Lie algebra. Let us say that two parabolic subalgebras $\mathfrak{p}, \mathfrak{p}'$ of $\mathfrak{g}$ are *weakly opposite*, if $\mathfrak{p} + \mathfrak{p}' = \mathfrak{g}$. Recall that the usual definition of *opposite* parabolic subalgebras requires a stronger condition that $\mathfrak{p} \cap \mathfrak{p}'$ is a Levi subalgebra in either of them.

2.1 Definition. Let $\mathfrak{p}$ and $\mathfrak{p}'$ be weakly opposite parabolic subalgebras of $\mathfrak{g}$. Then $\mathfrak{p} \cap \mathfrak{p}'$ is called a *seaweed* subalgebra of $\mathfrak{g}$.

Clearly, any seaweed is an algebraic Lie algebra. It is allowed that $\mathfrak{p}' = \mathfrak{g}$, hence each parabolic subalgebra is a seaweed. Since $\mathfrak{p} \cap \mathfrak{p}'$ always contains a Cartan subalgebra and hence the centre of $\mathfrak{g}$, we have

$$\mathfrak{p} \cap \mathfrak{p}' = \mathfrak{z}(\mathfrak{g}) \dotplus \Big((\mathfrak{p} \cap [\mathfrak{g}, \mathfrak{g}]) \cap (\mathfrak{p}' \cap [\mathfrak{g}, \mathfrak{g}]) \Big) .$$

Thus, it suffices to consider seaweeds in semisimple Lie algebras. (However, it will be technically easier to work with $\mathfrak{gl}(V)$ in place of $\mathfrak{sl}(V)$ in the following sections.) Fix a triangular decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}_-$, where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. The roots of $\mathfrak{t}$ on $\mathfrak{u}$ are proclaimed to be positive. Let Δ^+ denote the set of all positive roots and Π the corresponding set of simple roots. Set $\Delta = \Delta^+ \cup (-\Delta^+)$. The root space corresponding to $\gamma \in \Delta$ is denoted by $\mathfrak{g}_\gamma$. A seaweed subalgebra $\mathfrak{s} = \mathfrak{p} \cap \mathfrak{p}'$ is said to be *standard* if $\mathfrak{p} \supset \mathfrak{t} \oplus \mathfrak{u}$ and $\mathfrak{p}' \supset \mathfrak{t} \oplus \mathfrak{u}_-$. Assuming that $\mathfrak{p} \cap \mathfrak{p}'$ is standard, set $\Psi' = \{\alpha \in \Pi \mid \mathfrak{g}_\alpha \not\in \mathfrak{p}'\}$ and $\Psi = \{\alpha \in \Pi \mid \mathfrak{g}_{-\alpha} \not\in \mathfrak{p}\}$. Then $\mathfrak{p} \cap \mathfrak{p}'$ will be denoted by $\mathfrak{p}(\Psi \mid \Psi')$. Let G be a connected algebraic group with Lie algebra $\mathfrak{g}$. From the fact that all Borel subalgebras are G-conjugate, one easily deduces that any seaweed is G-conjugate to a standard one. It is therefore sufficient to consider only standard seaweeds.

Remark. Let w_0 be the longest element in the Weyl group W, i.e. $w_0(\Delta^+) = -\Delta^+$. Then the mapping $\alpha \mapsto \alpha^* = -w_0(\alpha)$ is an involution on Π. This also shows that $\mathfrak{p}(\Psi \mid \Psi')$ is G-conjugate to $\mathfrak{p}(\Psi'^* \mid \Psi^*)$. Hence G-conjugacy classes of seaweeds contain usually more than one standard representative.

2.2 Proposition. *Let Ψ, Ψ' be arbitrary subsets of Π.*
(1) $\mathfrak{p}(\Psi \mid \Psi') \simeq \mathfrak{p}(\Psi' \mid \Psi)$;
(2) $\mathfrak{p}(\Psi \mid \Psi')$ *is parabolic* $\Longleftrightarrow \Psi = \varnothing$ *or* $\Psi' = \varnothing$;
(3) $\mathfrak{p}(\Psi \mid \Psi')$ *is reductive* $\Longleftrightarrow \mathfrak{p}(\Psi \mid \Psi')$ *is a Levi subalgebra* $\Longleftrightarrow \Psi = \Psi'$;
(4) Suppose $\Psi \cap \Psi' \neq \varnothing$. *Then* $\mathfrak{p}(\Psi \mid \Psi')$ *is the direct sum of a 1-dimensional (central) subalgebra and a subalgebra* $\tilde{\mathfrak{p}}$, *which is a seaweed in a semisimple subalgebra* $\tilde{\mathfrak{g}} \subset \mathfrak{g}$.

Proof. 1. The group Aut $(\mathfrak{g})$ contains an involution θ that acts on $\mathfrak{t}$ as $-id$ (a so-called *Weyl involution*). Then $\theta(\mathfrak{p}(\Psi \mid \Psi')) = \mathfrak{p}(\Psi' \mid \Psi)$.
2, 3. Obvious.
4. Assume that $\alpha \in \Psi \cap \Psi'$. Deleting the node of the Dynkin diagram corresponding to α, one obtains the diagram for the required semisimple subalgebra $\tilde{\mathfrak{g}}$. □

Using the relationship between seaweeds in $\mathfrak{gl}_n$ and meander graphs, discovered in [3] (see also Section 3), one may deduce that $\operatorname{ind}\mathfrak{s} \le n$ for any seaweed $\mathfrak{s} \subset \mathfrak{gl}_n$. On the other hand, some computations of A.G. Elashvili provide a firm ground for the suggestion that $\operatorname{ind}\mathfrak{p} \le \operatorname{rk}\mathfrak{g}$ for any parabolic in $\mathfrak{g}$. It is therefore not too rash to state the following

2.3 Conjecture. Let $\mathfrak{s}$ be a seaweed subalgebra of a reductive $\mathfrak{g}$. Then $\operatorname{ind}\mathfrak{s} \le \operatorname{rk}\mathfrak{g}$. Furthermore, $\operatorname{ind}\mathfrak{s} = \operatorname{rk}\mathfrak{g}$ if and only if $\mathfrak{s}$ is reductive.

Making use of inductive formulas to be obtained below, we prove this conjecture for $\mathfrak{gl}(V)$ and $\mathfrak{sp}(V)$. (That is, for $\mathfrak{gl}(V)$ we exploit an approach different from that of [3].) For $\mathfrak{so}(V)$, the conjecture is reduced to the same conjecture for *parabolic* subalgebras. Although our results for orthogonal parabolics are not complete, some particular cases that we are able to handle support the conjecture.

3 Seaweed subalgebras of $\mathfrak{gl}_n$ and meander graphs

An ordered sequence of positive integers $(a_1, \dots, a_m)$ is called a *composition* of the number $\sum a_i$. The numbers a_i are said to be the *parts* or *coordinates* of the composition. Let V be an n-dimensional $\Bbbk$-vector space. It is well known that there exists a bijection between the conjugacy classes of parabolic subalgebras of $\mathfrak{gl}_n = \mathfrak{gl}(V)$ and the compositions of n. Under this bijection, a parabolic subalgebra corresponding to $(a_1, \dots, a_m)$ preserves a flag $\{0\} \subset V_1 \subset \cdots \subset V_{m-1} \subset V_m = V$, where $\dim V_i/V_{i-1} = a_i$. Following [3], we define a seaweed subalgebra of $\mathfrak{gl}(V)$ as the subalgebra preserving two "opposite" flags in V. This is a down-to-earth version of Definition 2.1 for $\mathfrak{gl}(V)$.

3.1 Definition. Let $\underline{a} = (a_1, \ldots, a_m)$ and $\underline{b} = (b_1, \ldots, b_t)$ be two compositions of $n = \dim V$. Fix a basis $(e_1, \ldots, e_n)$ for V, and consider two flags $\{0\} \subset V_1 \subset \cdots \subset V_{m-1} \subset V_m = V$ and $V = W_0 \supset W_1 \supset \cdots \supset W_{t-1} \supset W_t = \{0\}$, where $V_i = \langle e_1, \ldots, e_{a_1+\cdots+a_i} \rangle$ and $W_j = \langle e_{b_1+\cdots+b_j+1}, \ldots, e_n \rangle$. The subalgebra of $\mathfrak{gl}(V)$ preserving these two flags is called a seaweed subalgebra of $\mathfrak{gl}(V)$ or a seaweed of degree n. It will be denoted by $\mathfrak{p}(\underline{a} \mid \underline{b})$.

Remark. A basis-free exposition requires an intrinsic definition of "opposite" flags. Two flags $\{0\} \subset V_1 \subset \cdots \subset V_{m-1} \subset V_m = V$ and $V = W_0 \supset W_1 \supset \cdots \supset W_{t-1} \supset W_t = \{0\}$ are called opposite, if $\dim(V_i \cap W_j) = \max\{0, \dim V_i + \dim W_j - n\}$ for all i, j. It is not hard to show that two flags are opposite if and only if there exists a basis for V satisfying the properties of Definition 3.1.

