Indag. Mathem., N.S., 13 (1), 125–129 March 25, 2002

On covariants of reductive algebraic groups

by Dmitri I. Panyushev

Independent University of Moscow, Bol'shoi Vlasevskii per. 11, 121002 Moscow, Russia
e-mail: panyush@mccme.ru

Communicated by Prof. T.A. Springer at the meeting of February 25, 2002

1. AN EVALUATION MAP

Recently, Lehrer and Springer have proved the surjectivity of some natural map associated with covariants of a finite reflection group in a complex vector space, see [LS, Theorem A]. The aim of this note is to show that a similar statement is valid in a greater generality; namely, for an arbitrary action of a reductive algebraic group G on an affine variety X and for a sufficiently good G-orbit (e.g. closed) in X. We also demonstrate some invariant-theoretic applications of it.

The ground field $\Bbbk$ is algebraically closed and of characteristic zero. Let X be an affine variety, with coordinate ring $\Bbbk[X]$, which is acted upon by a reductive algebraic group G. For any (finite-dimensional) G-module M, the space $\Bbbk[X] \otimes M$ is being identified with the space of all polynomial morphisms from X to M, denoted by $\mathcal{P}(X, M)$. Under this identification, $f \otimes m$ ($f \in \Bbbk[X]$, $m \in M$) determines the mapping that takes $x \in X$ to $f(x)m \in M$. The group G acts on $\Bbbk[X]$ by $(g{\cdot}f)(x) = f(g^{-1}{\cdot}x)$. This yields a natural G-module structure on $\Bbbk[X] \otimes M$. Furthermore, the subset of G-invariant elements, denoted by $(\Bbbk[X] \otimes M)^G$ or $\mathcal{P}_G(X, M)$, is nothing but the set of G-equivariant polynomial morphisms from X to M. Clearly, $\mathcal{P}_G(X, M)$ is a module over $\Bbbk[X]^G$, the *module of covariants* (of type M).

For any $x \in X$, there is the 'evaluation' map:

$$\epsilon_x : \mathcal{P}_G(X, M) \to M^{G_x} ,$$

which is defined by $\epsilon_x(\phi) = \phi(x)$. Here $G_x \subset G$ is the stabiliser of x. Notice that the set $\mathcal{P}_G(X, M)$ can be considered for an arbitrary G-variety X. The main result of this note is the following

Theorem 1. *Suppose X is affine, the closure $\overline{G{\cdot}x}$ is normal, and* $\operatorname{codim}_{\overline{G\cdot x}}(\overline{G{\cdot}x} \setminus G{\cdot}x) \geq 2$. *Then ϵ_x is onto.*

Let us first recall a simple and well-known result.

Lemma. *Let Y be an arbitrary closed G-stable subvariety of X and $\bar{\varphi} : Y \to M$ an arbitrary G-equivariant morphism to a G-module. Then $\bar{\varphi}$ extends to a G-equivariant morphism $\varphi : X \to M$. In other words, the mapping $\mathcal{P}_G(X, M) \to \mathcal{P}_G(Y, M)$ is onto.*

Proof. The comorphism $\bar{\varphi}^* : \Bbbk[M] \to \Bbbk[Y]$ is fully determined by its values on $M^* = \Bbbk[M]_1$. (Conversely, any linear mapping $M^* \to \Bbbk[Y]$ determines an algebra homomorphism $\Bbbk[M] \to \Bbbk[Y]$.) Therefore, to construct a required extension, it suffices to find alinear mapping φ^* such that the following diagram be commutative:

$$\begin{tikzcd} & \Bbbk[X] \arrow[d] \\ M^* \arrow[ur, "\varphi^*"] \arrow[r, "\bar{\varphi}^*"] & \Bbbk[Y] \end{tikzcd} \; .$$

The existence of such φ^* follows from the fact that in view of complete reducibility of G-representations, the restriction homomorphism $\Bbbk[X] \to \Bbbk[Y]$ admits a G-equivariant section. $\square$

Proof of Theorem 1. The condition on codimension implies that $\Bbbk[\overline{G{\cdot}x}] = \Bbbk[G{\cdot}x]$, and therefore $\mathcal{P}_G(\overline{G{\cdot}x}, M) = \mathcal{P}_G(G{\cdot}x, M)$. Furthermore, it is clear that the mapping $\epsilon_x : \mathcal{P}_G(G{\cdot}x, M) \to M^{G_x}$ is bijective. It remains to apply the Lemma to $Y = \overline{G{\cdot}x}$. $\square$

Corollary. *If G is a finite group, then ϵ_x is onto for all $x \in X$.*

In case $X = V$ is a G-module and G is a reflection group in V, this corollary gives the aforementioned result of [LS].

