

# WEIGHT MULTIPLICITY FREE REPRESENTATIONS, $\mathfrak{g}$ -ENDOMORPHISM ALGEBRAS, AND DYNKIN POLYNOMIALS

DMITRI I. PANYUSHEV

## *Introduction*

Throughout this paper,  $G$  is a connected semisimple algebraic group defined over an algebraically closed field  $\mathbb{k}$  of characteristic zero, and  $\mathfrak{g}$  is its Lie algebra.

Recently, Kirillov introduced an interesting class of associative algebras connected with the adjoint representation of  $G$  [16]. In our paper, such algebras are called  $\mathfrak{g}$ -endomorphism algebras. Each  $\mathfrak{g}$ -endomorphism algebra is a module over the algebra of invariants  $\mathbb{k}[\mathfrak{g}]^G$ ; furthermore, it is a direct sum of modules of covariants. Hence it is a free graded finitely generated module over  $\mathbb{k}[\mathfrak{g}]^G$ . The aim of this paper is to show that commutative  $\mathfrak{g}$ -endomorphism algebras have intriguing connections with representation theory, combinatorics, commutative algebra, and equivariant cohomology.

Let  $\pi_\lambda : G \rightarrow \mathrm{GL}(\mathbb{V}_\lambda)$  be an irreducible representation, where  $\lambda$  stands for the highest weight of  $\mathbb{V}_\lambda$ . Following Kirillov, one can form an associative  $\mathbb{k}$ -algebra by taking the  $G$ -invariant elements in the  $G$ -module  $\mathrm{End} \mathbb{V}_\lambda \otimes \mathbb{k}[\mathfrak{g}]$ . That is, we set

$$\mathcal{C}_\lambda(\mathfrak{g}) = (\mathrm{End} \mathbb{V}_\lambda \otimes \mathbb{k}[\mathfrak{g}])^G.$$

This algebra will be referred to as the  $\mathfrak{g}$ -endomorphism algebra (of type  $\lambda$ ). We do not use Kirillov's term 'classical family algebra' for  $\mathcal{C}_\lambda(\mathfrak{g})$ ; nor we consider 'quantum family algebras' in our paper. It is proved in [16] that  $\mathcal{C}_\lambda(\mathfrak{g})$  is commutative if and only if all weight spaces in  $\mathbb{V}_\lambda$  are 1-dimensional. That paper also contains a description of  $\mathfrak{g}$ -endomorphism algebras for simplest representations of the classical Lie algebras. In our paper we do not attempt to dwell upon consideration of particular cases, but rather we try to investigate general properties of such algebras. As a tool for studying  $\mathfrak{g}$ -endomorphism algebras, we use  $\mathfrak{t}$ -endomorphism algebras. Let  $\mathfrak{t}$  be a Cartan subalgebra of  $\mathfrak{g}$  and let  $W$  be the corresponding Weyl group. Let  $\mathrm{End}_T(\mathbb{V}_\lambda)$  denote the set of  $T$ -equivariant endomorphisms of  $\mathbb{V}_\lambda$ , where  $T$  is the maximal torus with Lie algebra  $\mathfrak{t}$ . Then

$$\mathcal{C}_\lambda(\mathfrak{t}) = (\mathrm{End}_T(\mathbb{V}_\lambda) \otimes \mathbb{k}[\mathfrak{t}])^W$$

is called the  $\mathfrak{t}$ -endomorphism algebra of type  $\lambda$ . Both  $\mathcal{C}_\lambda(\mathfrak{t})$  and  $\mathcal{C}_\lambda(\mathfrak{g})$  are free graded  $\mathbb{k}[\mathfrak{g}]^G$ -modules of the same rank, and there exists an injective homomorphism of  $\mathbb{k}[\mathfrak{g}]^G$ -modules  $\hat{r}_\lambda : \mathcal{C}_\lambda(\mathfrak{g}) \rightarrow \mathcal{C}_\lambda(\mathfrak{t})$ . Moreover,  $\mathbb{k}[\mathfrak{g}]^G$  is a subalgebra in both  $\mathcal{C}_\lambda(\mathfrak{t})$  and  $\mathcal{C}_\lambda(\mathfrak{g})$ , and  $\hat{r}_\lambda$  is a monomorphism of  $\mathbb{k}[\mathfrak{g}]^G$ -algebras. We show that  $\hat{r}_\lambda$  becomes an isomorphism after inverting the discriminant  $D \in \mathbb{k}[\mathfrak{g}]^G$ .

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Most of our results concern the case in which  $\mathcal{C}_\lambda(\mathfrak{g})$  is commutative, that is,  $\mathbb{V}_\lambda$  is weight multiplicity free (**wmf**). A considerable amount of **wmf** representations are minuscule ones. We show that if  $\mathbb{V}_\lambda$  is minuscule, then  $\mathcal{C}_\lambda(\mathfrak{g}) \simeq \mathbb{k}[\mathfrak{t}]^{W_\lambda}$ . Here  $W_\lambda \subset W$  is the stabilizer of  $\lambda$ . The proof relies on a recent result of Broer concerning ‘small’  $G$ -modules [5]. If  $\mathbb{V}_\lambda$  is **wmf** but not minuscule, then both  $\mathcal{C}_\lambda(\mathfrak{t})$  and  $\mathcal{C}_\lambda(\mathfrak{g})$  have zero-divisors. We prove that  $\text{Spec } \mathcal{C}_\lambda(\mathfrak{t})$  is a disjoint union of affine spaces of dimension  $\text{rk } \mathfrak{g}$ , while  $\text{Spec } \mathcal{C}_\lambda(\mathfrak{g})$  is connected. However, both varieties have the same number of irreducible components, which is equal to the number of dominant weights of  $\mathbb{V}_\lambda$ . As a by-product, we obtain the assertion that  $\hat{r}_\lambda$  is an isomorphism if and only if  $\lambda$  is minuscule. Since  $\mathcal{C}_\lambda(\mathfrak{g})$  and  $\mathcal{C}_\lambda(\mathfrak{t})$  are graded  $\mathbb{k}$ -algebras, one may consider their Poincaré series. We explicitly compute these series for any  $\lambda$ .

The principal result of the paper is that any commutative algebra  $\mathcal{C}_\lambda(\mathfrak{g})$  is Gorenstein. The proof goes as follows. Any set  $f_1, \dots, f_l$  of algebraically independent homogeneous generators of  $\mathbb{k}[\mathfrak{g}]^G$  form a system of parameters for  $\mathcal{C}_\lambda(\mathfrak{g})$ . Therefore  $\mathcal{C}_\lambda(\mathfrak{g})$  is Gorenstein if and only if  $\mathcal{R}^{(\lambda)} := \mathcal{C}_\lambda(\mathfrak{g})/\mathcal{C}_\lambda(\mathfrak{g})f_1 + \dots + \mathcal{C}_\lambda(\mathfrak{g})f_l$  is. The finite-dimensional  $\mathbb{k}$ -algebra  $\mathcal{R}^{(\lambda)}$  is isomorphic with  $(\text{End } \mathbb{V}_\lambda)^A$ , where  $A \subset G$  is the connected centraliser of a regular nilpotent element. Using this fact, we prove that the socle of  $\mathcal{R}^{(\lambda)}$  is one-dimensional, that is,  $\mathcal{R}^{(\lambda)}$  is Gorenstein. It is also shown that the Poincaré polynomial of  $\mathcal{R}^{(\lambda)}$  is equal to the Dynkin polynomial for  $\mathbb{V}_\lambda$ . The Dynkin polynomial is defined for any  $\mathbb{V}_\lambda$ . It can be regarded as a  $q$ -analogue of  $\dim \mathbb{V}_\lambda$  that describes the distribution of weight spaces with respect to some level function. According to an old result of Dynkin [7], it is a symmetric unimodal polynomial with integral coefficients. Later on, Stanley observed that Dynkin polynomials have rich combinatorial applications and there is a multiplicative formula for them (see [20]). In our setting, the Dynkin polynomial of a **wmf**  $G$ -module  $\mathbb{V}_\lambda$  appears as the numerator of the Poincaré series of  $\mathcal{C}_\lambda(\mathfrak{g})$ . It is natural to suspect that any reasonable finite-dimensional Gorenstein  $\mathbb{k}$ -algebra is the cohomology algebra of a ‘good’ variety. Following this harmless idea, we construct for any **wmf** representation of a simple group  $G$  a certain variety  $X_\lambda \subset \mathbb{P}(\mathbb{V}_\lambda)$ . We conjecture that  $H^*(X_\lambda) \simeq \mathcal{R}^{(\lambda)}$  and  $H_{G_c}^*(X_\lambda) \simeq \mathcal{C}_\lambda(\mathfrak{g})$ , where  $G_c \subset G$  is a maximal compact subgroup. If  $\lambda$  is a minuscule dominant weight, then  $X_\lambda$  is nothing but  $G/P_\lambda$ , a generalised flag variety. In this case, the conjecture follows from the equality  $\mathcal{C}_\lambda(\mathfrak{g}) = \mathbb{k}[\mathfrak{t}]^{W_\lambda}$  (Theorem 2.6) and the well known description of  $H_{G_c}^*(G/P_\lambda)$ . We also verify (a part of) the conjecture for some non-minuscule weights.

The paper is organised as follows. In Section 1, we collect necessary information on modules of covariants. Section 2 is devoted to basic properties of endomorphism algebras. We describe the structure of these algebras in the commutative case, in particular, for the minuscule weights. Dynkin polynomials and their applications are discussed in Section 3. In Section 4, we give explicit formulas for the Poincaré series of endomorphism algebras. The Gorenstein property is considered in Section 5. Finally, in Section 6, we construct varieties  $X_\lambda \subset \mathbb{P}(\mathbb{V}_\lambda)$  and discuss connections between  $\mathfrak{g}$ -endomorphism algebras and (equivariant) cohomology of  $X_\lambda$ .

## 1. Generalities on modules of covariants

Fix a Borel subgroup  $B \subset G$  and a maximal torus  $T \subset B$ . We will always work with roots, simple roots, positive roots, and dominant weights that are determined

by this choice of the pair  $(B, T)$ . For instance, the roots of  $B$  are positive and a highest weight vector in some  $G$ -module is a  $B$ -eigenvector. More specifically, write  $\mathcal{P}$  for the  $T$ -weight lattice, and  $\mathcal{P}_+$  for the dominant weights in  $\mathcal{P}$ . Next,  $\Delta$  (respectively  $\Delta^+$ ) is the set of all (respectively positive) roots,  $\Pi$  is the set of simple roots, and  $\mathcal{Q}$  is the root lattice. For a  $G$ -module  $M$ , let  $M^\mu$  denote the  $\mu$ -weight space of  $M$  ( $\mu \in \mathcal{P}$ ). If  $\lambda \in \mathcal{P}_+$ , then  $\mathbb{V}_\lambda$  stands for the simple  $G$ -module with highest weight  $\lambda$ . Set  $m_\lambda^\mu = \dim \mathbb{V}_\lambda^\mu$ . The notation  $\mu \dashv \mathbb{V}_\lambda$  means that  $m_\lambda^\mu \neq 0$ . For instance, we have  $0 \dashv \mathbb{V}_\lambda$  if and only if  $\lambda \in \mathcal{Q}$ .