A standard seaweed algebra is depicted in Figure 1. We always think of seaweeds in $\mathfrak{gl}_n$ as

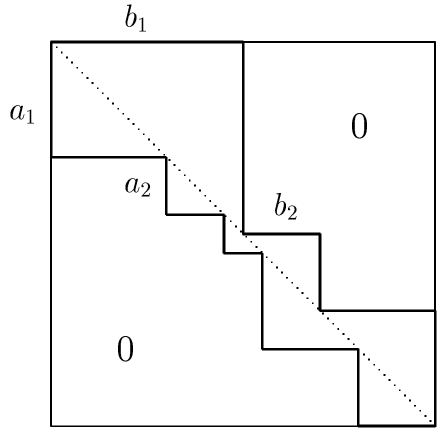


FIGURE 1. A seaweed subalgebra of $\mathfrak{gl}(V)$

matrix algebras of such form. That is, it is tacitly assumed that we have fixed a basis for V and the two "opposite" flags determined by two compositions of $\dim V$. The following is a specialisation of Proposition 2.2.

3.2 Proposition. *Set $\underline{a}^{-1} = (a_m, \ldots, a_1)$ and $\underline{b}^{-1} = (b_t, \ldots, b_1)$.*

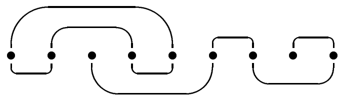
1. *$\mathfrak{p}(\underline{a} \mid \underline{b}) \simeq \mathfrak{p}(\underline{b} \mid \underline{a}) \simeq \mathfrak{p}(\underline{a}^{-1} \mid \underline{b}^{-1})$;*
2. *$\mathfrak{p}(\underline{a} \mid \underline{b})$ is parabolic $\Leftrightarrow$ $\underline{a} = (n)$ or $\underline{b} = (n)$;*
3. *$\mathfrak{p}(\underline{a} \mid \underline{b})$ is Levi $\Leftrightarrow$ $\underline{a} = \underline{b}$;*
4. *If $a_1 = b_1$, then $\mathfrak{p}(\underline{a} \mid \underline{b}) \simeq \mathfrak{gl}_{a_1} \dotplus \mathfrak{p}(a_2, \ldots, a_m \mid b_2, \ldots, b_t)$;*
5. *If $a_1 + \cdots + a_k = b_1 + \cdots + b_l$ for some $k < m$ and $l < t$, then $\mathfrak{p}(\underline{a} \mid \underline{b})$ is isomorphic to a direct sum of two proper subalgebras, either of which is a seaweed algebra (of smaller degree) in its own sense.* □

Since $\mathfrak{p}(\underline{a} \mid \underline{b})$ contains the centre of $\mathfrak{gl}(V)$, Lemma 1.1(2) implies that $\operatorname{ind} \mathfrak{p}(\underline{a} \mid \underline{b}) \geq 1$. One needn't expect a simple general expression for the index of seaweed subalgebras. For instance, it is known that the index of a Borel subalgebra of $\mathfrak{gl}_n$ equals $[(n+1)/2]$, see [12]. Therefore, as $\mathfrak{p}(1, \ldots, 1 \mid \underline{b})$ is a Borel subalgebra of $\mathfrak{gl}_{b_1} \oplus \cdots \oplus \mathfrak{gl}_{b_t}$, we obtain $\operatorname{ind} \mathfrak{p}(1, \ldots, 1 \mid$

$\underline{b}) = \sum_j [(b_j+1)/2]$. On the other hand, let $\mathfrak{p} = \mathfrak{p}(a, n-a \mid n)$ be a maximal parabolic subalgebra of $\mathfrak{gl}_n$. According to a result of Elashvili (see [3, sect. 10]), $\operatorname{ind}\mathfrak{p} = \gcd(a, n-a)$. It seems unlikely to find a closed formula which includes the previous two as particular cases.

A meander graph with n nodes is determined by two compositions of n. The word "meander" is taken from [3], but the following exposition is simpler than Definition 3.1 in [loc. cit]. As above, consider two compositions $\underline{a}, \underline{b}$. The corresponding meander graph is denoted by $M(\underline{a} \mid \underline{b})$. Its nodes are n consecutive points lying on a horizontal line. The parts of $\underline{a}$ determine arcs connecting pairs of different nodes. These arcs are located over the horizontal line. More precisely, the part a_1 determines $[a_1/2]$ consecutively imbedded arcs <u>*over*</u> the nodes $1, \dots, a_1$; the part a_2 determines $[a_2/2]$ imbedded arcs over the nodes $a_1+1, \dots, a_1+a_2$, etc. The arcs corresponding to $\underline{b}$ have to be drawn following the same rule, but <u>*under*</u> the line.

Example.

$M(5,2,2 \mid 2,4,3)=$ 

From the previous exposition it follows that any seaweed subalgebra of $\mathfrak{gl}_n$ determines a meander graph with n nodes and vice versa. In [3, Thm. 5.1], it is shown that $\operatorname{ind}\mathfrak{p}(\underline{a} \mid \underline{b})$ is equal to the number of connected components plus the number of circuits in $M(\underline{a} \mid \underline{b})$. It follows from this description that $\operatorname{ind}\mathfrak{p}(\underline{a} \mid \underline{b}) \le n$. However, we do not use meander graphs in the sequel, as our aim is to present an approach that generalise to other reductive Lie algebras.

4 An inductive formula for the index of seaweeds in $\mathfrak{gl}(V)$

We are going to apply Theorem 1.3 to computing the index of seaweeds. Given a seaweed subalgebra $\mathfrak{p} \subset \mathfrak{gl}_n$, we find an algebraic subalgebra $\mathfrak{q} \subset \mathfrak{p}$ and an Abelian ideal $W \subset \mathfrak{p}$ such that $\mathfrak{p} \simeq W \times_\rho \mathfrak{q}$. (Here $\rho : \mathfrak{q} \to \mathfrak{gl}(W)$). Then we show that $\mathfrak{q}{\cdot}\xi = W$ for some $\xi \in W^*$ (whence $\operatorname{ind}(\mathfrak{q}, W) = 0$) and that $\mathfrak{q}_\xi$ is isomorphic to a seaweed algebra of smaller degree. By Theorem 1.3 and Corollary 1.4, we then conclude that $\operatorname{ind}\mathfrak{p} = \operatorname{ind}\mathfrak{q}_\xi$.

Consider first a particular case of parabolic subalgebras in $\mathfrak{gl}_n$, i.e., assume that $\underline{b} = (n)$ and $a_1 + \dots + a_m = n$.

4.1 Proposition.

$$\operatorname{ind}\mathfrak{p}(\underline{a} \mid n) = \begin{cases} \operatorname{ind}\mathfrak{p}(a_2, \dots, a_m \mid n-2a_1, a_1) & \text{if } a_1 \le n/2; \\ \operatorname{ind}\mathfrak{p}(2a_1-n, a_2, \dots, a_m \mid a_1) & \text{if } a_1 > n/2. \end{cases}$$

Proof. Take $\mathfrak{q} = \mathfrak{p}(a_1, \dots, a_m \mid a_1, n-a_1) \simeq \mathfrak{gl}_{a_1} \dotplus \mathfrak{p}(a_2, \dots, a_m \mid n-a_1)$ and let W be the nilpotent radical of the maximal parabolic subalgebra $\mathfrak{p}^{max} := \mathfrak{p}(a_1, n-a_1 \mid n)$. Then W is Abelian, $W \subset \mathfrak{p}(\underline{a}, n)$, W is a $\mathfrak{q}$-module, and $\mathfrak{p}(\underline{a} \mid n) \simeq W \times_\rho \mathfrak{q}$. The standard Levi subalgebra of $\mathfrak{p}^{max}$ is $\mathfrak{l}^{max} := \mathfrak{p}(a_1, n-a_1 \mid a_1, n-a_1)$. Clearly, $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{l}^{max}$. The dual $\mathfrak{q}$-module W^* can also be regarded as $\mathfrak{l}^{max}$-module. We identify W^* with the nilpotent radical of the opposite maximal parabolic subalgebra $\mathfrak{p}(n \mid a_1, n-a_1)$. It is well known that W^* is a spherical L^{max}-module, i.e., for any Borel subalgebra $\mathfrak{b}$ of $\mathfrak{l}^{max}$ there exists a $\xi \in W^*$ such that $\mathfrak{b}{\cdot}\xi = W^*$. Therefore there exists a ξ such that $\mathfrak{q}{\cdot}\xi = W^*$. This means that $\operatorname{ind}(\mathfrak{q}, W) = 0$. We exhibit a special ξ with this property. The explicit matrix presentation will then show that $\mathfrak{q}_\xi$ is isomorphic to a standard seaweed algebra of smaller degree. One has to distinguish two possibilities.

(a) $a_1 \le n/2$.