As is known, $\mathcal{P}_G(X, M)$ is a finitely generated module over $\Bbbk[X]^G$, see [Kr]. If it is a free module, Theorem 1 admits a matrix interpretation (cf. [LS]). Namely, let $u_1, \ldots, u_r$ be a basis for the module and $m_1, \ldots, m_n$ a basis for M. Then $u_i = \sum_j A_{ij} \otimes m_j$, where A_{ij} are some functions on X ($1 \leq i \leq r$).

Theorem 2. *If $\mathcal{P}_G(X, M)$ is a free $\Bbbk[X]^G$-module and $x \in X$ satisfies the hypotheses of Theorem 1, then the rank of the $r \times n$-matrix$(A_{ij}(x))$ equals* $\dim M^{G_x}$.

Example. Let $X = M = \mathfrak{g}$ be Lie algebra of G. Then $\mathcal{P}_G(\mathfrak{g},\mathfrak{g})$ is a free $\Bbbk[\mathfrak{g}]^G$-module of rank $r = \operatorname{rk}\mathfrak{g}$. If $f_1,\dots,f_r \in \Bbbk[\mathfrak{g}]^G$ are basic invariants and $x_1,\dots,x_n$ are the coordinates on $\mathfrak{g}$, then one can take $A_{ij} = \partial f_i/\partial x_j$ $(1 \le i \le r,\ 1 \le j \le n)$. In this situation, the codimension condition in Theorem 1 is always satisfied and we arrive at the following conclusion:

$$\textit{If } \overline{G{\cdot}v} \textit{ is normal for } v \in \mathfrak{g}, \textit{ then } \operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(v)\right) = \dim \mathfrak{g}^{G_v}.$$

Notice also that if G_v is connected, then $\mathfrak{g}^{G_v}$ is the centre of Lie algebra $\mathfrak{g}_v$.

2. APPLICATIONS.

Let X be an irreducible normal G-variety and let H be a a generic stabiliser for the G-action on X. (The latter always exists, if X is smooth.) Suppose the action is stable and X^H is irreducible. The stability implies that H is reductive. From [Lu, Cor. 4] it then follows that $\Bbbk[X]^G \simeq \Bbbk[X^H]^{N_G(H)}$, where the isomorphism is induced by the restriction. Write J for this $\Bbbk$-algebra. More generally, for any G-module M, restricting G-equivariant morphisms to H-fixed points yields a homomorphism of J-modules

$$\rho_M : \mathcal{P}_G(X,M) \to \mathcal{P}_{N_G(H)}(X^H, M^H)\ .$$

Since $G{\cdot}X^H$ is dense in X, we have ρ_M is injective.

Consider the J-module $\mathcal{P}_G(X, \operatorname{End} M)$. Clearly, it is an associative $\Bbbk$-algebra, since $\operatorname{End} M$ is. Notice that $\mathcal{P}_G(X, \operatorname{End} M)$ contains J as commutative subalgebra, because $id_M \in \operatorname{End} M$.

Proposition 3. *The algebra $\mathcal{P}_G(X, \operatorname{End} M)$ is commutative if and only if $M|_H$ is a multiplicity free H-module.*

(Here 'multiplicity free' means that the multiplicity of any simple H-module in M is at most 1.)

Proof. As above, we have the injective homomorphism

$$\rho_{\operatorname{End} M} : \mathcal{P}_G(X, \operatorname{End} M) \to \mathcal{P}_{N_G(H)}(X^H, (\operatorname{End} M)^H)\ .$$

If $M|_H$ is multiplicity free, then $(\operatorname{End} M)^H$ is a commutative subalgebra of $\operatorname{End} M$. Hence $\mathcal{P}_G(X, \operatorname{End} M)$ is commutative as a subalgebra of the commutative algebra $\mathcal{P}_{N_G(H)}(X^H, (\operatorname{End} M)^H)$. Conversely, suppose $\mathcal{P}_G(X, \operatorname{End} M)$ is commutative. Take $x \in X^H$ such that $G{\cdot}x$ is closed and $G_x = H$. By Theorem 1, we have $\epsilon_x(\mathcal{P}_G(X, \operatorname{End} M)) = (\operatorname{End} M)^H$. Therefore, $(\operatorname{End} M)^H$ is commutative, and hence M is multiplicity free as H-module. $\square$

This generalises a recent result of Kirillov concerning commutativity of 'family' algebras, see [Ki, Cor. 1].