Given  $\lambda \in \mathcal{P}_+$ , the space  $(\mathbb{V}_\lambda \otimes \mathbb{k}[\mathfrak{g}])^G$  is called the *module of covariants* (of type  $\lambda$ ). We will write  $J_\lambda(\mathfrak{g})$  for it. Clearly,  $J_\lambda(\mathfrak{g})$  is a module over  $J_0(\mathfrak{g}) = \mathbb{k}[\mathfrak{g}]^G$ , and  $J_\lambda(\mathfrak{g}) \neq 0$  if and only if  $\lambda \in \mathcal{Q}$ . The elements of  $J_\lambda(\mathfrak{g})$  can be identified with the  $G$ -equivariant morphisms from  $\mathfrak{g}$  to  $\mathbb{V}_\lambda$ . More precisely, an element  $\sum v_i \otimes f_i \in J_\lambda(\mathfrak{g})$  defines the morphism that takes  $x \in \mathfrak{g}$  to  $\sum f_i(x)v_i \in \mathbb{V}_\lambda$ . This interpretation of  $J_\lambda(\mathfrak{g})$  will freely be used in the sequel. Since  $\mathbb{k}[\mathfrak{g}]^G$  is a graded algebra, each  $J_\lambda(\mathfrak{g})$  is a graded module too:

$$J_\lambda(\mathfrak{g}) = \bigoplus_{n \geq 0} J_\lambda(\mathfrak{g})_n,$$

the component of grade  $n$  being  $(\mathbb{V}_\lambda \otimes \mathbb{k}[\mathfrak{g}]_n)^G$ . The *Poincaré series* of  $J_\lambda(\mathfrak{g})$  is the formal power series

$$\mathcal{F}(J_\lambda(\mathfrak{g}); q) = \sum_{n \geq 0} \dim J_\lambda(\mathfrak{g})_n q^n \in \mathbb{Z}[[q]].$$

More generally,  $C$  being an arbitrary graded object, we write  $\mathcal{F}(C; q)$  for its Poincaré series.

The following fundamental result is due to Kostant [17, Theorem 11].

**THEOREM 1.1.**  *$J_\lambda(\mathfrak{g})$  is a free graded  $J_0(\mathfrak{g})$ -module of rank  $m_\lambda^0$ .*

Let  $d_1, \dots, d_l$  be the degrees of basic invariants in  $\mathbb{k}[\mathfrak{g}]^G$ , where  $l = \text{rk } \mathfrak{g}$ . It follows from the theorem that  $\mathcal{F}(J_\lambda(\mathfrak{g}); q)$  is a rational function of the form

$$\mathcal{F}(J_\lambda(\mathfrak{g}); q) = \frac{\sum_j q^{e_j(\lambda)}}{\prod_{i=1}^l (1 - q^{d_i})}.$$

The numbers  $\{e_j(\lambda)\}$  ( $1 \leq j \leq m_\lambda^0$ ), which are merely the degrees of a set of free homogeneous generators of  $J_\lambda(\mathfrak{g})$ , are called the *generalised exponents* for  $\mathbb{V}_\lambda$ . Another interpretation of generalised exponents is obtained as follows. Let  $f_1, \dots, f_l \in \mathbb{k}[\mathfrak{g}]^G$  be a set of basic invariants, with  $\deg f_i = d_i$ . It is a homogeneous system of parameters for  $J_\lambda(\mathfrak{g})$ . Therefore  $J_\lambda(\mathfrak{g}) = (f_1, \dots, f_l)J_\lambda(\mathfrak{g}) \oplus \mathcal{H}_\lambda$ , where  $\mathcal{H}_\lambda$  is a graded finite-dimensional  $\mathbb{k}$ -vector space such that  $\dim \mathcal{H}_\lambda = m_\lambda^0$ . Any homogeneous  $\mathbb{k}$ -basis for  $\mathcal{H}_\lambda$  is also a basis for  $J_\lambda(\mathfrak{g})$  as  $\mathbb{k}[\mathfrak{g}]^G$ -module. Let  $\mathcal{N} \subset \mathfrak{g}$  denote the set of nilpotent elements of  $\mathfrak{g}$  (the nilpotent cone). Since  $\mathbb{k}[\mathfrak{g}]^G/(f_1, \dots, f_l) \simeq \mathbb{k}[\mathcal{N}]$ , we see that

$$\mathcal{H}_\lambda \simeq (\mathbb{V}_\lambda \otimes \mathbb{k}[\mathcal{N}])^G \quad \text{and} \quad \mathcal{F}(\mathcal{H}_\lambda; q) = \sum_{i=1}^{m_\lambda^0} q^{e_j(\lambda)}.$$

The polynomial  $\mathcal{F}(\mathcal{H}_\lambda; q)$  has non-negative integral coefficients and  $\mathcal{F}(\mathcal{H}_\lambda; q)|_{q=1} = m_\lambda^0$ . It is a  $q$ -analogue of  $m_\lambda^0$ . A combinatorial formula for  $\mathcal{F}(\mathcal{H}_\lambda; q)$  was found by Hesselink [13] and Peterson, independently.

Now we describe another approach to computing  $\mathcal{F}(\mathcal{H}_\lambda; q)$ , which is due to Brylinski. Let  $e \in \mathcal{N}$  be a regular nilpotent element. Then  $G \cdot e$  is dense in  $\mathcal{N}$  and  $\mathbb{k}[\mathcal{N}] \simeq \mathbb{k}[G]^{G_e}$  [17]. Hence  $\mathcal{H}_\lambda \simeq (\mathbb{V}_\lambda)^{G_e}$ , and in particular,  $\dim(\mathbb{V}_\lambda)^{G_e} = m_\lambda^0$ . Thus the space  $(\mathbb{V}_\lambda)^{G_e}$  is equipped with a grading coming from the above isomorphism. A direct description of this grading can be obtained in terms of ‘jump polynomials’. Fix a principal  $\mathfrak{sl}_2$ -triple  $\{e, h, f\}$  such that  $G_h = T$  and  $e$  is a sum of root vectors corresponding to the simple roots. We have  $G_e \subset B$  and  $G_e \simeq Z(G) \times A$ , where  $Z(G)$  is the centre of  $G$  and  $A$  is a connected commutative unipotent group. As  $[h, e] = 2e$ , the space  $(\mathbb{V}_\lambda)^A$  is ad  $h$ -stable. It follows from the  $\mathfrak{sl}_2$ -theory that ad  $h$ -eigenvalues on  $(\mathbb{V}_\lambda)^A$  are nonnegative. Moreover, since  $\lambda \in \mathcal{Q}$ , we see that  $Z(G)$  acts trivially on  $\mathbb{V}_\lambda$ ,  $(\mathbb{V}_\lambda)^{G_e} = (\mathbb{V}_\lambda)^A$ , and these eigenvalues are even. Therefore it is convenient to consider  $\tilde{h} = \frac{1}{2}h$  and its eigenvalues. Set

$$(\mathbb{V}_\lambda)_i^A = \{x \in (\mathbb{V}_\lambda)^A \mid [\tilde{h}, x] = ix\}$$

and  $\mathcal{J}_{\mathbb{V}_\lambda}(q) = \sum_i \dim(\mathbb{V}_\lambda)_i^A q^i$ . This polynomial is called the *jump polynomial* for  $\mathbb{V}_\lambda$ .

**THEOREM 1.2** [4, Theorem 2.4; 6, Theorem 3.4]. *For any  $\lambda \in \mathcal{Q} \cap \mathcal{P}_+$ , we have  $\mathcal{J}_{\mathbb{V}_\lambda}(q) = \mathcal{F}(\mathcal{H}_\lambda; q)$ .*

Let  $\mathfrak{t}$  denote the Lie algebra of  $T$  and let  $W$  be the Weyl group of  $T$ . For any  $\lambda \in \mathcal{Q}$ , the space  $\mathbb{V}_\lambda^0$  is a  $W$ -module. Therefore, one can form the space  $J_\lambda(\mathfrak{t}) = (\mathbb{V}_\lambda^0 \otimes \mathbb{k}[\mathfrak{t}])^W$ . It is a module over  $J_0(\mathfrak{t}) = \mathbb{k}[\mathfrak{t}]^W$ . By Chevalley’s theorem, the restriction homomorphism  $\mathbb{k}[\mathfrak{g}] \rightarrow \mathbb{k}[\mathfrak{t}]$  induces an isomorphism of  $J_0(\mathfrak{g})$  and  $J_0(\mathfrak{t})$ , so that this common algebra will be denoted by  $J$ . Since  $W$  is a finite reflection group in  $\mathfrak{t}$ , we have  $J_\lambda(\mathfrak{t})$  is a free graded  $J$ -module of rank  $m_\lambda^0$ . Restricting a  $G$ -equivariant morphism  $\mathfrak{g} \rightarrow \mathbb{V}_\lambda$  to  $\mathfrak{t} \subset \mathfrak{g}$  yields a  $W$ -equivariant morphism  $\mathfrak{t} \rightarrow \mathbb{V}_\lambda^0$ . In other words, we obtain a map  $\text{res}_\lambda : J_\lambda(\mathfrak{g}) \rightarrow J_\lambda(\mathfrak{t})$ , which, in view of Chevalley’s theorem, is a homomorphism of  $J$ -modules. Since  $G \cdot \mathfrak{t}$  is dense in  $\mathfrak{g}$ , the homomorphism  $\text{res}_\lambda$  is injective. It is not, however, always surjective. The following elegant result is due to Broer [5, Theorem 1].

**THEOREM 1.3.** *Suppose that  $\lambda \in \mathcal{Q} \cap \mathcal{P}_+$ . Then the homomorphism  $\text{res}_\lambda$  is onto  $\iff m_\lambda^{2\mu} = 0$  for all  $\mu \in \Delta$ .*

In other words,  $\text{res}_\lambda$  is an isomorphism if and only if twice a root is not a weight for  $\mathbb{V}_\lambda$ . The  $G$ -modules satisfying the last condition are said to be *small*.

Looking at the elements of  $J_\lambda(\mathfrak{g})$  as  $G$ -equivariant morphisms

$$\mathfrak{g} \xrightarrow{\varphi} \mathbb{V}_\lambda,$$

one can consider the evaluation map

$$\epsilon_x : J_\lambda(\mathfrak{g}) \rightarrow (\mathbb{V}_\lambda)^{G_x}$$

for any  $x \in \mathfrak{g}$ . Namely, set  $\epsilon_x(\varphi) = \varphi(x)$ . The following is a particular case of a more general statement [18, Theorem 1], which applies to arbitrary  $G$ -actions.

**THEOREM 1.4.** *Suppose that  $\overline{G \cdot x}$  is normal. Then  $\epsilon_x$  is onto.*

2.  $\mathfrak{g}$ -endomorphism and  $\mathfrak{t}$ -endomorphism algebras

Following Kirillov, define the  $\mathfrak{g}$ -endomorphism algebra of type  $\lambda$  by the formula

$$\mathcal{C}_\lambda(\mathfrak{g}) = (\text{End } \mathbb{V}_\lambda \otimes \mathbb{k}[\mathfrak{g}])^G.$$

It is immediate that  $\mathcal{C}_\lambda(\mathfrak{g})$  is a  $J$ -module and an associative  $\mathbb{k}$ -algebra. Since  $\text{End } \mathbb{V}_\lambda \simeq \mathbb{V}_\lambda \otimes \mathbb{V}_\lambda^* = \bigoplus c_\nu \mathbb{V}_\nu$ , one sees that  $\mathcal{C}_\lambda(\mathfrak{g})$  is a direct sum of modules of covariants (possibly with multiplicities). Hence  $\mathcal{C}_\lambda(\mathfrak{g}) \simeq \bigoplus c_\nu J_\nu(\mathfrak{g})$ . It is important that all  $\nu$  belong to  $\mathcal{Q}$ . Therefore  $\dim(\text{End } \mathbb{V}_\lambda)^A = \dim(\text{End } \mathbb{V}_\lambda)^0 = \sum c_\nu m_\nu^0$ . Notice that  $\mathcal{C}_\lambda(\mathfrak{g})$  is not only a  $J$ -module, but it also contains  $J$  as subalgebra, since  $\text{id}_{\mathbb{V}_\lambda} \in \text{End } \mathbb{V}_\lambda$ . In particular,  $\mathcal{C}_\lambda(\mathfrak{g})$  is a  $J$ -algebra. Clearly,  $\mathcal{C}_\lambda(\mathfrak{g})$  is a graded  $\mathbb{k}$ -algebra, the component of grade  $n$  being  $(\text{End } \mathbb{V}_\lambda \otimes \mathbb{k}[\mathfrak{g}]_n)^G$ .