In this case ξ is represented by the identity matrix of order a_1 which is placed in the left lower corner of $\mathfrak{gl}_n$, see the left picture in Figure 2. The stripped region in the left picture

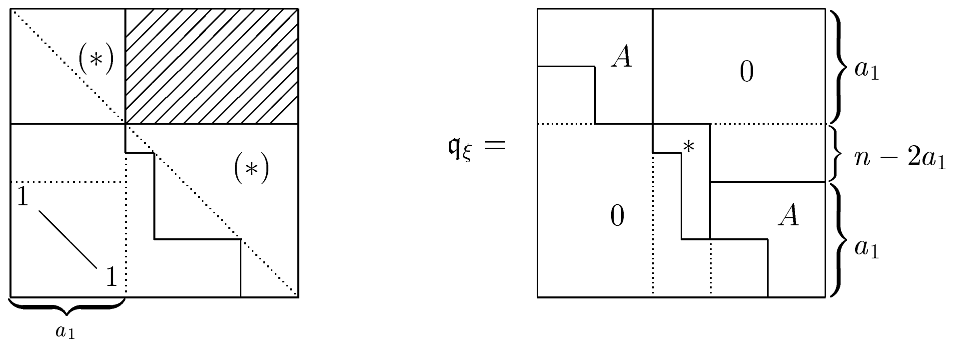


FIGURE 2. $a_1 \le n/2$

represents W, whereas its transpose is W^*. The regions marked by '$(*)$' depict $\mathfrak{q}$. An easy calculation with matrices shows that $\mathfrak{q}_\xi$ consists of all matrices depicted in the right picture in Figure 2, where the fragment marked by '$*$' is arbitrary and two symbols A mean that the corresponding fragments must be equal. It follows that $\mathfrak{q}_\xi$ is isomorphic to the seaweed subalgebra $\mathfrak{p}(a_2, \dots, a_m \mid n - 2a_1, a_1)$ in $\mathfrak{gl}_{n-a_1}$, where the latter is realised as the right lower square in $\mathfrak{gl}_n$.

(b) $a_1 > n/2$.

In this case we take ξ as the identity matrix of order $n - a_1$ which is placed in W^* as indicated in the left picture on Figure 3. The structure of $\mathfrak{q}_\xi$ is depicted on the right picture in Figure 3. In both pictures we follow the same conventions as in part (a). It follows that

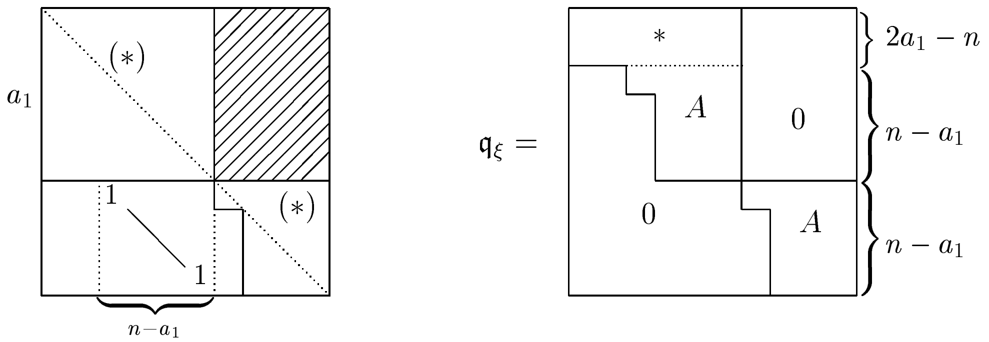


FIGURE 3. $a_1 > n/2$

$\mathfrak{q}_\xi$ is isomorphic to the parabolic subalgebra $\mathfrak{p}(2a_1 - n, a_2, \dots, a_m \mid a_1)$ in $\mathfrak{gl}_{a_1}$, where the latter is realised as the left upper square in $\mathfrak{gl}_n$.

Making use of Theorem 1.3 and Corollary 1.4, we conclude that $\operatorname{ind}\mathfrak{p}(\underline{a} \mid n) = \operatorname{ind}\mathfrak{q}_\xi$ in both (a) and (b). $\square$

Note that Proposition 4.1 is not sufficient for inductive computing the index of parabolics. For, $\mathfrak{q}_\xi$ is not a parabolic subalgebra, if $a_1 < n/2$. Fortunately, Raïs' theorem can be applied in the general situation as well, and now we come back to arbitrary seaweed algebras.

Let $\underline{a} = (a_1, \dots, a_m)$ and $\underline{b} = (b_1, \dots, b_t)$ be two compositions of $\dim V$. Without loss of generality, we assume that $a_1 \le b_1$.

4.2 Theorem.
(i) *Suppose $a_1 = b_1$. Then*

$$\operatorname{ind}\mathfrak{p}(\underline{a} \mid \underline{b}) = a_1 + \operatorname{ind}\mathfrak{p}(a_2, \dots, a_m \mid b_2, \dots, b_t)\ ;$$

(ii) *Suppose $a_1 < b_1$. Then*

$$\operatorname{ind}\mathfrak{p}(\underline{a} \mid \underline{b}) = \begin{cases} \operatorname{ind}\mathfrak{p}(a_2, \dots, a_m \mid b_1 - 2a_1, a_1, b_2, \dots, b_t) & \text{if } a_1 \le b_1/2; \\ \operatorname{ind}\mathfrak{p}(2a_1 - b_1, a_2, \dots, a_m \mid a_1, b_2, \dots, b_t) & \text{if } a_1 > b_1/2. \end{cases}$$

Proof. (i) This follows from Lemma 1.1(3) and Proposition 3.2(4).
(ii) Consider $\mathfrak{gl}_{b_1}$ as the subalgebra of $\mathfrak{gl}(V)$ corresponding to the first b_1 basis vectors in V. The intersection $\mathfrak{p}(\underline{a} \mid \underline{b}) \cap \mathfrak{gl}_{b_1}$ is a parabolic subalgebra of $\mathfrak{gl}_{b_1}$. More precisely, if k is the maximal number such that $a_1 + \dots + a_k \le b_1$, then the corresponding composition is $(a_1, \dots, a_k, b_1 - \sum_{i=1}^k a_i)$. Inside of this subalgebra we may perform the same manipulation as in Proposition 4.1. It is easily seen that this does not affect the part of $\mathfrak{p}(\underline{a} \mid \underline{b})$ lying outside of $\mathfrak{gl}_{b_1}$.
Alternatively, this can be explained as follows. Consider the subalgebra $\mathfrak{q} := \mathfrak{p}(\underline{a} \mid a_1, b_1 - a_1, b_2, \dots, b_t) \subset \mathfrak{p}(\underline{a} \mid \underline{b})$. There exists a unique $\mathfrak{q}$-stable complement to $\mathfrak{q}$ in $\mathfrak{p}(\underline{a} \mid \underline{b})$, say W. As in Proposition 4.1, W is the Abelian nilpotent radical of the maximal parabolic subalgebra $\mathfrak{p}(a_1, b_1 - a_1 \mid b_1)$ of $\mathfrak{gl}_{b_1}$. Therefore $\mathfrak{p}(\underline{a} \mid \underline{b}) \simeq W \times_\rho \mathfrak{q}$ and $\operatorname{ind}(\mathfrak{q}, W) = 0$. Choosing the $\mathfrak{q}$-regular element $\xi \in W^*$ as in 4.1(a) or (b), we find that $\mathfrak{q}_\xi$ is isomorphic to the seaweed subalgebra of required shape. $\square$

Thus, we have already obtained an inductive formula for the index of an arbitrary seaweed subalgebra. However the formula in 4.2(ii) has a drawback that it includes consideration of cases and the decrease of degree depends on cases. Now we show that the consecutive application of formulas 4.2(ii) enables us to obtain a unified expression for $\operatorname{ind}\mathfrak{p}(\underline{a} \mid \underline{b})$, where the decrease of degree is always equal to a_1.

4.3 Theorem. *Suppose $a_1 < b_1$. Then there exists a unique nonnegative integer p such that $\dfrac{p}{p+1} b_1 < a_1 \le \dfrac{p+1}{p+2} b_1$ and in this case*

$$\operatorname{ind}\mathfrak{p}(\underline{a} \mid \underline{b}) = \operatorname{ind}\mathfrak{p}(a_2, \dots, a_m \mid (p+1)b_1 - (p+2)a_1, (p+1)a_1 - pb_1, b_2, \dots, b_t)\ .$$

If $(p+1)b_1 = (p+2)a_1$, then the zero coordinate in the right-hand side must be omitted.

Proof. Notice that p is also being computed as the least integer such that $a_1/(b_1-a_1)\le p+1$.