For yet another application of Theorem 1, consider the following situation.

Let V be a G-module and H a generic stabiliser for the G-action on V. Suppose H is reductive and $W := N_G(H)/H$ is finite. By [Lu], we then have $G{\cdot}x$ is closed for any $x \in V^H$ and $\Bbbk[V]^G \simeq \Bbbk[V^H]^W (= J)$. Moreover, the quotient map $\pi : V \to V/\!\!/G$ is equidimensional, and W is a reflection group in V^H, see [Pa]. It follows that J is a polynomial algebra and $\Bbbk[V]$ is a free J-module. Furthermore, any module of covariants $\mathcal{P}_G(V, M)$ is a free J-module of rank $\dim M^H$. Similarly, M^H is a W-module, and $\mathcal{P}_W(V^H, M^H)$ is a free J-module of rank $\dim M^H$. As above, we have the injective homomorphism of J-modules

$$\rho_M : \mathcal{P}_G(V, M) \to \mathcal{P}_W(V^H, M^H) .$$

As both J-modules are free and of the same rank, ρ_M becomes an isomorphism over the fraction field of J. Using Theorem 1, we show that it suffices to invert one specific function, which is valid for all M. Let $D \in J$ be the discriminant of the reflection group W. Recall that the zero locus of D in V^H/W is the ramification divisor for the finite morphism $V^H \to V^H/W$. For $u \in V^H$, we have $D(u) \neq 0$ if and only if $W_u = \{1\}$ if and only if $G_u = H$.

Proposition 4. *For any G-module M, the homomorphism ρ_M becomes an isomorphism after inverting $D \in J$.*

Proof. Let $u_1, \dots, u_n$ (resp. $\tilde{u}_1, \dots, \tilde{u}_n$) be a basis for the J-module $\mathcal{P}_G(V, M)$ (resp. $\mathcal{P}_W(V^H, M^H)$), where $n = \dim V^H$. Since ρ_M is injective, $u_i = \sum_j f_{ij}\tilde{u}_j$ for some $f_{ij} \in J$. For any $x \in V^H$, we have two evaluation maps

$$\tilde{\epsilon}_x : \mathcal{P}_W(V^H, M^H) \to M^H \text{ and } \epsilon_x : \mathcal{P}_G(V, M) \to M^H.$$

By Theorem 1, if $D(x) \neq 0$ then both ϵ_x and $\tilde{\epsilon}_x$ are onto. This means that $\det(f_{ij}(x)) \neq 0$ for all such x. Thus, $\det(f_{ij})$ becomes invertible in $J[1/D]$. $\square$

It may happen that ρ_M is already an isomorphism for some M. For instance, this is always the case, if $M = V^*$, see [Vu]. Later, Broer [Br] obtained an elegant description of all such M for the adjoint representation of G, i.e., for $V = \mathfrak{g}$. It might be interesting to extend Broer's results to the case of representations accosiated with periodic automorphisms of semisimple Lie algebras.

REFERENCES

[Br] Broer, A. – The sum of generalized exponents and Chevalley's restriction theorem, Indag. Math. **6**, 385–396 (1995).

[Ki] Kirillov, A.A. – Introduction to family algebras, Moscow Math. J. **1**, 49–63 (2001).

[Kr] Kraft, H. – 'Geometrische Methoden in der Invariantentheorie'. Braunschweig-Wiesbaden, Vieweg & Sohn, 1984.

[LS] Lehrer, G. and Springer, T.A. – A note concerning fixed points of parabolic subgroups of unitary reflection groups, Indag. Math. **10**, 549–553 (1999).

[Lu] Luna, D. – Adhérences d'orbite et invariants, Invent. Math. **29**, 231–238 (1975).

[Pa] Panyushev, D. – On orbit spaces of finite and connected linear groups, Math. USSR-Izv. **20**, 97–101 (1983).

[Vu] Vust, Th. – Covariants de groupes algébriques réductifs, Thèse no. 1671, Université de Genève, (1974).

(Received December 2001)