The zero-weight space in the  $G$ -module  $\text{End } \mathbb{V}_\lambda$  is the set of  $T$ -equivariant endomorphisms of  $\mathbb{V}_\lambda$ ; that is, we have  $\text{End}_T(\mathbb{V}_\lambda) = (\text{End } \mathbb{V}_\lambda)^0$ . Define the  $\mathfrak{t}$ -endomorphism algebra of type  $\lambda$  by the formula

$$\mathcal{C}_\lambda(\mathfrak{t}) = (\text{End}_T(\mathbb{V}_\lambda) \otimes \mathbb{k}[\mathfrak{t}])^W.$$

Using the above notation, one sees that  $\mathcal{C}_\lambda(\mathfrak{t}) = \bigoplus c_\nu J_\nu(\mathfrak{t})$ . It follows that, patching together the homomorphisms  $\text{res}_\nu$ , one obtains the monomorphism of  $J$ -algebras  $\hat{r}_\lambda : \mathcal{C}_\lambda(\mathfrak{g}) \rightarrow \mathcal{C}_\lambda(\mathfrak{t})$ . Thus we have two associative  $J$ -algebras such that both are free graded  $J$ -modules of the same rank,  $\dim \text{End}_T(\mathbb{V}_\lambda)$ .

LEMMA 2.1. *Let  $D \in J$  be the discriminant. Then  $\mathcal{C}_\lambda(\mathfrak{g})_D$  and  $\mathcal{C}_\lambda(\mathfrak{t})_D$  are isomorphic as  $J_D$ -modules.*

*Proof.* Both  $\mathcal{C}_\lambda(\mathfrak{t})$  and  $\mathcal{C}_\lambda(\mathfrak{g})$  are built of modules of covariants. Therefore the result stems from the analogous statement for the modules of covariants, which was proved in [5, Lemma 1(iii)]; see also [18, Proposition 4] for another proof in a more general context. □

We will primarily be interested in commutative  $\mathfrak{g}$ - and  $\mathfrak{t}$ -endomorphism algebras. The following proposition contains a criterion of commutativity. Part (i) has been proved in [16, Corollary 1]. However, we give a somewhat different proof for it, which has the potential to be applied in more general situations; cf. [18, Proposition 3].

PROPOSITION 2.2. (i)  $\mathcal{C}_\lambda(\mathfrak{g})$  is commutative  $\iff m_\lambda^\mu = 1$  for all  $\mu \dashv \mathbb{V}_\lambda$ .  
 (ii)  $\mathcal{C}_\lambda(\mathfrak{g})$  is commutative  $\iff \mathcal{C}_\lambda(\mathfrak{t})$  is commutative.

*Proof.* (i) ‘ $\Leftarrow$ ’: Because  $\text{End}_T(\mathbb{V}_\lambda) \simeq \bigoplus_\mu \text{End}(V_\lambda^\mu)$ , the algebra  $\mathcal{C}_\lambda(\mathfrak{t})$  is commutative whenever  $m_\lambda^\mu = 1$  for all  $\mu \dashv \mathbb{V}_\lambda$ . Since  $\hat{r}_\lambda$  is injective, we are done.

‘ $\Rightarrow$ ’: Interpreting elements of  $\mathcal{C}_\lambda(\mathfrak{g})$  as  $G$ -equivariant morphisms, consider the evaluation map

$$\epsilon_x : \mathcal{C}_\lambda(\mathfrak{g}) \rightarrow (\text{End } \mathbb{V}_\lambda)^{G_x} \quad (x \in \mathfrak{g}).$$

Taking  $x \in \mathfrak{t}$  with  $G_x = T$  and applying Theorem 1.4, we obtain a surjective  $\mathbb{k}$ -algebra homomorphism  $\mathcal{C}_\lambda(\mathfrak{g}) \rightarrow \text{End}_T(\mathbb{V}_\lambda) = \bigoplus_\mu \text{End}(V_\lambda^\mu)$ . Thus commutativity of  $\mathcal{C}_\lambda(\mathfrak{g})$  forces that of  $\text{End}(V_\lambda^\mu)$  for all  $\mu \vdash \mathbb{V}_\lambda$ , whence  $m_\lambda^\mu = 1$ .

(ii) This readily follows from part (i) and from the fact that  $\mathcal{C}_\lambda(\mathfrak{g})$  is a subalgebra of  $\mathcal{C}_\lambda(\mathfrak{t})$ . □

**DEFINITION 2.3.** A  $G$ -module  $\mathbb{V}_\lambda$  is called *weight multiplicity free* (wmf) if  $m_\lambda^\mu = 1$  for all  $\mu \vdash \mathbb{V}_\lambda$ .

Although we do not need this directly, it is worth mentioning that a classification of the wmf  $G$ -modules with  $G$  simple is contained in [14, 4.6].

**LEMMA 2.4.** *Suppose that  $\mathbb{V}_\lambda$  is wmf. Then  $\text{End } \mathbb{V}_\lambda$  is a multiplicity free  $G$ -module; that is, in the decomposition  $\text{End } \mathbb{V}_\lambda = \bigoplus c_\nu \mathbb{V}_\nu$ , all nonzero coefficients  $c_\nu$  are equal to 1.*

*Proof.* An explicit formula for the multiplicities in tensor products can be found in [19, Theorem 2.1]. In particular, that formula says that  $c_\nu \leq m_\lambda^{\nu-\lambda^*} (\leq 1)$ . Here  $\lambda^*$  is the highest weight of the dual  $G$ -module  $\mathbb{V}_\lambda^*$ . □

We regard roots and weights as elements of the  $\mathbb{Q}$ -vector space  $\mathcal{P} \otimes_{\mathbb{Z}} \mathbb{Q}$  sitting in  $\mathfrak{t}^*$ ; next,  $(,)$  is a fixed  $W$ -invariant bilinear form on  $\mathfrak{t}^*$  that is positive-definite on  $\mathcal{P} \otimes_{\mathbb{Z}} \mathbb{Q}$ . Then, as usual,  $\gamma^\vee = 2\gamma/(\gamma, \gamma)$  for all  $\gamma \in \Delta$ .

Of all wmf  $G$ -modules, the minuscule ones occupy a distinguished position. Recall that  $\mathbb{V}_\lambda$  is called *minuscule* if  $\lambda$  is minuscule. Some properties of minuscule dominant weights are presented in the following proposition, where either of two items can be taken as a definition of a minuscule dominant weight.

**PROPOSITION 2.5.** *For  $\lambda \in \mathcal{P}_+$ , the following conditions are equivalent.*

- (i) *If  $\mu \vdash \mathbb{V}_\lambda$ , then  $\mu \in W\lambda$ .*
- (ii)  *$(\lambda, \alpha^\vee) \leq 1$  for all  $\alpha \in \Delta^+$ .*

It follows from Proposition 2.5(i) that the corresponding simple  $G$ -module is wmf, while Proposition 2.5(ii) implies that if  $\mathfrak{g}$  is simple, then a minuscule dominant weight is fundamental.

**THEOREM 2.6.** *If  $\lambda$  is minuscule, then  $\mathcal{C}_\lambda(\mathfrak{g}) \simeq \mathcal{C}_\lambda(\mathfrak{t}) \simeq \mathbb{k}[\mathfrak{t}]^{W_\lambda}$ .*

*Proof.* (1) Let us look again at the decomposition

$$\mathbb{V}_\lambda \otimes \mathbb{V}_\lambda^* = \bigoplus_{\nu \in I} c_\nu \mathbb{V}_\nu = \mathbb{V}_{\lambda+\lambda^*} \oplus \left( \bigoplus_{\nu < \lambda+\lambda^*} c_\nu \mathbb{V}_\nu \right).$$

By Lemma 2.4, all  $c_\nu = 1$ .

(2) We claim that each  $\mathbb{V}_\nu$  ( $\nu \in I$ ) is small in the sense of Broer. Indeed, assume that  $m_\nu^{2\gamma} \neq 0$  for some dominant root  $\gamma$ . By a standard property of weights of  $\mathbb{V}_\lambda \otimes \mathbb{V}_\lambda^*$ , we have  $\gamma = \lambda + \lambda^* - \sum_{\alpha \in \Pi} n_\alpha \alpha$ , where  $n_\alpha \geq 0$ . Then  $4 = (2\gamma, \gamma^\vee) \leq (\lambda + \lambda^*, \gamma^\vee) \leq 2$ , a contradiction!

(3) It follows from the previous part and Theorem 1.3 that  $\hat{r}_\lambda = \bigoplus_{\nu \in \mathcal{I}} \text{res}_\nu : \mathcal{C}_\lambda(\mathfrak{g}) \longrightarrow \mathcal{C}_\lambda(\mathfrak{t})$  is an isomorphism of  $J$ -algebras. Since all weight spaces in  $\mathbb{V}_\lambda$  are one-dimensional and all weights are  $W$ -conjugate,

$$\text{End}_T(\mathbb{V}_\lambda) = \bigoplus_{\mu \in W\lambda} \text{End}(V_\lambda^\mu) \simeq \mathbb{k}[W/W_\lambda]$$

as  $W$ -modules. Thus

$$\mathcal{C}_\lambda(\mathfrak{t}) = (\mathbb{k}[W/W_\lambda] \otimes \mathbb{k}[\mathfrak{t}])^W \simeq \mathbb{k}[\mathfrak{t}]^{W_\lambda}.$$

The last equality is a manifestation of the transfer principle in invariant theory; see [10, Chapter 2]. □

REMARK 2.7. Another proof of this result, which does not appeal to Broer’s theorem, follows from the description of Poincaré series given in Theorem 4.1.

By a result of Steinberg,  $W_\lambda$  is a reflection group in  $\mathfrak{t}$ . Hence  $\mathcal{C}_\lambda(\mathfrak{g})$  is a polynomial algebra if  $\mathbb{V}_\lambda$  is minuscule. However, for the other wmf  $G$ -modules, the situation is not so good. Since  $\mathcal{C}_\lambda(\mathfrak{g})$  and  $\mathcal{C}_\lambda(\mathfrak{t})$  are commutative  $\mathbb{k}$ -algebras in the wmf-case, one can consider the varieties  $M_\lambda(\mathfrak{t}) := \text{Spec } \mathcal{C}_\lambda(\mathfrak{t})$  and  $M_\lambda(\mathfrak{g}) := \text{Spec } \mathcal{C}_\lambda(\mathfrak{g})$ . The chain of algebras  $J \subset \mathcal{C}_\lambda(\mathfrak{g}) \subset \mathcal{C}_\lambda(\mathfrak{t})$  yields the following commutative diagram.

$$\begin{array}{ccc} M_\lambda(\mathfrak{t}) & & \\ \tau \downarrow & \searrow & \\ M_\lambda(\mathfrak{g}) & \longrightarrow & \mathfrak{g} // G \simeq \mathfrak{t} / W \end{array}$$

All maps here are finite flat morphisms.