1. If $a_1 \le b_1/2$, then $p=0$ and the required formula for the index is nothing but the first expression in 4.2(ii).
2. If $b_1/2 < a_1 < b_1$, then the second expression in 4.2(ii) yields the seaweed algebra corresponding to the compositions $(2a_1-b_1, a_2,\dots,a_m)$ and $(a_1,b_2,\dots,b_t)$. Here $2a_1-b_1<a_1$, so that we may apply the same inductive procedure.
   (a) If $2a_1-b_1\le a_1/2$, then $a_1\le 2b_1/3$, i.e., $p=1$. Here the application of the first case in 4.2(ii) yields $\mathfrak{p}(a_2,\dots,a_m \mid 2b_1-3a_1, 2a_1-b_1, b_2,\dots,b_t)$, which is just the required formula with $p=1$.
   (b) If $2a_1-b_1>a_1/2$ (i.e., $a_1>2b_1/3$), then applying the second case in 4.2(ii) we obtain $\mathfrak{p}(3a_1-2b_1, a_2,\dots,a_m \mid 2a_1-b_1,b_2,\dots,b_t)$. Here $3a_1-2b_1\le 2a_1-b_1$, so that 4.2(ii) again applies.

   If $3a_1-2b_1\le(2a_1-b_1)/2$ (i.e., $a\le 3b_1/4$), then the application of the first case in 4.2(ii) yields the required formula with $p=2$, and the procedure terminates. Otherwise we continue inductively, using the second case in 4.2(ii). Eventually, we reach the number p such that
   $(p+1)a_1-pb_1\le(pa_1-(p-1)b_1)/2$, i.e., $a_1\le(p+1)b_1/(p+2)$. Thereafter the application of the first case in 4.2(ii) yields the required formula.

Summarising, we see that if $a_1<b_1$ and p is the least integer such that $a_1\le(p+1)b_1/(p+2)$, then the formula of the Theorem is being obtained after p times application of the second case in 4.2(ii) and then a single application of the first case in 4.2(ii). □

The meaning of Theorem 4.3 is that one may just drop the first coordinate of $\underline{a}$.

4.4 Corollary.

(i) *Let $\underline{a},\underline{b}$ be arbitrary compositions of* $\dim V$. *Then* $1\le \operatorname{ind}\mathfrak{p}(\underline{a}\mid\underline{b})\le\dim V$. *Moreover,* $\operatorname{ind}\mathfrak{p}(\underline{a}\mid\underline{b})=\dim V$ *if and only if* $\underline{a}=\underline{b}$;

(ii) *Suppose d is a common divisor for all a_i, b_j. Set $\underline{a}'=\underline{a}/d$ and $\underline{b}'=\underline{b}/d$. Then* $\operatorname{ind}\mathfrak{p}(\underline{a}\mid\underline{b})=d{\cdot}\operatorname{ind}\mathfrak{p}(\underline{a}'\mid\underline{b}')$.

Proof. Argue by induction on $\dim V$, using Theorems 4.2(i) and 4.3. For part (ii), observe also that both inductive formulas preserve the divisibility on d for all coordinates of the compositions in question. □

4.5 Remark. It is natural to say that the inductive formulas in 4.2 and 4.3 are *left* ones, since only the left ends of compositions are disturbed. One can similarly write down *right* inductive formulas for the index, i.e., the ones dealing with a_m and b_t. The proofs are essentially the same or, formally, one may refer to Proposition 3.2(1). The explicit expressions are left as an exercise for the reader.

Examples. 1. Let $\dim V=2^n-1$, $\underline{a}=(2^{n-1},2^{n-2},\dots,1)$, and $\underline{b}=(2^n-1)$. Let i_n $(n\ge 1)$ denote the index of this parabolic subalgebra of $\mathfrak{gl}(V)$. In the following computations we write $i\left\{\frac{\underline{a}}{\underline{b}}\right\}$ for the index of $\mathfrak{p}(\underline{a}\mid\underline{b})$. The symbol $\overset{right}{=}$ means that we apply the right inductive formula, and likewise for 'left'.

$$i\left\{\frac{(2^{n-1},2^{n-2},\dots,1)}{(2^n-1)}\right\}\overset{left}{=}i\left\{\frac{(2^{n-2},2^{n-3},\dots,1)}{(2^{n-1}-2,1)}\right\}\overset{right}{=}1+i\left\{\frac{(2^{n-2},2^{n-3},\dots,2)}{(2^{n-1}-2)}\right\}\overset{4.4(\mathrm{ii})}{=}$$

$$=1+2\cdot i\left\{\frac{(2^{n-3},2^{n-4},\dots,1)}{(2^{n-2}-1)}\right\}.$$

Whence $i_n=1+2i_{n-2}$. Since $i_1=i_2=1$, we obtain $i_n=2^{[(n+1)/2]}-1$.

2. (a) $\dim V$ is odd, $\underline{a}=(2,\dots,2,1)$, and $\underline{b}=(1,2,\dots,2)$.
   (b) $\dim V$ is even, $\underline{a}=(1,2,\dots,2,1)$, and $\underline{b}=(2,\dots,2)$.

Applying left or right inductive formulas for the index switches between (a) and (b), and eventually kills all '2'. Since $\operatorname{ind}\mathfrak{p}(1\mid 1)=1$, we conclude that the index of all these seaweed algebras is equal to 1.

Remark. All seaweed subalgebras of $\mathfrak{sl}(V)$ are of the form $\mathfrak{p}'(\underline{a}\mid\underline{b})=\mathfrak{p}(\underline{a}\mid\underline{b})\cap\mathfrak{sl}(V)$. Here we have $\operatorname{ind}\mathfrak{p}'(\underline{a}\mid\underline{b})=\operatorname{ind}\mathfrak{p}(\underline{a}\mid\underline{b})-1$ and hence $0\le\operatorname{ind}\mathfrak{p}'(\underline{a}\mid\underline{b})\le\dim V-1=\operatorname{rk}\mathfrak{sl}(V)$.

An interesting open problem is to characterise the seaweeds of index 1 for $\mathfrak{gl}(V)$, i.e., the Frobenius seaweeds of $\mathfrak{sl}(V)$. Indeed, while the maximal index case is settled in Corollary 4.4(i), no explicit formulas are known for the case of index 1. A lower bound for the index that we give below provides only a necessary condition for $\mathfrak{p}(\underline{a}\mid\underline{b})$ to have index 1. Since $\sum a_i=\sum b_j$, the total number of the odd parts in $\underline{a}$ and $\underline{b}$ is even.

4.6 Proposition. *Suppose $\underline{a}$ and $\underline{b}$ contain together $2k$ odd parts.*

(i) *If $k=0$, then $\operatorname{ind}\mathfrak{p}(\underline{a}\mid\underline{b})\ge 2$;*

(ii) *If $k>0$, then $\operatorname{ind}\mathfrak{p}(\underline{a}\mid\underline{b})\ge k$.*

Proof. (i) Since all parts are even, Corollary 4.4(ii) applies. We also know that the index is positive.
(ii) Computing the index consecutively, we apply on each step either 4.2(i) or 4.2(ii). These transformation gradually reduce the pair of initial compositions. Finally, we get the pair $\underline{a}'=\underline{b}'=(0)$, while the sum of those numbers that arise in connection with all applications of transformations 4.2(i) gives the index. As transformations 4.2(ii) preserve the number of odd parts and transformations 4.2(i) may reduce the number of odd parts by 2, we see that 4.2(i) has been applied at least k times. Thus, the index is at least k. □

As an immediate consequence, the following necessary condition is obtained. It was observed earlier, in the context of meander graphs, by A.G. Elashvili.

4.7 Corollary. *Suppose $\operatorname{ind}\mathfrak{p}(\underline{a}\mid\underline{b})=1$. Then $\underline{a}$ and $\underline{b}$ have exactly two odd parts.*

Simple examples show that this condition is not sufficient, e.g., $\operatorname{ind}\mathfrak{p}(1,5\mid 4,2)=3$.

It seems that there is no a simple closed formula for the index of parabolics in $\mathfrak{gl}_n$. But such a formula exists for the symmetric parabolics. Recall that a parabolic subalgebra is called *symmetric*, if it is conjugate to an opposite one. In case of $\mathfrak{gl}_n$, we have

$$\mathfrak{p}(a_1,\dots,a_m \mid n) \text{ is symmetric} \Leftrightarrow a_i = a_{m+1-i} \text{ for all } i.$$

4.8 Proposition. *Let* $\mathfrak{p} = \mathfrak{p}(a_1,\dots,a_m \mid n) \subset \mathfrak{gl}_n$ *be symmetric. Then* $\operatorname{ind}\mathfrak{p} = \sum\limits_{i=1}^{[m+1/2]} a_i$.

Proof. If $m = 1$, then $\mathfrak{p} = \mathfrak{gl}_n$ and the assertion is clear.
If $m > 1$, then we argue by induction on m. Since $a_1 = a_m$, we have $a_1 \le n/2$. Applying Proposition 4.1 yields

$$\operatorname{ind}\mathfrak{p} = \operatorname{ind}\mathfrak{p}(a_2,\dots,a_m \mid n-2a_1, a_1) = a_1 + \operatorname{ind}\mathfrak{p}(a_2,\dots,a_{m-1} \mid n-2a_1)\ .$$

This completes the inductive step. □

4.9 Corollary (cf. [12])**.** *The index of a Borel subalgebra of* $\mathfrak{gl}_n$ *is equal to* $[n+1/2]$.

Proof. We have $\underline{a} = (1,\dots,1)$ (n times) for a Borel subalgebra. □

5 On seaweed subalgebras of symplectic and orthogonal Lie algebras

There are three other series of classical Lie algebras: B_n, C_n, D_n. To a great extent, it is convenient to treat them together, so that we begin with introducing a unified notation.