THEOREM 2.8. *Let  $\mathbb{V}_\lambda$  be a wmf  $G$ -module. Then the following hold.*

- (i)  $\mathcal{C}_\lambda(\mathfrak{t})$  and  $\mathcal{C}_\lambda(\mathfrak{g})$  are commutative Cohen–Macaulay  $\mathbb{k}$ -algebras.
- (ii)  $\mathcal{C}_\lambda(\mathfrak{t})$  and  $\mathcal{C}_\lambda(\mathfrak{g})$  are reduced.
- (iii)  $M_\lambda(\mathfrak{t})$  is a disjoint union of affine spaces of dimension  $l$ . The connected (= irreducible) components of  $M_\lambda(\mathfrak{t})$  are parametrised by the dominant weights in  $\mathbb{V}_\lambda$ .
- (iv) The morphism  $\tau$  yields a one-to-one correspondence between the irreducible components of  $M_\lambda(\mathfrak{t})$  and  $M_\lambda(\mathfrak{g})$ . The variety  $M_\lambda(\mathfrak{g})$  is connected.

*Proof.* (1) Cohen–Macaulayness follows, since both  $\mathcal{C}_\lambda(\mathfrak{t})$  and  $\mathcal{C}_\lambda(\mathfrak{g})$  are graded free modules over the polynomial ring  $J$ .

(2) Since  $\mathcal{C}_\lambda(\mathfrak{g})$  is a subalgebra of  $\mathcal{C}_\lambda(\mathfrak{t})$  via  $\hat{r}_\lambda$ , it is enough to prove that  $\mathcal{C}_\lambda(\mathfrak{t})$  has no nilpotent elements. Let  $\varphi \in \mathcal{C}_\lambda(\mathfrak{t})$ . Regard  $\varphi$  as a  $T$ -equivariant morphism from  $\mathfrak{t}$  to  $\bigoplus_\mu \text{End}(V_\lambda^\mu)$ . If  $\varphi \neq 0$ , then there is  $\mu \dashv \mathbb{V}_\lambda$  such that  $\varphi(t)(v_\mu) = c_\mu v_\mu$  for  $t \in \mathfrak{t}$ ,  $0 \neq v_\mu \in V_\lambda^\mu$ , and some  $c_\mu \in \mathbb{k} \setminus \{0\}$ . Hence  $\varphi^n(t)(v_\mu) = (c_\mu)^n v_\mu \neq 0$ .

(3) (Cf. the proof of Theorem 2.6.) Let  $\mu_1, \dots, \mu_k$  be all dominant weights in  $\mathbb{V}_\lambda$ . Then

$$\bigoplus_\mu \text{End } V_\lambda^\mu = \bigoplus_{i=1}^k \left( \bigoplus_{w \in W} \text{End}(\mathbb{V}_\lambda^{w\mu_i}) \right) = \bigoplus_{i=1}^k \mathbb{k}[W/W_{\mu_i}]$$

as  $W$ -modules. Hence

$$\mathcal{C}_\lambda(\mathfrak{t}) = \left( \left( \bigoplus_{i=1}^k \mathbb{k}[W/W_{\mu_i}] \right) \otimes \mathbb{k}[\mathfrak{t}] \right)^W = \bigoplus_{i=1}^k (\mathbb{k}[W/W_{\mu_i}] \otimes \mathbb{k}[\mathfrak{t}])^W = \bigoplus_{i=1}^k \mathbb{k}[\mathfrak{t}]^{W_{\mu_i}}.$$

(4) Because  $\mathcal{C}_\lambda(\mathfrak{t})$  is a free  $J$ -module, none of the elements of  $J$  becomes a zero-divisor in  $\mathcal{C}_\lambda(\mathfrak{t})$  (and hence in  $\mathcal{C}_\lambda(\mathfrak{g})$ ). Therefore, given any  $f \in J$ , the principal open subset  $M_\lambda(\mathfrak{t})_f$  (respectively  $M_\lambda(\mathfrak{g})_f$ ) has the same irreducible components as  $M_\lambda(\mathfrak{t})$  (respectively  $M_\lambda(\mathfrak{g})$ ). Let us apply this to  $f = D \in J$ , the discriminant. In the commutative case, one can consider  $\mathcal{C}_\lambda(\mathfrak{t})_D$  not only as  $J_D$ -module, which is a localisation of a  $J$ -module, but also as  $\mathbb{k}$ -algebra in its own right. By Lemma 2.1, we then conclude that the  $\mathbb{k}$ -algebras  $\mathcal{C}_\lambda(\mathfrak{t})_D$  and  $\mathcal{C}_\lambda(\mathfrak{g})_D$  are isomorphic.

That  $M_\lambda(\mathfrak{g})$  is connected follows from the fact that  $\mathcal{C}_\lambda(\mathfrak{g})$  is graded, and the component of grade 0 is just  $\mathbb{k}$ . (Recall that  $\mathcal{C}_\lambda(\mathfrak{g})_n = (\text{End } \mathbb{V}_\lambda \otimes \mathbb{k}[\mathfrak{g}]_n)^G$ .)  $\square$

We have shown that  $\mathcal{C}_\lambda(\mathfrak{t}) \simeq \bigoplus_{i=1}^k \mathbb{k}[\mathfrak{t}]^{W_{\mu_i}}$ . Since  $J \simeq \mathbb{k}[\mathfrak{t}]^W$  and  $\mathbb{k}[\mathfrak{t}]^W \subset \mathbb{k}[\mathfrak{t}]^{W_{\mu_i}}$ , it is not hard to realise that  $J$  embeds diagonally in the above sum.

COROLLARY 2.9.  $\mathcal{C}_\lambda(\mathfrak{g}) \simeq \mathcal{C}_\lambda(\mathfrak{t})$  if and only if  $\lambda$  is minuscule.

*Proof.* One implication is proved in Theorem 2.6. Conversely, if the algebras are isomorphic, then parts (iii) and (iv) show that  $k = 1$ , that is,  $\lambda$  is minuscule.  $\square$

In view of Theorem 1.3, the corollary states that if  $\lambda$  is not minuscule, then at least one simple  $\mathfrak{g}$ -module occurring in  $\mathbb{V}_\lambda \otimes \mathbb{V}_\lambda^*$  is not small.

The discrepancy between  $\mathcal{C}_\lambda(\mathfrak{t})$  and  $\mathcal{C}_\lambda(\mathfrak{g})$  can be seen on the level of Poincaré series. We give below precise formulae for  $\mathcal{F}(\mathcal{C}_\lambda(\mathfrak{t}); q)$  and  $\mathcal{F}(\mathcal{C}_\lambda(\mathfrak{g}); q)$ .

### 3. Dynkin polynomials

In 1950, Dynkin showed that to any simple  $G$ -module  $\mathbb{V}_\lambda$ , one can attach a symmetric unimodal polynomial [7]. This polynomial represents the distribution of the weight spaces in  $\mathbb{V}_\lambda$  with respect to some level function. In Dynkin’s paper, the properties of symmetry and unimodality were also expressed in terms of the weight system of  $\mathbb{V}_\lambda$  being ‘spindle-like’. To prove this, Dynkin introduced what is now called ‘a principal  $\mathfrak{sl}_2$ -triple in  $\mathfrak{g}$ ’. We say that the resulting polynomial is the *Dynkin polynomial* (of type  $\lambda$ ). In this section, we give some formulae for Dynkin polynomials and some applications of them.

The lowest weight vector in  $\mathbb{V}_\lambda$  is  $-\lambda^*$ . Recall that  $\Pi$  is the set of simple roots. Given  $\mu \vdash \mathbb{V}_\lambda$ , let us say that  $\mu$  is on the  $n$ th floor if  $\mu - (-\lambda^*) = \sum_{\alpha \in \Pi} n_\alpha \alpha$  with  $\sum n_\alpha = n$ . Thus the lowest weight is on the zero (ground) floor and the highest weight is on the highest floor.

DEFINITION 3.1 (cf. [7, p. 222]). Letting  $(\mathbb{V}_\lambda)_n = \sum_{\mu: \text{floor}(\mu)=n} \dim V_\lambda^\mu$  and  $a_n(\lambda) = \dim(\mathbb{V}_\lambda)_n$ , define the Dynkin polynomial  $\mathcal{D}_\lambda(q)$  to be  $\sum_n a_n(\lambda)q^n$ . (If we wish to indicate explicitly the dependence of this polynomial on  $\mathfrak{g}$ , we write  $\mathcal{D}_\lambda(\mathfrak{g})(q)$ .)

To obtain a more formal presentation, consider the element  $\rho^\vee = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha^\vee$ . Since  $(\alpha, \rho^\vee) = 1$  for any  $\alpha \in \Pi$ , we have  $(\gamma, \rho^\vee) = \text{ht} \gamma$ , the height of  $\gamma$ , for any  $\gamma \in \Delta$ . If  $\mu \in \mathcal{Q}$ , then  $(\mu, \rho^\vee) \in \mathbb{Z}$ . Since  $\lambda - (-\lambda^*) \in \mathcal{Q}$  and  $(\lambda, \rho^\vee) = (\lambda^*, \rho^\vee)$ , we see that  $\text{ht}(\mu) := (\mu, \rho^\vee) \in \frac{1}{2}\mathbb{Z}$  for an arbitrary  $\mu \in \mathcal{P}$ . In view of these properties



of  $\rho^\vee$ , the following is an obvious reformulation of the previous definition:

$$\mathcal{D}_\lambda(q) = q^{(\lambda, \rho^\vee)} \sum_{\mu \in \mathbb{V}_\lambda} m_\lambda^\mu q^{(\mu, \rho^\vee)} = \sum_{\mu \in \mathbb{V}_\lambda} m_\lambda^\mu q^{\text{ht}(\lambda + \mu)}. \tag{3.1}$$

Thus  $\rho^\vee$  defines a grading of  $\mathbb{V}_\lambda$ , and  $\mathcal{D}_\lambda(q)$  is nothing but the shifted Poincaré polynomial of this grading.

**THEOREM 3.2** [7, Theorem 4]. *For any  $\lambda \in \mathcal{P}_+$ , the polynomial  $\mathcal{D}_\lambda$  is symmetric (that is,  $a_i(\lambda) = a_{m-i}(\lambda)$ ,  $m = \deg \mathcal{D}_\lambda$ ) and unimodal (that is,  $a_0(\lambda) \leq a_1(\lambda) \leq \dots \leq a_{\lfloor m/2 \rfloor}(\lambda)$ ).*

*Idea of proof.* Having identified  $\mathfrak{t}$  and  $\mathfrak{t}^*$ , we see that  $2\rho^\vee$  becomes  $h$ , the semisimple element of our fixed principal  $\mathfrak{sl}_2$ -triple. Therefore  $(\mathbb{V}_\lambda)_n = \{v \in \mathbb{V}_\lambda \mid h \cdot v = 2(n - \text{ht}(\lambda))v\}$ . Hence, up to a shift of degree,  $\mathcal{D}_\lambda(q^2)$  gives the character of  $\mathbb{V}_\lambda$  as  $\mathfrak{sl}_2$ -module. From (3.1), one also sees that  $\deg \mathcal{D}_\lambda = 2(\rho^\vee, \lambda) = 2 \text{ht}(\lambda)$ .

The next proposition provides a multiplicative formula for the Dynkin polynomials. Apparently, this formula was first proved, in a slightly different form, in [20]. It was Stanley who realised that Dynkin’s result has numerous combinatorial applications. Of course, the proof exploits Weyl’s character formula.

**PROPOSITION 3.3.**

$$\mathcal{D}_\lambda(q) = \prod_{\alpha \in \Delta^+} \frac{1 - q^{(\rho + \lambda, \alpha^\vee)}}{1 - q^{(\rho, \alpha^\vee)}},$$

where  $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ .

**REMARK 3.4.** A similar formula also appears in [22, Lemma 2.5] (and probably in many other places) as a  $q$ -analogue of Weyl’s dimension formula (or a specialization of Weyl’s character formula). However, Stembridge did not make the degree shift  $q^{(\rho^\vee, \lambda)}$  and did not mention a connection of this  $q$ -analogue with Dynkin’s results.

**EXAMPLE 3.5.** Suppose that  $\mathfrak{g} = \mathfrak{sl}_{n+1}$ , and let  $\varphi_i$  be the  $i$ th fundamental weight of it. Then a direct calculation based on Proposition 3.3 gives

$$\begin{aligned} \mathcal{D}_{m\varphi_1}(\mathfrak{sl}_{n+1})(q) &= \frac{(1 - q^{m+1}) \dots (1 - q^{m+n})}{(1 - q) \dots (1 - q^n)} =: \begin{bmatrix} m+n \\ n \end{bmatrix} = \begin{bmatrix} m+n \\ m \end{bmatrix}, \\ \mathcal{D}_{\varphi_m}(\mathfrak{sl}_{n+1})(q) &= \begin{bmatrix} n+1 \\ m \end{bmatrix}. \end{aligned}$$

As a consequence of this example, one can deduce the following (well known) assertions.

**PROPOSITION 3.6.** *Suppose that  $\dim V = 2$ . Then there are two isomorphisms of  $\mathfrak{sl}(V)$ -modules:*

$$\begin{aligned} S^n(S^m(V)) &\simeq S^m(S^n(V)) \quad (\text{Hermite’s reciprocity}) \\ \wedge^m(S^{n+m-1}(V)) &= S^m(S^n(V)). \end{aligned}$$

*Proof.* The first formula follows from the equality  $\mathcal{D}_{m\varphi_1}(\mathfrak{sl}_{n+1}) = \mathcal{D}_{n\varphi_1}(\mathfrak{sl}_{m+1})$  and the fact that  $\mathbb{V}_{m\varphi_1}(\mathfrak{sl}_{n+1})|_{\mathfrak{sl}_2} = S^m(S^n(V))$ , where  $\mathfrak{sl}_2 = \mathfrak{sl}(V)$  is a principal  $\mathfrak{sl}_2$  in  $\mathfrak{sl}_{n+1}$ . Similarly, the second formula follows from the equality  $\mathcal{D}_{m\varphi_1}(\mathfrak{sl}_{n+1}) = \mathcal{D}_{\varphi_m}(\mathfrak{sl}_{m+n})$ .  $\square$

Now we show that Dynkin polynomials arise in connection with  $\mathfrak{g}$ -endomorphism algebras for the minuscule dominant weights.

For any  $\lambda \in \mathcal{P}_+$ , we have  $W_\lambda$  is a parabolic subgroup of  $W$ . Let  $d_i(W_\lambda)$ ,  $1 \leq i \leq l$ , be the degrees of basic invariants in  $\mathbb{k}[\mathfrak{t}]^{W_\lambda}$ . In particular,  $d_i = d_i(W_0) = d_i(W)$ . Note that if  $\lambda \neq 0$ , then some  $d_i(W_\lambda)$  are equal to 1. Let  $n : W \rightarrow \mathbb{N} \cup \{0\}$  be the length function with respect to the set of simple reflections. Set  $t_\lambda(q) = \sum_{w \in W_\lambda} q^{n(w)}$ . It is well known (see for example [15, 3.15]) that

$$t_\lambda(q) = \prod_{i=1}^l \frac{1 - q^{d_i(W_\lambda)}}{1 - q}. \tag{3.2}$$

These polynomials will appear frequently in the following exposition.

**PROPOSITION 3.7.** *Suppose that  $\lambda \in \mathcal{P}_+$  is minuscule. Then*

$$\mathcal{D}_\lambda(q) = \frac{t_0(q)}{t_\lambda(q)} = \prod_{i=1}^l \frac{1 - q^{d_i}}{1 - q^{d_i(W_\lambda)}}.$$

*Proof.* Set  $\Delta(\lambda) = \{\alpha \in \Delta \mid (\alpha, \lambda) = 0\}$ . Using Propositions 3.3 and 2.5(ii), we obtain

$$\mathcal{D}_\lambda(q) = \prod_{\alpha \in \Delta^+ \setminus \Delta(\lambda)^+} \frac{1 - q^{(\rho, \alpha^\vee)+1}}{1 - q^{(\rho, \alpha^\vee)}} = \prod_{\alpha \in \Delta^+} \frac{1 - q^{(\rho, \alpha^\vee)+1}}{1 - q^{(\rho, \alpha^\vee)}} \cdot \prod_{\alpha \in \Delta(\lambda)^+} \frac{1 - q^{(\rho, \alpha^\vee)}}{1 - q^{(\rho, \alpha^\vee)+1}}.$$

Note that  $(\rho, \alpha^\vee) = \text{ht}\alpha^\vee$ , the height of  $\alpha^\vee$  in the dual root system  $\Delta^\vee$ . By a result of Kostant (see [15, 3.20]),

$$t_0(q) = \prod_{\alpha \in \Delta^+} \frac{1 - q^{\text{ht}(\alpha)+1}}{1 - q^{\text{ht}(\alpha)}}.$$

Applying this formula to  $\Delta^\vee$  and  $\Delta(\lambda)^\vee$  and substituting in the previous expression for  $\mathcal{D}_\lambda$ , we complete the proof.  $\square$

Let us look at the Poincaré series of  $\mathcal{C}_\lambda(\mathfrak{g})$ , where  $\lambda$  is minuscule. Since  $\mathcal{C}_\lambda(\mathfrak{g}) = \mathbb{k}[\mathfrak{t}]^{W_\lambda}$  (Theorem 2.6) and  $\mathcal{F}(J; q) = \prod_{i=1}^l (1/(1 - q^{d_i}))$ , we deduce from Proposition 3.7 that

$$\mathcal{F}(\mathcal{C}_\lambda(\mathfrak{g}); q) = \mathcal{D}_\lambda(q) \cdot \mathcal{F}(J; q). \tag{3.3}$$

It is shown in Section 5 that this relation holds for all wmf  $G$ -modules.

#### 4. The Poincaré series of endomorphism algebras

In this section we find explicit formulas for the Poincaré series  $\mathcal{F}(\mathcal{C}_\lambda(\mathfrak{g}); q)$  and  $\mathcal{F}(\mathcal{C}_\lambda(\mathfrak{t}); q)$  with arbitrary  $\lambda \in \mathcal{P}_+$ .

Let  $m_\lambda^\mu(q)$  be Lusztig's  $q$ -analogue of weight multiplicity. It is a certain polynomial in  $q$ , with integral coefficients, such that  $m_\lambda^\mu(1) = m_\lambda^\mu$ . Defining these  $q$ -analogues

consists of two steps. First, one defines a  $q$ -analogue of Kostant’s partition function  $\mathcal{P}$  by

$$\prod_{\alpha \in \Delta^+} \frac{1}{1 - e^\alpha q} = \sum_{\nu \in \mathcal{Q}} \mathcal{P}_q(\nu) e^\nu.$$

Then we set  $m_\lambda^\mu(q) = \sum_{w \in W} \det(w) \mathcal{P}_q(w(\lambda + \rho) - \mu - \rho)$ . Proofs of the next properties of polynomials  $m_\lambda^\mu(q)$  can be found in [4, 11].

(1) If both  $\lambda, \mu$  are dominant, then the following hold.

(i) The coefficients of  $m_\lambda^\mu(q)$  are nonnegative.

(ii)  $m_\lambda^\mu(q) \neq 0 \Leftrightarrow \mu \preceq \mathbb{V}_\lambda$ .

(iii)  $\deg m_\lambda^\mu(q) = (\lambda - \mu, \rho^\vee) = \text{ht}(\lambda - \mu)$ .

(2) If  $\lambda \in \mathcal{Q}$ , then  $m_\lambda^0(q)$  is the numerator of the Poincaré series for the module of covariants  $J_\lambda(\mathfrak{g})$ ; that is, in the notation of Section 1, we have  $\mathcal{F}(\mathcal{H}_\lambda; q) = \mathcal{F}_{\mathbb{V}_\lambda}(q) = m_\lambda^0(q)$ .

**THEOREM 4.1.** *Let  $\lambda$  be an arbitrary dominant weight.*

(i)

$$\begin{aligned} \mathcal{F}(\mathcal{C}_\lambda(\mathfrak{g}); q) &= \sum_{\nu \in \mathcal{P}_+} (m_\lambda^\nu(q))^2 \frac{t_0(q)}{t_\nu(q)} \bigg/ \prod_{i=1}^l (1 - q^{d_i}) \\ &= \sum_{\nu \in \mathcal{P}_+} (m_\lambda^\nu(q))^2 \frac{t_0(q)}{t_\nu(q)} \cdot \mathcal{F}(J; q). \end{aligned}$$

(ii)

$$\mathcal{F}(\mathcal{C}_\lambda(\mathfrak{t}); q) = \sum_{\nu \preceq \mathbb{V}_\lambda, \nu \in \mathcal{P}_+} \frac{t_0(q)}{t_\nu(q)} \bigg/ \prod_{i=1}^l (1 - q^{d_i}) = \sum_{\nu \preceq \mathbb{V}_\lambda, \nu \in \mathcal{P}_+} \frac{t_0(q)}{t_\nu(q)} \cdot \mathcal{F}(J; q).$$

*Proof.* (i) Recall that  $J = \mathbb{k}[f_1, \dots, f_l]$  and  $\deg f_i = d_i$ . Since  $f_1, \dots, f_l$  is a homogeneous system of parameters for  $\mathcal{C}_\lambda(\mathfrak{g})$  as  $J$ -module, we have

$$\mathcal{C}_\lambda(\mathfrak{g})/\mathcal{C}_\lambda(\mathfrak{g})f_1 + \dots + \mathcal{C}_\lambda(\mathfrak{g})f_l \simeq (\text{End } \mathbb{V}_\lambda \otimes \mathbb{k}[\mathcal{M}])^G \simeq (\text{End } \mathbb{V}_\lambda)^A$$

(cf. Section 1). Hence  $\mathcal{F}(\mathcal{C}_\lambda(\mathfrak{g}); q) = \mathcal{F}((\text{End } \mathbb{V}_\lambda)^A; q) / \prod_{i=1}^l (1 - q^{d_i})$ . Using the decomposition  $\text{End } \mathbb{V}_\lambda = \bigoplus_{\nu \in I} c_\nu \mathbb{V}_\nu$  and Theorem 1.2, we see that the numerator is just the jump polynomial corresponding to the (reducible)  $G$ -module  $\text{End } \mathbb{V}_\lambda$ . That is,  $\mathcal{F}((\text{End } \mathbb{V}_\lambda)^A; q) = \sum_{\nu \in \mathcal{P}_+} c_\nu m_\nu^0(q)$ . It is remarkable, however, that, for the  $G$ -modules of the form  $\mathbb{V}_\lambda \otimes \mathbb{V}_\mu^*$ , there is a formula for the jump polynomial that does not appeal to the explicit decomposition of this tensor product. Namely, by [12, Corollary 2.4], we have

$$\mathcal{F}_{\mathbb{V}_\lambda \otimes \mathbb{V}_\mu^*}(q) = \sum_{\nu \in \mathcal{P}_+} m_\lambda^\nu(q) m_\mu^\nu(q) \frac{t_0(q)}{t_\nu(q)}. \tag{4.1}$$

Letting  $\mu = \lambda$ , one obtains the required formula.