In this section, V is an N-dimensional $\Bbbk$-vector space equipped with a non-degenerate bilinear form β which is either symmetric or alternate. Set $n = [N/2]$. Of course, N is even whenever β is alternate. We have $\beta(v,w) = \varepsilon\cdot\beta(w,v)$, where $\varepsilon \in \{+1,-1\}$. The Lie algebra preserving this form is denoted by $\mathfrak{g}_\varepsilon(V)$. Thus, $\mathfrak{g}_{+1}(V) = \mathfrak{so}(V)$ and $\mathfrak{g}_{-1}(V) = \mathfrak{sp}(V)$, and in both cases one has $\operatorname{rk}\mathfrak{g}_\varepsilon(V) = n$. A flag $V_1 \subset \dots \subset V_m$ in V is called *isotropic*, if $\beta\mid_{V_m} \equiv 0$. Then $\dim V_m \le n$. It is well-known that the stabiliser of any isotropic flag is a parabolic subalgebra of $\mathfrak{g}_\varepsilon(V)$ and that each parabolic subalgebra is obtained in this way. Note that V_m is not necessarily maximal isotropic.

To describe explicitly seaweeds in $\mathfrak{g}_\varepsilon(V)$, we first fix a maximal isotropic subspace $V^0 \subset V$. Consider two opposite flags in V^0:

$$\{0\} \subset V_1 \subset \dots \subset V_{m-1} \subset V_m \text{ and } W_0 \supset W_1 \supset \dots \supset W_{t-1} \supset W_t = \{0\}\ .$$

(The definition of opposite flags is given after Definition 3.1.) It is easily seen that the corresponding parabolic subalgebras of $\mathfrak{g}_\varepsilon(V)$ are weakly opposite in the sense of Section 2. Set $a_i = \dim V_i/V_{i-1}$, $i = 1,\dots,m$, and $b_j = \dim W_{j-1}/W_j$, $j = 1,\dots,t$. It is assumed that the spaces of each flag are different so that $a_i > 0$ and $b_j > 0$. The seaweed which is intersection of these two parabolic subalgebras is denoted by $\mathfrak{p}_n(\underline{a} \mid \underline{b})$. If we need to indicate explicitly the Cartan type of $\mathfrak{g}_\varepsilon(V)$, then this notation to be supplied by the corresponding superscript; e.g., $\mathfrak{p}_n^D(\underline{a} \mid \underline{b})$ is a seaweed in $\mathfrak{so}_{2n}$. This superscript (even if it is omitted) is called the *type* of a seaweed.

Remark. If $\varepsilon = 1$ and $N = 2n$, then the variety of isotropic n-dimensional subspaces consists of two SO_{2n}-orbits, which form together a sole O_{2n}-orbit. In this case, a fixed choice of V_0 means that we consider not all parabolic and seaweed subalgebras. This is, however, harmless, since an outer automorphism of $\mathfrak{so}_{2n}$ takes these skipped subalgebras to the ones that we do consider.

Convention. Whenever we write in the sequel an expression relating two different seaweeds whose type is not indicated, this means that the type can be arbitrary but the same.

To summarise, there exists a one-to-one correspondence between the conjugacy classes of seaweeds in $\mathfrak{g}_\varepsilon(V)$ and the unordered pairs of strings of positive integers $\underline{a} = (a_1, \dots, a_m)$, $\underline{b} = (b_1, \dots, b_t)$, where $\sum a_i \le n$ and $\sum b_j \le n$. It is not necessarily the case that $\sum a_i = \sum b_j$.

The following is obvious (cf. Prop. 3.2).

5.1 Proposition. *Let $\mathfrak{p}_n(\underline{a} \mid \underline{b})$ be a seaweed in $\mathfrak{g}_\varepsilon(V)$. Then*

1. $\mathfrak{p}_n(\underline{a} \mid \underline{b}) \simeq \mathfrak{p}_n(\underline{b} \mid \underline{a})$;
2. $\mathfrak{p}_n(\underline{a} \mid \underline{b})$ *is parabolic* $\Leftrightarrow$ $\underline{a} = \varnothing$ *or* $\underline{b} = \varnothing$;
3. $\mathfrak{p}_n(\underline{a} \mid \underline{b})$ *is Levi* $\Leftrightarrow$ $\underline{a} = \underline{b}$;
4. *If* $a_1 = b_1$, *then* $\mathfrak{p}_n(\underline{a} \mid \underline{b}) \simeq \mathfrak{gl}_{a_1} \dotplus \mathfrak{p}_{n-a_1}(a_2, \dots, a_m \mid b_2, \dots, b_t)$;
5. *Suppose* $a_1 + \dots + a_k = b_1 + \dots + b_l =: r$ *for some* k, t, *where* $r < n$. *Then* $\mathfrak{p}(\underline{a} \mid \underline{b})$ *is isomorphic to a direct sum of two proper subalgebras. One of them is a seaweed in* $\mathfrak{gl}_r$ *and the other is a seaweed in* $\mathfrak{g}_\varepsilon(V')$, *where* $\dim V' = N - 2r$. □

Note that unlike 3.2(1), there is no symmetry here between the ends of $\underline{a}$ and $\underline{b}$; that is, $\mathfrak{p}_n(\underline{a} \mid \underline{b}) \not\simeq \mathfrak{p}_n(\underline{a}^{-1} \mid \underline{b}^{-1})$.

• For explicit calculations in $\mathfrak{sp}_{2n}$, it is convenient to use a basis for the V, where the matrix of β is equal to $\begin{pmatrix} 0 & -\Upsilon_n \\ \Upsilon_n & 0 \end{pmatrix}$ and Υ_n is the $n \times n$-matrix whose the only nonzero entries are 1's along the antidiagonal. For any $A \in Mat_n(\Bbbk)$, let $\widehat{A}$ denote the matrix $\Upsilon_n(A^t)\Upsilon_n$, where A^t is the usual transpose of A. The transformation $A \mapsto \widehat{A}$ is the transpose relative to the antidiagonal. In the above basis for V, the algebra $\mathfrak{sp}_{2n}$ has the following block form:

$$\mathfrak{sp}_{2n} = \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid B = \widehat{B}, C = \widehat{C}, D = -\widehat{A} \} ,$$

where A, B, C, D are $n \times n$ matrices.

• For explicit calculations in $\mathfrak{so}_N$, it is convenient to use a basis for the V, where the matrix of β is equal to Υ_N. Then $\mathfrak{so}_N = \{A \in Mat_N(\Bbbk) \mid A = -\widehat{A}\}$.

Associated with these nice bases, one has a natural triangular decomposition of $\mathfrak{g}_\varepsilon(V)$. Namely, $\mathfrak{t}$ to be the set of all diagonal matrices in $\mathfrak{g}_\varepsilon(V)$ and $\mathfrak{u}$ (resp. $\mathfrak{u}_-$) to be the set of all strictly upper (resp. lower) triangular matrices. It is also assumed in all cases that V^0 is the linear span of the first n basis vectors. Then each seaweed $\mathfrak{p}_n(\underline{a} \mid \underline{b})$ is standard relative to the above triangular decomposition. In particular, $\mathfrak{p}_n(n \mid n) \simeq \mathfrak{gl}_n$ is a standard Levi subalgebra of $\mathfrak{g}_\varepsilon(V)$. For $\mathfrak{sp}_{2n}$, it is just the set of all block matrices with $B = C = 0$.

Our plan is as follows. First, we show that if a seaweed in $\mathfrak{g}_\varepsilon(V)$ is not a parabolic subalgebra, then the inductive procedure from Section 4 applies and yields a seaweed in a $\mathfrak{g}_\varepsilon(V')$ with the same ε and $\dim V' < \dim V$. This procedure terminates when, at some stage, one obtains a parabolic subalgebra. Then we discuss the problem of computing the index of parabolic subalgebras.

5.2 Theorem. *Let $\mathfrak{p}_n(\underline{a} \mid \underline{b})$ be a seaweed in $\mathfrak{g}_\varepsilon(V)$. Suppose $\underline{a} \ne \varnothing$ and $\underline{b} \ne \varnothing$ and assume for definiteness that $a_1 \le b_1$.*

(i) *If $a_1 = b_1$, then $\operatorname{ind}\mathfrak{p}_n(\underline{a} \mid \underline{b}) = a_1 + \operatorname{ind}\mathfrak{p}_{n-a_1}(a_2, \dots, a_m \mid b_2, \dots, b_t)$;*

(ii) *If $a_1 < b_1$, then*

$$\operatorname{ind}\mathfrak{p}_n(\underline{a} \mid \underline{b}) = \begin{cases} \operatorname{ind}\mathfrak{p}_{n-a_1}(a_2, \dots, a_m \mid b_1 - 2a_1, a_1, b_2, \dots, b_t) & \text{if } a_1 \le b_1/2; \\ \operatorname{ind}\mathfrak{p}_{n-b_1+a_1}(2a_1 - b_1, a_2, \dots, a_m \mid a_1, b_2, \dots, b_t) & \text{if } a_1 > b_1/2. \end{cases}$$

Proof. (i) This follows from Lemma 1.1(3) and Proposition 5.1(4).
(ii) The proofs of Proposition 4.1 and Theorem 4.2 apply here as well, since all essential events happen inside of the standard Levi subalgebra $\mathfrak{p}_n(n \mid n) = \mathfrak{gl}_n$. Indeed, consider the seaweed $\mathfrak{q} := \mathfrak{p}_n(\underline{a} \mid a_1, b_1 - a_1, b_2, \dots, b_t)$. It is a subalgebra of $\mathfrak{p}_n(\underline{a} \mid \underline{b})$ having a unique $\mathfrak{q}$-stable complement, say W. The space W lies inside of $\mathfrak{p}_n(n \mid n)$. It is an Abelian ideal of $\mathfrak{p}_n(\underline{a} \mid \underline{b})$, and therefore $\mathfrak{p}_n(\underline{a} \mid \underline{b}) \simeq W \times_\rho \mathfrak{q}$. Taking a $\mathfrak{q}$-regular point $\xi \in W^*$ as prescribed in the proof of Proposition 4.1, one finds that $\mathfrak{q}{\cdot}\xi = W^*$ and $\mathfrak{q}_\xi$ is isomorphic to a seaweed of required form. It is clear that this procedure preserves both ε and the parity of $\dim V$,

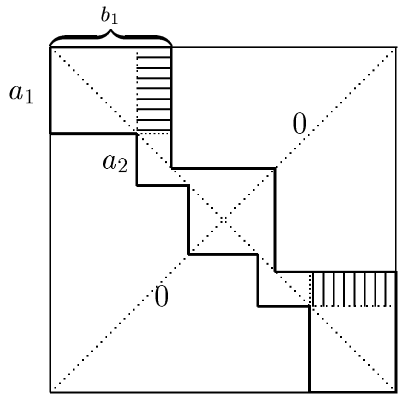


FIGURE 4. A standard seaweed in $\mathfrak{g}_\varepsilon(V)$

i.e., preserves the type of the seaweed. A typical seaweed in $\mathfrak{g}_\varepsilon(V)$ is depicted in Figure 4, where, as before, the stripped region represents W. This figure corresponds to the case, where $m = 2$ and $t = 1$. $\square$

Repeating verbatim the proof of Theorem 4.3, we obtain the following consequence of Theorem 5.2(ii).

5.3 Theorem. *Suppose $a_1 < b_1$ and let p be the unique nonnegative integer such that $\dfrac{p}{p+1}b_1 < a_1 \le \dfrac{p+1}{p+2}b_1$. Then*

$$\operatorname{ind}\mathfrak{p}_n(\underline{a} \mid \underline{b}) = \operatorname{ind}\mathfrak{p}_{n-a_1}(a_2,\dots,a_m \mid (p+1)b_1-(p+2)a_1,(p+1)a_1-pb_1,b_2,\dots,b_t)\ .$$

If $(p+1)b_1=(p+2)a_1$, then the zero coordinate in the right-hand side has to be omitted.

One may apply Theorems 5.2 and 5.3 until one of the two strings will vanish. Assume for definiteness that $\sum a_i \ge \sum b_j =: q$. Then the $\underline{b}$-part will eventually vanish and we obtain an equality of the form

(5.4) $$\operatorname{ind}\mathfrak{p}_n(\underline{a} \mid \underline{b}) = d + \operatorname{ind}\mathfrak{p}_{n-q}(\underline{a}' \mid \varnothing).$$

Here $\underline{a}'$ is a new string whose sum of coordinates is equal to $\sum a_i - q$. The quantity d depends on the number of applications of transformation 5.2(i). At least, one may assert that $d \le q$ and, moreover, $d = q$ if and only if $\underline{b}$ is a left substring of $\underline{a}$. In the last case, $\underline{a}'$ is the right "remainder" of $\underline{a}$, i.e., $\underline{a} = (\underline{b}, \underline{a}')$.

5.5 Example. Consider the seaweed $\mathfrak{p}_{11}(3,3 \mid 4,5)$. It can be a subalgebra in $\mathfrak{sp}_{22}$ or $\mathfrak{so}_{22}$ or $\mathfrak{so}_{23}$. This diversity of possibilities does not affect next computations. We compute the index applying Theorem 5.3:

$$\operatorname{ind}\mathfrak{p}_{11}(3,3 \mid 4,5) = \operatorname{ind}\mathfrak{p}_{10}(2,3 \mid 3,5) = \operatorname{ind}\mathfrak{p}_{9}(1,3 \mid 2,5) = \operatorname{ind}\mathfrak{p}_{8}(3 \mid 1,5) =$$

$$\operatorname{ind}\mathfrak{p}_{7}(1,1 \mid 5) = \operatorname{ind}\mathfrak{p}_{6}(1 \mid 3,1) = \operatorname{ind}\mathfrak{p}_{5}(\varnothing \mid 1,1,1).$$

The last term is the index of a parabolic subalgebra in $\mathfrak{sp}_{10}$ or $\mathfrak{so}_{10}$ or $\mathfrak{so}_{11}$, according to the initial choice for $\mathfrak{g}$.

Thus, we are left with the problem of computing the index of parabolic subalgebras of $\mathfrak{sp}(V)$ and $\mathfrak{so}(V)$. At this point orthogonal and symplectic Lie algebras have to be treated separately. The symplectic case, which is relatively easy, is considered in the following theorem. As the orthogonal case requires more work, it will be considered in a separate section.

5.6 Theorem. *For parabolic subalgebras in $\mathfrak{g} = \mathfrak{sp}_{2n}$, we have*

$$\operatorname{ind}\mathfrak{p}^C_n(\underline{a} \mid \varnothing) = \sum_{i=1}^{m}\left[\frac{a_i}{2}\right] + (n - \textstyle\sum a_i)\ .$$

Proof. 1. Suppose $\underline{a} = (n)$; then $\mathfrak{p}^C_n(n \mid \varnothing) = \{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mid B = \widehat{B},\ D = -\widehat{A}\}$. Here we have $\mathfrak{p}^C_n(n \mid \varnothing) \simeq W \times_\rho \mathfrak{q}$, where $\mathfrak{q} = \mathfrak{p}^C_n(n|n) \simeq \mathfrak{gl}_n$ and $W = \{A = D = 0\}$ is the Abelian nilpotent radical of $\mathfrak{p}^C_n(n \mid \varnothing)$. Therefore the index can be computed by Theorem 1.3. It is easily seen that the $\mathfrak{q}$-module W is isomorphic to the second symmetric power of the tautological representation of $\mathfrak{gl}_n$. It follows that $\operatorname{ind}(\mathfrak{q}, W) = 0$ and $\mathfrak{q}_\xi \simeq \mathfrak{so}_n$. Thus $\operatorname{ind}\mathfrak{p}^C_n(n \mid \varnothing) = \operatorname{rk}\mathfrak{so}_n = [n/2]$.

2. In all other cases we have $a_1 < n$, and the following inductive procedure applies. Let us endow $\mathfrak p = \mathfrak p_n^C(\underline{a} \mid \varnothing)$ with a $\mathbb Z$-grading satisfying the hypothesis of Proposition 1.5. To this end, take the simple root α that has number a_1 in the standard numeration of the simple roots of $\mathfrak{sp}_{2n}$. For any $\gamma \in \Delta \cup \{0\}$, let $\mathrm{ht}_{\alpha}\gamma$ denote the coefficient of α in the expression of γ through the simple roots. Since $\mathrm{ht}_{\alpha}\Lambda = 2$ for the highest root Λ, we have $-2 \le \mathrm{ht}_{\alpha}\gamma \le 2$ for all γ. Letting $\mathfrak g(i)$ to be the sum of all $\mathfrak g_\gamma$ with $\mathrm{ht}_{\alpha}\gamma = i$, one obtains a $\mathbb Z$-grading of $\mathfrak g$ (if $\gamma = 0$, then $\mathfrak g_0$ is Cartan). Notice that $\sum_{i\ge 0}\mathfrak g(i)$ is the maximal parabolic subalgebra corresponding to $\alpha \in \Pi$. This grading induces a nonnegative grading on $\mathfrak p$: $\quad \mathfrak p = \mathfrak p(0) \oplus \mathfrak p(1) \oplus \mathfrak p(2)$. Here $\mathfrak p(i) = \mathfrak g(i)$ for $i = 1, 2$ and

$$\mathfrak p(0) = \mathfrak p_n^C(\underline{a} \mid a_1) \simeq \mathfrak{gl}_{a_1} \dotplus \mathfrak p_{n-a_1}^C(a_2, \dots, a_m \mid \varnothing)\ .$$