(ii) Recall from Section 2 that  $\mathcal{C}_\lambda(\mathfrak{t}) \simeq \bigoplus_{\nu} \mathbb{k}[\mathfrak{t}]^{W_\nu}$ , where  $\nu$  ranges over all dominant weights of  $\mathbb{V}_\lambda$ . Making use of equation (3.2), we obtain

$$\mathcal{F}(\mathbb{k}[\mathfrak{t}]^{W_\nu}; q) = 1 \bigg/ \prod_{i=1}^l (1 - q^{d_i(W_\nu)}) = \frac{t_0(q)}{t_\nu(q)} \bigg/ \prod_{i=1}^l (1 - q^{d_i}),$$

which completes the proof. □

In the case when  $\lambda$  is minuscule, Theorem 4.1 shows that

$$\mathcal{F}(\mathcal{C}_\lambda(\mathfrak{g}); q) = \mathcal{F}(\mathcal{C}_\lambda(\mathfrak{t}); q) = \frac{t_0(q)}{t_\lambda(q)} \cdot \mathcal{F}(J; q),$$

since  $m_\lambda^\lambda(q) = 1$ . Thus we recover in this way equation (3.3) and the fact that  $\mathcal{C}_\lambda(\mathfrak{g}) = \mathcal{C}_\lambda(\mathfrak{t})$ . From the last equality, we deduce the following claim. (This yields another proof for part of Broer’s results.)

**COROLLARY 4.2.** *Suppose that  $\lambda$  is minuscule, and let  $\mathbb{V}_\nu$  be any irreducible constituent of  $\mathbb{V}_\lambda \otimes \mathbb{V}_\lambda^*$ . Then the restriction homomorphism  $\text{res}_\nu : J_\nu(\mathfrak{g}) \rightarrow J_\nu(\mathfrak{t})$  is onto.*

Let  $\mathcal{F}_\lambda(q)$  denote the right-hand side of equation (4.1) with  $\lambda = \mu$ , that is, the jump polynomial for the  $G$ -module  $\text{End } \mathbb{V}_\lambda$ .

- LEMMA 4.3.** (i)  $\deg \mathcal{F}_\lambda(q) = 2 \text{ht}(\lambda)$ .  
 (ii)  $\mathcal{F}_\lambda(1) = \sum_{\mu \in \mathbb{V}_\lambda} (m_\lambda^\mu)^2$ .  
 (iii) If  $\mathbb{V}_\lambda$  is wmf, then

$$\mathcal{F}_\lambda(q) = \sum_{\mu \in \mathbb{V}_\lambda, \mu \in \mathcal{P}_+} q^{2\text{ht}(\lambda - \mu)} \cdot \frac{t_0(q)}{t_\mu(q)}.$$

*Proof.* (1)  $\mathcal{F}_\lambda(q)$  is a sum (with multiplicities) of the jump polynomials for the irreducible constituents of  $\text{End } \mathbb{V}_\lambda$ . The jump polynomial for  $\mathbb{V}_{\lambda + \lambda^*}$  is of degree  $\text{ht}(\lambda + \lambda^*) = 2\text{ht}(\lambda)$ . For all other simple  $G$ -submodules in  $\text{End } \mathbb{V}_\lambda$ , the height of the highest weight is strictly less.

(2)  $\mathcal{F}_\lambda(1) = \dim(\text{End } \mathbb{V}_\lambda)^A = \dim(\text{End } \mathbb{V}_\lambda)^T = \sum_\mu \dim \text{End}(V_\lambda^\mu)$ .

(3) In view of Theorem 4.1(i), it suffices to prove that  $m_\lambda^\mu(q) = q^{\text{ht}(\lambda - \mu)}$ . Because  $m_\lambda^\lambda = 1$  and the coefficients of  $m_\lambda^\mu(q)$  are nonnegative integers,  $m_\lambda^\mu(q) = q^a$ . Since  $\deg m_\lambda^\mu(q) = \text{ht}(\lambda - \mu)$ , we are done.  $\square$

The last expression demonstrates an advantage of using equation (4.1) in the wmf case. We obtain a closed formula for the jump polynomial of a reducible representation that requires no bulky computations.

5. *The Gorenstein property for the commutative  $\mathfrak{g}$ -endomorphism algebras*

The goal of this section is to prove that if  $\mathbb{V}_\lambda$  is wmf, then  $\mathcal{C}_\lambda(\mathfrak{g})$  is a Gorenstein algebra. We also give another expression for the Poincaré series of  $\mathcal{C}_\lambda(\mathfrak{g})$ , which includes the Dynkin polynomial of type  $\lambda$ .

To begin with, we recall some facts about graded Gorenstein algebras. A nice exposition of relevant material is found in [21]. Let  $\mathcal{C} = \bigoplus_{n \geq 0} \mathcal{C}_n$  be a graded Cohen–Macaulay  $\mathbb{k}$ -algebra with  $\mathcal{C}_0 = \mathbb{k}$ . Suppose that the Krull dimension of  $\mathcal{C}$  is  $n$  and let  $f_1, \dots, f_n$  be a homogeneous system of parameters. Then  $\bar{\mathcal{C}} = \mathcal{C}/(f_1, \dots, f_n)$  is a graded Artinian Cohen–Macaulay  $\mathbb{k}$ -algebra. We have  $\bar{\mathcal{C}} = \bigoplus_{i=0}^d \bar{\mathcal{C}}_i$  for some  $d$ , and  $\mathfrak{m} = \bigoplus_{i=1}^d \bar{\mathcal{C}}_i$  is the unique maximal ideal in  $\bar{\mathcal{C}}$ . The annihilator of  $\mathfrak{m}$  in  $\bar{\mathcal{C}}$  is called the *socle* of  $\bar{\mathcal{C}}$ :

$$\text{soc}(\bar{\mathcal{C}}) = \{c \in \bar{\mathcal{C}} \mid c \cdot \mathfrak{m} = 0\}.$$

Then the following are true.

- (1)  $\mathcal{C}$  is Gorenstein if and only if  $\bar{\mathcal{C}}$  is.
- (2)  $\bar{\mathcal{C}}$  is Gorenstein if and only if  $\text{soc}(\bar{\mathcal{C}})$  is one-dimensional.

In the rest of the section,  $\mathbb{V}_\lambda$  is a wmf  $G$ -module, and hence  $\mathcal{C}_\lambda(\mathfrak{g})$  is commutative. Consider the  $\mathbb{k}$ -algebra  $(\text{End } \mathbb{V}_\lambda)^A =: \mathcal{R}^{(\lambda)}$ . Being a homomorphic image of  $\mathcal{C}_\lambda(\mathfrak{g})$ , it is commutative as well. We also have  $\dim \mathcal{R}^{(\lambda)} = \dim \mathbb{V}_\lambda$ .

PROPOSITION 5.1. *Let  $v_{-\lambda^*}$  be a lowest weight vector in  $\mathbb{V}_\lambda$ . Then*

$$\mathcal{R}^{(\lambda)}(v_{-\lambda^*}) = \mathbb{V}_\lambda.$$

*Proof.* Set  $N = \mathcal{R}^{(\lambda)}(-v_{\lambda^*})$ , and consider  $N^\perp$ , the annihilator of  $N$  in the dual space  $\mathbb{V}_{\lambda^*}$ . Let  $v_{\lambda^*}$  be a highest weight vector in  $\mathbb{V}_{\lambda^*}$ . By definition,  $N$  contains  $v_{-\lambda^*}$  and therefore  $v_{\lambda^*} \notin N^\perp$ . Since  $A \subset G \subset \text{End } \mathbb{V}_\lambda$  and  $A$  is commutative,  $N$  is an  $A$ -module and hence  $N^\perp$  is an  $A$ -module too. Assume that  $N^\perp \neq 0$ . Since  $A$  is unipotent,  $N^\perp$  must contain a non-trivial  $A$ -fixed vector. By a result of Graham (see [9, 1.6]),  $\dim(\mathbb{V}_{\lambda^*})^A = 1$  in the wmf case. As  $A \subset B$ , we conclude that  $(\mathbb{V}_{\lambda^*})^A = \mathbb{k}v_{\lambda^*}$ . Thus  $N^\perp \cap (\mathbb{V}_{\lambda^*})^A = 0$ . This contradiction shows that  $N^\perp = 0$ .  $\square$

PROPOSITION 5.2.  *$\mathcal{F}_\lambda(q) = \mathcal{D}_\lambda(q)$  if and only if  $\mathbb{V}_\lambda$  is wmf. In particular,  $\mathcal{F}_\lambda(q)$  is symmetric and unimodal in the wmf case.*

*Proof.* (1) ‘ $\Rightarrow$ ’: Suppose that  $\mathcal{F}_\lambda(q) = \mathcal{D}_\lambda(q)$ . Then

$$\sum_{\mu \in \mathbb{V}_\lambda} (m_\lambda^\mu)^2 = \mathcal{F}_\lambda(1) = \mathcal{D}_\lambda(1) = \dim \mathbb{V}_\lambda = \sum_{\mu \in \mathbb{V}_\lambda} m_\lambda^\mu,$$

whence  $\mathbb{V}_\lambda$  is wmf.

(2) ‘ $\Leftarrow$ ’: The polynomial  $\mathcal{F}_\lambda(q)$  is determined via the  $\frac{1}{2}h$ -grading in  $\mathcal{R}^{(\lambda)} = (\text{End } \mathbb{V}_\lambda)^A$ , whereas  $\mathcal{D}_\lambda(q)$  is determined via the shifted ‘ $\rho^\vee$ -grading’ in  $\mathbb{V}_\lambda$ . Identifying  $\mathfrak{t}$  and  $\mathfrak{t}^*$ , we obtain  $h = 2\rho^\vee$ . Obviously, the bijective linear map  $\mathcal{R}^{(\lambda)} \rightarrow \mathbb{V}_\lambda, x \mapsto x(v_{-\lambda^*})$ , respects both gradings and has degree zero. Hence

$$\mathcal{R}_i^{(\lambda)} \xrightarrow{\sim} (\mathbb{V}_\lambda)_i. \quad \square$$

REMARK 5.3. If  $\mathbb{V}_\lambda$  is not wmf, then  $\mathcal{F}_\lambda(q)$  can be neither symmetric nor unimodal. For instance, if  $\mathfrak{g} = \mathfrak{sp}_6$  and  $\lambda = \varphi_2$ , the second fundamental weight, then  $\mathcal{F}_{\varphi_2}(q) = 1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 3q^6 + q^7 + q^8$ .

By a theorem of Stanley [21, 12.7], if  $\mathcal{C}$  is a Cohen–Macaulay domain, then the symmetricity of the Poincaré polynomial of  $\mathcal{C}/(f_1, \dots, f_n)$  implies the Gorenstein property for  $\mathcal{C}$ . In our situation,  $\mathcal{C}_\lambda(\mathfrak{g})$  is not a domain unless  $\lambda$  is minuscule; see Theorem 2.8. Therefore we still cannot conclude that  $\mathcal{C}_\lambda(\mathfrak{g})$  is always Gorenstein.