Another way to introduce this $\mathbb Z$-grading is as follows. Take the diagonal matrix $h = \mathrm{diag}(\underbrace{1, \dots, 1}_{a_1}, \underbrace{0, \dots, 0}_{2n-2a_1}, \underbrace{-1, \dots, -1}_{a_1}) \in \mathfrak t \subset \mathfrak{sp}_{2n}$. Then $\mathfrak g(i)$ is the eigenspace of $\mathrm{ad}\, h$ belonging to the eigenvalue i. This $\mathbb Z$-grading of $\mathfrak g$ is a particular instance of the construction described in Example 1.6. (For, it is easy to indicate a quadratic nilpotent element of $\mathfrak{sp}_{2n}$ producing this grading. The weighted Dynkin diagram of this element has label 1 over the node corresponding to α and zero labels over all other nodes.) Therefore $\mathrm{ind}\,\mathfrak p = \mathrm{ind}\,\mathfrak p(0)_\xi$, where ξ is a regular element in the $\mathfrak p(0)$-module $\mathfrak g(-2)$. It remains to observe that $\mathfrak p_{n-a_1}^C(a_2, \dots, a_m \mid \varnothing) \subset \mathfrak p(0)$ acts trivially on $\mathfrak g(-2)$ and that $\mathfrak g(-2)$ as $\mathfrak{gl}_{a_1}$-module is isomorhpic to the second symmetric power of the tautological representation of $\mathfrak{gl}_{a_1}$. Thus, $\mathfrak p(0)_\xi \simeq \mathfrak{so}_{a_1} \dotplus \mathfrak p_{n-a_1}^C(a_2, \dots, a_m \mid \varnothing)$ and hence $\mathrm{ind}\,\mathfrak p_n^C(\underline{a} \mid \varnothing) = [a_1/2] + \mathrm{ind}\,\mathfrak p_{n-a_1}^C(a_2, \dots, a_m \mid \varnothing)$. This completes the inductive step.

3. If $\sum a_i = n$, then the base of induction is the case considered in part 1. If $\sum a_i < n$, then the base of induction is the case $\mathfrak p_n^C(\varnothing \mid \varnothing) = \mathfrak{sp}_{2n}$. Since in both these cases the index is given by the required formula, we are done. □

The following is immediate:

5.7 Corollary.

1. $\mathrm{ind}\,\mathfrak p_n^C(\underline{a} \mid \varnothing) \in \{0, 1 \dots, n\}$*;*

2. $\mathrm{ind}\,\mathfrak p_n^C(\underline{a} \mid \varnothing) = 0 \iff \underline{a} = (1, \dots, 1)$ *(n times), i.e. for a Borel subalgebra;*

3. $\mathrm{ind}\,\mathfrak p_n^C(\underline{a} \mid \varnothing) = n \iff \underline{a} = \varnothing$*.*

Combining this Corollary with Eq. (5.4), we conclude that

(5.8) *The index of any seaweed in $\mathfrak{sp}_{2n}$ is at most n. Moreover, $\mathrm{ind}\,\mathfrak p_n^C(\underline{a} \mid \underline{b}) = n$ if and only if $\underline{a} = \underline{b}$, i.e., for Levi subalgebras.*

Here we have again the situation that the description of the seaweeds of minimal index is an open problem.

Example. Now we are able to complete, in the symplectic case, the computations of Example 5.5:

$$\mathrm{ind}\,\mathfrak p_{11}^C(3, 3 \mid 4, 5) = \mathrm{ind}\,\mathfrak p_5^C(\varnothing \mid 1, 1, 1) = 5 - 3 = 2\ .$$

6 Parabolic subalgebras of $\mathfrak{so}_N$

In this section, $\mathfrak g = \mathfrak{so}_N$ and $n = [N/2]$. The situation with parabolics in $\mathfrak{so}_N$ seems to be more complicated. The main difficulty of the 'orthogonal' situation is that the inductive step described in the proof of Theorem 5.6 goes through only if a_1 is even. The notation $\mathfrak p_n^\star(\underline a \mid \varnothing)$ used below means that $\star$'=D if $N = 2n$, and '$\star$'=B if $N = 2n + 1$.

6.1 Theorem. *Let $\mathfrak p_n^\star(\underline a \mid \varnothing)$ be a parabolic subalgebra of $\mathfrak g = \mathfrak{so}_N$. Suppose $a_1 + \cdots + a_k$ is even for some $k \le m$. Set $n' = a_1 + \cdots + a_k$, $\underline a' = (a_1, \ldots, a_k)$, and $\underline a'' = (a_{k+1}, \ldots, a_m)$. Then*

$$\operatorname{ind} \mathfrak p_n^\star(\underline a \mid \varnothing) = \operatorname{ind} \mathfrak p_{n'}^D(\underline a' \mid \varnothing) + \operatorname{ind} \mathfrak p_{n-n'}^\star(\underline a'' \mid \varnothing).$$

Proof. If $N = 2n$ and $n' = n$, the assertion is vacuous. Otherwise, consider the $\mathbb Z$-grading of $\mathfrak g$ determined by $h = \operatorname{diag}(\underbrace{1, \ldots, 1}_{n'}, \underbrace{0, \ldots, 0}_{N-2n'}, \underbrace{-1, \ldots, -1}_{n'})$. This yields a $\mathbb Z$-grading of height 2 on $\mathfrak g$ and the induced nonnegative $\mathbb Z$-grading on $\mathfrak p := \mathfrak p_n^\star(\underline a \mid \varnothing)$ with the following properties: $\mathfrak p = \mathfrak p(0) \oplus \mathfrak p(1) \oplus \mathfrak p(2)$, where $\mathfrak p(i) = \mathfrak g(i)$ for $i = 1, 2$ and

$$\mathfrak p(0) = \mathfrak p(\underline a' \mid n') \dotplus \mathfrak p_{n-n'}^\star(\underline a'' \mid \varnothing) \ .$$

The first summand in the RHS is a parabolic in $\mathfrak{gl}_{n'}$. For n' even, this grading arises from a quadratic nilpotent element. It then follows from Proposition 1.5 and Example 1.6 that $\operatorname{ind} \mathfrak p = \operatorname{ind} \mathfrak p^{ev} = \operatorname{ind}(\mathfrak p(0) \oplus \mathfrak g(2))$. Since $\mathfrak p_{n-n'}^\star(\underline a'' \mid \varnothing)$ acts trivially on $\mathfrak g(2)$ and $\mathfrak p(\underline a' \mid n') \oplus \mathfrak g(2) \simeq \mathfrak p_{n'}^D(\underline a' \mid \varnothing)$, we conclude that $\mathfrak p^{ev} \simeq \mathfrak p_{n'}^D(\underline a' \mid \varnothing) \dotplus \mathfrak p_{n-n'}^\star(\underline a'' \mid \varnothing)$.

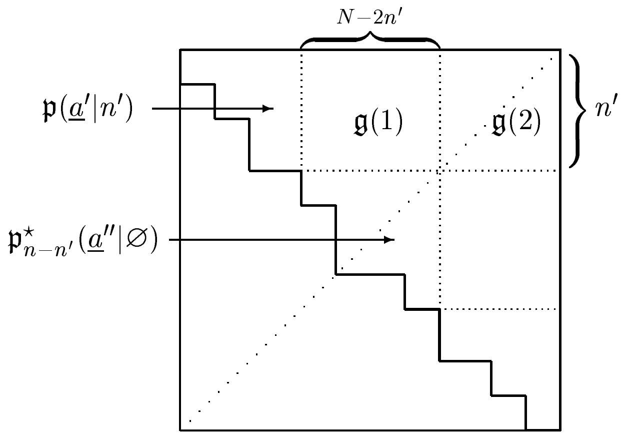


FIGURE 5. A parabolic subalgebra in $\mathfrak{so}_N$, with n' even

All this is depicted in Figure 5. Recall that the matrices in $\mathfrak{so}_N$ are skew-symmetric with respect to the antidiagonal. $\square$

This rather particular result is already sufficient for considering an interesting example.

Example: index of Borel subalgebras of $\mathfrak{so}_N$.

Let $\underline a = (\underbrace{1, \ldots, 1}_{n}) =: 1^n$. Then $\mathfrak p_n^\star(1^n \mid \varnothing)$ is a Borel subalgebra. Since $a_1 + a_2$ is even, Theorem 6.1 applies. We obtain

$$\operatorname{ind}\mathfrak{p}_n^\star(1^n \mid \varnothing) = \operatorname{ind}\mathfrak{p}_2^D(1,1 \mid \varnothing) + \mathfrak{p}_{n-2}^\star(1^{n-2} \mid \varnothing).$$

The first term in the RHS is the index of a Borel subalgebra in $\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, i.e., it is zero. That is,

$$(\text{index of a Borel in } \mathfrak{so}_N) = (\text{index of a Borel in } \mathfrak{so}_{N-4}).$$

Looking at the basic cases $N = 1, 2, 3$, we recover the well-known result (see [12]) that

$$\operatorname{ind}\mathfrak{p}_n^B(1^n \mid \varnothing) = 0 \quad \text{and} \quad \operatorname{ind}\mathfrak{p}_n^D(1^n \mid \varnothing) = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

6.2 Corollary. *Let $\mathfrak{p}_n^\star(\underline{a} \mid \varnothing)$ be a parabolic subalgebra of $\mathfrak{g} = \mathfrak{so}_N$.*
1. Suppose a_1 is even; then

$$\operatorname{ind}\mathfrak{p}_n^\star(\underline{a} \mid \varnothing) = a_1/2 + \operatorname{ind}\mathfrak{p}_{n-a_1}^\star(a_2, \ldots, a_m \mid \varnothing).$$

2. If all the coordinates a_i are even, then

$$\operatorname{ind}\mathfrak{p}_n^\star(\underline{a} \mid \varnothing) = \sum_{i=1}^m \frac{a_i}{2} + (n - \textstyle\sum a_i)\ .$$

Proof. 1. Let us exploit Theorem 6.1 with $\underline{a}' = (a_1)$ and $n' = a_1$. We have to only compute $\operatorname{ind}\mathfrak{p}_{a_1}^D(a_1 \mid \varnothing)$. Since we have $\mathfrak{p}_{a_1}^D(a_1 \mid \varnothing) \simeq \wedge^2 W \times_\rho \mathfrak{gl}(W)$ with $\dim W = a_1$, Theorem 1.3 applies. Here $\mathfrak{q} = \mathfrak{gl}(W)$, $V = \wedge^2 W$, $\operatorname{ind}(\mathfrak{q}, V) = 0$, and $\mathfrak{q}_\xi \simeq \mathfrak{sp}(W)$. Thus, the index is equal to $\operatorname{rk}\mathfrak{sp}(W) = a_1/2$.

2. If all the coordinates of $\underline{a}$ are even, we argue by induction, using part 1. The base of induction is the case $\mathfrak{p} = \mathfrak{p}_n^\star(\varnothing \mid \varnothing)$, where the index equals n. □

We conclude with some formulas for the index, where a_1 odd. They show that a general expression for the index in the orthogonal case cannot be so simple as in the symplectic case.

6.3 Proposition.
1. If n is odd and $n \geq 3$, then $\operatorname{ind}\mathfrak{p}_n^D(n \mid \varnothing) = [n/2] - 1$;
2. $\operatorname{ind}\mathfrak{p}_n^D(1 \mid \varnothing) = n - 2$ $(n \geq 2)$ and $\operatorname{ind}\mathfrak{p}_1^D(1 \mid \varnothing) = 1$,
$\operatorname{ind}\mathfrak{p}_n^B(1 \mid \varnothing) = n - 1$.

Proof. 1. Recall from section 5 a maximal isotropic subspace V^0. We have $\mathfrak{p}_n^D(n \mid \varnothing) \simeq \wedge^2 V^0 \times_\rho \mathfrak{gl}(V^0)$, where $\dim V^0 = n = 2s + 1$ and $s \geq 1$. Here $\operatorname{ind}(\mathfrak{gl}(V^0), \wedge^2 V^0) = 0$ and the stabiliser $\mathfrak{gl}(V^0)_\xi$ has a "Raïs structure"; more precisely, $\mathfrak{gl}(V^0)_\xi \simeq W \times_\rho (\mathfrak{sp}(W) \dotplus \Bbbk)$, where $\dim W = 2s$ and 1-dimensional reductive algebra $\Bbbk$ acts scalarly on W. Applying Theorem 1.3 again, we obtain $\operatorname{ind}\mathfrak{gl}(V^0)_\xi = \operatorname{ind}(\mathfrak{sp}(W) \dotplus \Bbbk)_\eta$, where $\eta \in W^*$ is a regular element. Explicit (easy) calculations shows that $(\mathfrak{sp}(W) \dotplus \Bbbk)_\eta$ is isomorphic to a parabolic subalgebra of $\mathfrak{sp}(W) = \mathfrak{sp}_{2s}$. Namely, $(\mathfrak{sp}(W) \dotplus \Bbbk)_\eta \simeq \mathfrak{p}_s^C(1 \mid \varnothing)$. Making use of Theorem 5.6, we finally obtain $\operatorname{ind}\mathfrak{p}_s^C(1 \mid \varnothing) = s - 1$.
2. If $N \geq 3$, then $\mathfrak{p}_n^\star(n \mid \varnothing) \simeq W \times_\rho (\mathfrak{so}(W) \dotplus \Bbbk)$, where $\dim W = N - 2$. The conclusion

follows by Theorem 1.3.
For $N=2$, we have $\mathfrak{p}_1^D(1 \mid \varnothing)$ is 1-dimensional. Therefore its index equals 1. □

Example. Now, we are able to complete, in the orthogonal case, the computation of Example 5.5:

$$\operatorname{ind}\mathfrak{p}_{11}^{\star}(3,3 \mid 4,5) = \operatorname{ind}\mathfrak{p}_5^{\star}(\varnothing \mid 1,1,1) = \operatorname{ind}\mathfrak{p}_3^{\star}(\varnothing \mid 1) = \begin{cases} 3-2=1, & \text{if} \quad \star = D \\ 3-1=2, & \text{if} \quad \star = B \end{cases} .$$

Remark. If $a_1 = 1$, then $\mathfrak{p}_n^{\star}(\underline{a} \mid \varnothing)$ has always a "Raïs structure", and Theorem 1.3 applies. Unfortunately, for $\underline{a} \neq (a_1)$, the subalgebra $\mathfrak{q}_\xi$ appears to be not a seaweed, in general, and the induction fails.
If $a_1 \geq 3$ is odd, a nonnegative $\mathbb{Z}$-grading of height 2 can still be constructed; but it does not satisfy hypotheses of Proposition 1.5.

Although we have no general result for parabolics in $\mathfrak{so}_N$, Theorem 6.1 and Corollary 6.2 reduce the index problem to considering parabolics $\mathfrak{p}_n^{\star}(\underline{a} \mid \varnothing)$, where a_1 is odd, or $\mathfrak{p}_n^D(\underline{a} \mid \varnothing)$, where a_1 and a_m are odd and $\sum a_i = n$; all other coordinates of $\underline{a}$ are even.

Furthermore, Equation 5.4 shows that if the index of any parabolic in $\mathfrak{so}_N$ is less than or equal to $[N/2]$ for all N, then the same holds for all seaweeds.

7 Concluding remarks

A (partial) success of our approach to computing the index can be explained, a posteriori, as follows. We maintain the notation of Section 2.

Let $\mathfrak{p} = \mathfrak{p}(\Psi \mid \varnothing)$ be a standard parabolic subalgebra of $\mathfrak{g}$. Take any $\alpha \in \Psi$ and consider the $\mathbb{Z}$-grading of $\mathfrak{g}$ determined by α; that is, $\mathfrak{g} = \oplus_i \mathfrak{g}(i)$, where $\mathfrak{g}(i) = \displaystyle\bigoplus_{\mathrm{ht}_\alpha \gamma = i} \mathfrak{g}_\gamma$ (cf. the proof of Theorem 5.6). Then $\mathfrak{p} = \oplus_{i\geq 0}\mathfrak{p}(i)$, where $\mathfrak{p}(0) \subset \mathfrak{g}(0)$ and $\mathfrak{p}(i) = \mathfrak{g}(i)$ for $i > 0$. The number $d_\alpha = \max\{i \mid \mathfrak{p}(i) \neq 0\}$ is determined as $\mathrm{ht}_\alpha\Lambda$, where Λ is the highest root. If $d_\alpha = 1$, then we may apply Theorem 1.3 to $\mathfrak{p}$. If $d_\alpha = 2$ and the $\mathbb{Z}$-grading in question is associated with a quadratic nilpotent element, then we may apply Proposition 1.5. All this appears to be rather helpful in the classical case, because here $d_\alpha \leq 2$ for *any* simple root α. In the exceptional Lie algebras, there always exists an $\alpha \in \Pi$ such that $d_\alpha \geq 3$. Therefore, in case $\Psi = \{\beta\}$ with $d_\beta \geq 3$, we are forced to consider this root. This leads to a grading with at least four summands, and the problem becomes more complicated. Yet, the algebra of type G_2 is small enough to compute all indices by hand. Here $\Pi = \{\alpha, \beta\}$, where $d_\alpha = 3$ and $d_\beta = 2$. There are three proper standard parabolic subalgebras and one non-reductive non-parabolic seaweed. Their indices are:

$$\operatorname{ind}\mathfrak{p}(\alpha \mid \varnothing) = \operatorname{ind}\mathfrak{p}(\beta \mid \varnothing) = 1 \quad \text{and} \quad \operatorname{ind}\mathfrak{p}(\alpha,\beta \mid \varnothing) = \operatorname{ind}\mathfrak{p}(\alpha \mid \beta) = 0 \ ,$$

which confirmes Conjecture 2.3.

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Independent University of Moscow
Bol'shoi Vlasievskii per. 11
Moscow 121002 Russia
dmitri@panyushev.mccme.ru