PROPOSITION 5.4. *The socle of  $\mathcal{R}^{(\lambda)}$  is one-dimensional.*

*Proof.* Write  $\mathcal{R}$  for  $\mathcal{R}^{(\lambda)}$  in this proof. Recall from Section 3 that  $\mathbb{V}_\lambda$  is a disjoint union of ‘floors’, the weight space  $\mathbb{V}_\lambda^{-\lambda^*}$  being the zero floor and  $\mathbb{V}_\lambda^\lambda$  being the  $2 \text{ht}(\lambda)$ th floor. We know that  $\mathcal{R}$  is commutative and  $\mathcal{R} = \bigoplus_{i=0}^{2\text{ht}(\lambda)} \mathcal{R}_i$ . By Propositions 5.1 and 5.2, we have  $\mathcal{R}_i(v_{-\lambda^*}) = (\mathbb{V}_\lambda)_i$ , the  $i$ th floor in  $\mathbb{V}_\lambda$ , and  $\dim \mathcal{R}_{2\text{ht}(\lambda)} = 1$ . Clearly,  $\mathcal{R}_{2\text{ht}(\lambda)}$  takes  $\mathbb{V}_\lambda^{-\lambda^*}$  to  $\mathbb{V}_\lambda^\lambda$ , and  $\mathcal{R}_{2\text{ht}(\lambda)}(V_\lambda^\mu) = 0$

for  $\mu \neq -\lambda^*$ . We wish to show that  $\text{soc}(\mathcal{R}) = \mathcal{R}_{2\text{ht}(\lambda)}$ . Suppose  $x \in \mathcal{R}_i$ , then  $xv_{-\lambda^*} \in (\mathbb{V}_\lambda)_i$ . Let  $\langle, \rangle$  denote the natural pairing of  $\mathbb{V}_\lambda$  and  $\mathbb{V}_{\lambda^*}$ . Using the  $h$ -invariance, we see that  $\langle (\mathbb{V}_\lambda)_i, (\mathbb{V}_{\lambda^*})_j \rangle = 0$  unless  $i + j = 2\text{ht}(\lambda)$ . Hence there exists  $\xi \in (\mathbb{V}_{\lambda^*})_j$ , where  $j = 2\text{ht}(\lambda) - i$ , such that  $\langle \xi, xv_{-\lambda^*} \rangle \neq 0$ . Applying Proposition 5.1 to  $\mathbb{V}_{\lambda^*}$ , we obtain  $\xi = y(v_{-\lambda})$  for some  $y \in \mathcal{R}_j$ . Here  $v_{-\lambda}$  is a lowest weight vector in  $\mathbb{V}_{\lambda^*}$ , and  $\mathcal{R}$  is identified with  $(\text{End } \mathbb{V}_{\lambda^*})^A$ . Hence  $\langle v_{-\lambda}, yx(v_{-\lambda^*}) \rangle \neq 0$ . Thus, for any  $x \in \mathcal{R}_i$ , there exists  $y \in \mathcal{R}_{2\text{ht}(\lambda)-i}$  such that  $xy \neq 0$ , which is exactly what we need.  $\square$

Combining all previous results of this section, we obtain the following.

**THEOREM 5.5.** *Let  $\mathbb{V}_\lambda$  be a wmf  $G$ -module. Then the following hold.*

- (i) *The map  $(\text{End } \mathbb{V}_\lambda)^A \rightarrow \mathbb{V}_\lambda, x \mapsto x(v_{-\lambda^*})$ , is bijective.*
- (ii)  *$(\text{End } \mathbb{V}_\lambda)^A$  is an Artinian Gorenstein  $\mathbb{k}$ -algebra.*
- (iii)  *$\mathcal{D}_\lambda(q) = \mathcal{F}_\lambda(q)$ .*
- (iv)  *$\mathcal{C}_\lambda(\mathfrak{g})$  is a Gorenstein  $\mathbb{k}$ -algebra.*

It is worth mentioning the following property of wmf  $G$ -modules whose proof also uses Graham’s result; cf. Proposition 5.1.

**PROPOSITION 5.6.** *For any  $n \in \mathbb{N}$ , the vector  $e^n(v_{-\lambda^*})$  has nonzero projections to all weight spaces in  $(\mathbb{V}_\lambda)_n$ .*

*Proof.* Let  $\mathcal{E}$  be the  $\mathbb{k}$ -subalgebra of  $\mathcal{R}^{(\lambda)}$  generated by  $e$ . Then  $M := (\mathcal{E}(v_{-\lambda^*}))^\perp$  is an  $\mathcal{E}$ -stable subspace of  $\mathbb{V}_{\lambda^*}$ , and, obviously,  $v_{\lambda^*} \notin M$ . Assume that  $\mathcal{E}(v_{-\lambda^*})$  has the zero projection to some weight space, say  $\mathbb{V}_{\lambda^*}^{-\nu}$ . Then  $M$  contains the weight space  $\mathbb{V}_{\lambda^*}^\nu$ . Let  $v_\nu$  be a nonzero vector in  $\mathbb{V}_{\lambda^*}^\nu$  and let  $k$  be the maximal integer such that  $e^k(v_\nu) \neq 0$ . By [6, 2.6],  $e^k(v_\nu) \in (\mathbb{V}_{\lambda^*})^A$ , and by [9, 1.6],  $(\mathbb{V}_{\lambda^*})^A = \mathbb{k}v_{\lambda^*}$ . As  $e^k(v_\nu) \in M$ , we obtain a contradiction.  $\square$

All wmf representations of simple algebraic groups were found by Howe in [14, 4.6]. This information is included in the first two columns of Table 1; the last column gives the corresponding Dynkin polynomial. We write  $\varphi_i$  for the  $i$ th fundamental weight of  $G$ , according to the numbering of [23].

One can observe that each  $\lambda$  in Table 1 is a multiple of a fundamental weight. Using polynomials  $m_\lambda^\mu(q)$ , one can give a conceptual proof of this fact. This will appear elsewhere.

### 6. A connection with (equivariant) cohomology

Let  $\bar{\mathcal{C}}$  be an Artinian graded commutative associative  $\mathbb{k}$ -algebra,  $\bar{\mathcal{C}} = \bigoplus_{i=0}^d \bar{\mathcal{C}}_i$ . Suppose that  $\dim \bar{\mathcal{C}}_d = 1$ , and let  $\xi$  be a nonzero linear form on  $\bar{\mathcal{C}}$  that annihilates the space  $\bar{\mathcal{C}}_0 \oplus \dots \oplus \bar{\mathcal{C}}_{d-1}$ . Then  $\bar{\mathcal{C}}$  is called a *Poincaré duality algebra* if the bilinear form  $(x, y) \mapsto \xi(xy)$ ,  $x, y \in \bar{\mathcal{C}}$ , is nondegenerate. This name suggests that  $\bar{\mathcal{C}}$  looks very much as if it were the cohomology algebra of some ‘good’ manifold. It is easily seen that  $\bar{\mathcal{C}}$  is a Poincaré duality algebra if and only if  $\dim \text{soc}(\bar{\mathcal{C}}) = 1$ , that is,  $\bar{\mathcal{C}}$  is Gorenstein.

In the previous section, we proved that  $\mathcal{R}^{(\lambda)} = (\text{End } \mathbb{V}_\lambda)^A$  is an Artinian Gorenstein  $\mathbb{k}$ -algebra if  $\mathbb{V}_\lambda$  is wmf. (Here  $d = 2\text{ht}(\lambda)$ , which is not necessarily

TABLE 1. *Dynkin polynomials for the weight multiplicity free representations.*

Type	$\lambda$	Minuscule	$\mathcal{D}_\lambda(q)$
$A_n$	$\varphi_i$	Yes	See Example 3.5
	$m\varphi_1, m\varphi_n$ ( $m \geq 2$ )	No	See Example 3.5
$B_n$	$\varphi_n$	Yes	$(1+q)(1+q^2)\dots(1+q^n)$
	$\varphi_1$	No	$1+q+\dots+q^{2n}$
$C_n$	$\varphi_1$	Yes	$1+q+\dots+q^{2n-1}$
	$\varphi_3$ ( $n=3$ )	No	$1+q+q^2+2(q^3+\dots+q^6)+q^7+q^8+q^9$
$D_n$	$\varphi_1$	Yes	$(1+q^{n-1})(1+q+\dots+q^{n-1})$
	$\varphi_{n-1}, \varphi_n$	Yes	$(1+q)(1+q^2)\dots(1+q^{n-1})$
$E_6$	$\varphi_1$	Yes	$(1+q^4+q^8)(1+q+\dots+q^8)$
$E_7$	$\varphi_1$	Yes	$(1+q^5)(1+q^9)(1+q+\dots+q^{13})$
$G_2$	$\varphi_1$	No	$1+q+\dots+q^6$

even.) In this situation, one has more evidence in favour of the assertion that  $\mathcal{R}^{(\lambda)}$  can be a cohomology algebra. Recall that  $e$  is a sum of root vectors corresponding to the simple roots. Hence  $e(v_{-\lambda^*}) \in (\mathbb{V}_\lambda)_1$ , that is,  $e \in \mathcal{R}_1^{(\lambda)}$ .

PROPOSITION 6.1. *The multiplication operator  $e : \mathcal{R}_i^{(\lambda)} \rightarrow \mathcal{R}_{i+1}^{(\lambda)}$  is injective for  $i \leq [(2\text{ht}(\lambda) - 1)/2]$  and surjective for  $i \geq [\text{ht}(\lambda)]$ .*

*Proof.* The proof follows from the  $\mathfrak{sl}_2$ -theory and the equality  $\mathcal{D}_\lambda(q) = \mathcal{F}_\lambda(q)$ . □

Hence, if  $\mathcal{R}^{(\lambda)} = H^*(X)$  for some algebraic variety  $X$ , then  $e \in \mathcal{R}_1^{(\lambda)}$  can be regarded as the class of a hyperplane section, and the hard Lefschetz theorem holds for  $X$ . If  $G$  is simple, then  $\lambda$  is necessarily fundamental. Therefore the subspace  $(\mathbb{V}_\lambda)_1 \subset \mathbb{V}_\lambda$  is one-dimensional. Indeed, if  $\alpha \in \Pi$  is the unique root such that  $(\alpha, \lambda^*) \neq 0$ , then  $(\mathbb{V}_\lambda)_1 = \mathbb{V}_\lambda^{\alpha-\lambda^*}$ . In terms of  $\mathcal{R}^{(\lambda)}$ , this means that  $\mathcal{R}_1^{(\lambda)} = \mathbb{k}e$ . That is, such  $X$  should satisfy the condition  $b_2(X) = 1$ .

In the rest of the paper, we assume that  $\mathbb{k} = \mathbb{C}$ , and consider cohomology with complex coefficients. Now we give a hypothetical description of such  $X$  in the case when  $G$  is simple. More precisely, we conjecture that there exists  $X_\lambda \subset \mathbb{P}(\mathbb{V}_\lambda)$  such that odd cohomology of  $X_\lambda$  vanishes and  $\mathcal{R}^{(\lambda)} \simeq H^*(X_\lambda)$ ; moreover, the  $\mathfrak{g}$ -endomorphism algebra  $\mathcal{C}_\lambda(\mathfrak{g})$  gives the equivariant cohomology of  $X_\lambda$ . We refer to [3] for a nice introduction to equivariant cohomology. As usual,  $\mathbb{V}_\lambda$  is a wmf  $G$ -module. The variety we are seeking should satisfy the constraints  $\chi(X_\lambda) = \dim \mathbb{V}_\lambda$

and  $\dim X_\lambda = 2 \operatorname{ht}(\lambda)$ . Notice that the set of  $T$ -fixed points in  $\mathbb{P}(\mathbb{V}_\lambda)$  is finite:

$$\mathbb{P}(\mathbb{V}_\lambda)^T = \bigcup_{\nu \vdash \mathbb{V}_\lambda} \langle v^\nu \rangle,$$

where  $\langle v^\nu \rangle$  is the image of  $v^\nu \in \mathbb{V}_\lambda^\nu$  in the projective space. Since  $\chi(Y) = \chi(Y^T)$  for any algebraic variety  $Y$  acted upon by a torus  $T$  [1], we see that our variety  $X_\lambda$  must contain all  $\langle v^\nu \rangle$ ,  $\nu \vdash \mathbb{V}_\lambda$ . This provides some explanation for the following description. Let  $\mu \vdash \mathbb{V}_\lambda$  be the unique dominant minuscule weight. For instance,  $\mu = 0$  if and only if  $\lambda \in \mathcal{Q}$ .

(1) If  $(G, \lambda) \neq (A_n, m\varphi_1)$  or  $(A_n, m\varphi_n)$  with  $m > n + 1$ , then we define  $X_\lambda$  to be the closure of the  $G$ -orbit of the line  $\langle v_\mu \rangle$  in the projective space  $\mathbb{P}(\mathbb{V}_\lambda)$ .

(2) Alternatively, for  $G = A_n$  and  $\lambda = m\varphi_1$  with any  $m \geq 1$ , let  $X_\lambda$  be the projectivisation of the variety of decomposable forms of degree  $m$  in  $\mathbb{V}_{m\varphi_1} = S^m(\mathbb{V}_{\varphi_1})$ , where a form of degree  $m$  is said to be *decomposable* if it is a product of  $m$  linear forms. (A similar definition applies to  $\lambda = m\varphi_n$ .)

As is easily seen, the two constructions coincide if they both apply, that is, for  $(A_n, m\varphi_1$  or  $m\varphi_n)$  with  $m \leq n + 1$ . However, I do not see how to give a uniform description of  $X_\lambda$  in all cases. Direct calculations show that  $\dim X_\lambda = 2 \operatorname{ht}(\lambda)$ ; for example, for  $(A_n, m\varphi_1)$ , we have  $\dim X_\lambda = mn$ .

CONJECTURE 6.2. *Let  $\mathbb{V}_\lambda$  be a wmf  $G$ -module. Then the following hold.*

(1) *The variety  $X_\lambda \subset \mathbb{P}(\mathbb{V}_\lambda)$  is rationally smooth, odd cohomology of  $X_\lambda$  vanishes, and  $H^*(X_\lambda) \simeq \mathcal{R}^{(\lambda)}$ . In particular, the Poincaré polynomial of  $H^*(X_\lambda)$  is equal to  $\mathcal{D}_\lambda(q^2)$ .*

(2) *Let  $G_c \subset G$  be a maximal compact subgroup of  $G$ . Then  $H_{G_c}^*(X_\lambda) \simeq \mathcal{C}_\lambda(\mathfrak{g})$ .*

Let  $J_+ \subset J$  be the augmentation ideal. Since  $\mathcal{C}_\lambda(\mathfrak{g})/J_+\mathcal{C}_\lambda(\mathfrak{g}) \simeq \mathcal{R}^{(\lambda)}$ ,  $J \simeq H_{G_c}^*(\{pt\})$ , and  $H_{G_c}^*(X_\lambda)/(J_+) \simeq H^*(X_\lambda)$  (see [3, Proposition 2]), the first part of the conjecture follows from the second one. Actually, the conjecture is true for most of items in Table 1, and in particular for all minuscule weights. We list below all the known results supporting the conjecture.

(1) If  $\lambda$  is minuscule, then  $\lambda = \mu$ , and therefore  $X_\lambda = G/P_\lambda$ , a generalised flag variety. Here the conjecture follows from Theorem 2.6 and the well known description of  $H_{G_c}^*(G/P_\lambda)$ . Indeed, if  $P_\lambda$  is an arbitrary parabolic subgroup (that is,  $\lambda$  is not necessarily minuscule), then  $H_{G_c}^*(G/P_\lambda) \simeq \mathbb{k}[t]^{W_\lambda}$ .

(2) For the simplest representations of  $B_n$  and  $G_2$ , we have  $\mu = 0$  and  $X_\lambda = \mathbb{P}(\mathbb{V}_\lambda)$ . On the other hand, the formulae for  $\mathcal{D}_\lambda(q)$  in Table 1 shows that  $(\operatorname{End} \mathbb{V}_\lambda)^A$  is generated by  $e$  as  $\mathbb{k}$ -algebra, and the equality  $H^*(X_\lambda) = \mathcal{R}^{(\lambda)}$  follows. This also shows that  $\mathcal{C}_\lambda(\mathfrak{g})$  and  $H_{G_c}^*(X_\lambda)$  have the same Poincaré series.

(3) We have essentially four cases with nonminuscule  $\lambda$ :  $(B_n, \varphi_1)$ ,  $(C_3, \varphi_3)$ ,  $(G_2, \varphi_1)$ , and  $(A_n, m\varphi_1)$ ,  $m \geq 2$ . For the first three cases and for the last one with  $m = 2$ ,  $X_\lambda$  is a compact multiplicity-free  $G_c$ -space; in other words,  $X_\lambda$  is a spherical  $G$ -variety. Therefore, making use of [3, Theorem 9], one obtains the description of  $H_{G_c}^*(X_\lambda)$ . In these ‘spherical’ cases, the number of dominant weights in  $\mathbb{V}_\lambda$  equals 2. Therefore the structure of  $\mathcal{C}_\lambda(\mathfrak{g})$  is not too complicated, and this can be used for proving that  $\mathcal{C}_\lambda(\mathfrak{g}) = H_{G_c}^*(X_\lambda)$ .

(4) The projectivisation of the variety of decomposable forms of degree  $m$  in  $n + 1$  variables is isomorphic with  $(\mathbb{P}^n)^m / \Sigma_m$ , where  $\Sigma_m$  is the symmetric group



permuting factors in  $\mathbb{P}^n \times \dots \times \mathbb{P}^n$  ( $m$  times); see [2, Theorem 1.3; 8, 4.2]. Notice that

$$H^*((\mathbb{P}^n)^m/\Sigma_m) \simeq H^*((\mathbb{P}^n)^m)^{\Sigma_m} \simeq (\mathbb{K}[x_1, \dots, x_m]/(x_1^{n+1}, \dots, x_m^{n+1}))^{\Sigma_m},$$

the algebra of truncated symmetric polynomials. It is easily seen that its dimension is equal to

$$\binom{m+n}{m} = \dim \mathbb{V}_{m\varphi_1}.$$

A somewhat more bulky but still elementary calculation shows that the Poincaré polynomial of  $H^*((\mathbb{P}^n)^m/\Sigma_m)$  is equal to the Dynkin polynomial for  $(A_n, m\varphi_1)$ . Our proof is purely combinatorial. We establish a natural one-to-one correspondence between suitably chosen bases of  $H^*((\mathbb{P}^n)^m/\Sigma_m)$  and  $\mathbb{V}_{m\varphi_1}$  such that  $H^i((\mathbb{P}^n)^m/\Sigma_m)$  corresponds to  $(\mathbb{V}_{m\varphi_1})_i$ . It is not, however, clear how to compare the multiplicative structure of  $H^*((\mathbb{P}^n)^m/\Sigma_m)$  and  $(\text{End } \mathbb{V}_{m\varphi_1})^A$ .

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References

1. A. BIALYNICKI-BIRULA, ‘On fixed points schemes of actions of multiplicative and additive groups’, *Topology* 12 (1973) 99–103.
2. M. BRION, ‘Stable properties of plethysm: on two conjectures of Foulkes’, *Manuscripta Math.* 80 (1993) 347–371.
3. M. BRION, ‘Equivariant cohomology and equivariant intersection theory’, *Representation theories and algebraic geometry* (ed. A. Broer, Kluwer, 1998) 1–37.
4. A. BROER, ‘Line bundles on the cotangent bundle of the flag variety’, *Invent. Math.* 113 (1993) 1–20.
5. A. BROER, ‘The sum of generalized exponents and Chevalley’s restriction theorem’, *Indag. Math.* 6 (1995) 385–396.
6. R. K. BRYLINSKI, ‘Limits of weight spaces, Lusztig’s  $q$ -analogs, and fiberings of adjoint orbits’, *J. Amer. Math. Soc.* 2 (1989) 517–533.
7. E. B. DYNKIN, ‘Some properties of the weight system of a linear representation of a semisimple Lie group’, *Doklady Akad. Nauk SSSR* 71 (1950) 221–224 (Russian).
8. I. M. GELFAND, M. KAPRANOV and A. ZELEVINSKY, *Discriminants, resultants, and multidimensional determinants* (Birkhäuser, 1994).
9. W. GRAHAM, ‘Functions on the universal cover of the principal nilpotent orbit’, *Invent. Math.* 108 (1992) 15–27.
10. F. GROSSHANS, *Algebraic homogeneous spaces and invariant theory*, Lecture Notes in Mathematics 1673 (Springer, 1997).
11. R. K. GUPTA, ‘Characters and  $q$ -analog of weight multiplicity’, *Bull. London Math. Soc.* (2) 36 (1987) 68–76.
12. R. K. GUPTA, ‘Generalized exponents via Hall–Littlewood symmetric functions’, *Bull. Amer. Math. Soc.* 16 (1987) 287–291.
13. W. HESSELINK, ‘Characters of the nullcone’, *Math. Ann.* 252 (1980) 179–182.
14. R. HOWE, ‘Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond’, *The Schur lectures* (1992), Israel Mathematical Conference Proceedings 8 (Bar-Ilan University, Ramat Gan, 1995) 1–182.
15. J. E. HUMPHREYS, *Reflection groups and Coxeter groups* (Cambridge University Press, 1992).
16. A. A. KIRILLOV, ‘Introduction to family algebras’, *Moscow Math. J.* 1 (2001) 49–64.
17. B. KOSTANT, ‘Lie group representations in polynomial rings’, *Amer. J. Math.* 85 (1963) 327–404.
18. D. PANYUSHEV, ‘On covariants of reductive algebraic groups’, *Indag. Math.* 13 (2002) 125–129.
19. K. R. PARTHASARATHY, R. RANGA RAO and V. S. VARADARAJAN, ‘Representations of complex semisimple Lie groups and Lie algebras’, *Ann. of Math.* 85 (1967) 383–429.
20. R. P. STANLEY, ‘Unimodal sequences arising from Lie algebras’, *Combinatorics, representation theory and statistical methods in groups* (ed. T. V. Narayana et al., Marcel Dekker, 1980) 127–136.

21. R. P. STANLEY, *Combinatorics and commutative algebra*, 2nd edn (Birkhäuser, 1996).
22. J. STEMBRIDGE, 'On minuscule representations, plane partitions and involutions in complex Lie groups', *Duke Math. J.* 73 (1994) 469–490.
23. E. B. VINBERG and A. L. ONISHCHIK (eds.), *Seminar on Lie groups and algebraic groups* (Nauka, Moscow, 1988) (Russian), *Lie groups and algebraic groups* (ed. A. L. Onishchik and E. B. Vinberg, Springer, 1990) (English).

*Dmitri I. Panyushev*  
*Independent University of Moscow*  
*Bol'shoi Vlasevskii per. 11*  
*121002 Moscow*  
*Russia*  
panyush@mccme.ru