The index of a Lie algebra, the centraliser of a nilpotent element, and the normaliser of the centraliser

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Introduction

Let $\mathfrak q$ be a Lie algebra and ξ a linear form on $\mathfrak q$. Denote by $\mathfrak q_\xi$ the set of all $x \in \mathfrak q$ such that $\xi([\mathfrak q, x]) = 0$. In other words, $\mathfrak q_\xi = \{x \in \mathfrak q \mid (\mathrm{ad}^* x){\cdot}\xi = 0\}$, where $\mathrm{ad}^* : \mathfrak q \to \mathfrak{gl}(\mathfrak q^*)$ is the coadjoint representation of $\mathfrak q$. The *index* of $\mathfrak q$, $\mathrm{ind}\,\mathfrak q$, is defined by

$$\mathrm{ind}\,\mathfrak q = \min_{\xi \in \mathfrak q^*} \dim \mathfrak q_\xi \ .$$

This definition goes back to J. Dixmier, see [**Di**, 11.1.6]. He considered index because of its importance in Representation Theory. The problem of computing the index may also be treated as part of Invariant Theory. For, if $\mathfrak q$ is an algebraic Lie algebra and Q is a corresponding algebraic group, then $\mathrm{ind}\,\mathfrak q$ equals the transcendence degree of the field of Q-invariant rational functions on $\mathfrak q^*$. If $\mathfrak q$ is reductive, then $\mathfrak q$ and $\mathfrak q^*$ are isomorphic as $\mathfrak q$-modules and hence $\mathrm{ind}\,\mathfrak q = \mathrm{rk}\,\mathfrak q$. It is therefore interesting to study index for non-reductive Lie algebras. On the other hand, studying the index for all Lie algebras is too pretentious. Therefore I think that the most promising approach is to look at the index for various natural classes of non-reductive subalgebras of semisimple Lie algebras. There are at least two such classes: a) parabolic subalgebras and their 'relatives' (nilpotent radicals, seaweeds, etc.); b) centralisers of elements and their 'relatives'. Some recent results on a) are found in [**Pa4**], whereas the present paper deals mainly with b).

The ground field $\Bbbk$ is algebraically closed and of characteristic 0. Let $\mathfrak g$ be a semisimple Lie algebra and G its adjoint group. Given $x \in \mathfrak g$, let $\mathfrak z_{\mathfrak g}(x)$ denote the centraliser of x. The normaliser in $\mathfrak g$ of $\mathfrak z_{\mathfrak g}(x)$ is denoted by $\mathfrak n_{\mathfrak g}(x)$. The goal of this paper is to study $\mathrm{ind}\,\mathfrak z_{\mathfrak g}(x)$ and $\mathrm{ind}\,\mathfrak n_{\mathfrak g}(x)$ for x nilpotent. By a result of R. Brylinski and B. Kostant [**BK1**], $\mathfrak n_{\mathfrak g}(x)$ is determined by $\mathfrak z_{\mathfrak g}(x)$ and the double centraliser of x, denoted by $\mathfrak d_{\mathfrak g}(x)$. For this reason, we include a discussion of some properties of $\mathfrak d_{\mathfrak g}(x)$. We also prove some general results on the index, which are of independent interest. The following brief exposition of our results is not exhaustive.

In Section 1, we collect some old and new general results for the index of an arbitrary Lie algebra. Actually, to deal with index of Lie algebras, a more general concept of the index of a representation is required. Using this, we prove an inequality relating the index of an algebra and an ideal in it. More precisely, let $\mathfrak{q}$ be an ideal in $\tilde{\mathfrak{q}}$. Then

$$\operatorname{ind}\mathfrak{q} + \operatorname{ind}\tilde{\mathfrak{q}} \le \dim(\tilde{\mathfrak{q}}/\mathfrak{q}) + 2\operatorname{ind}(\tilde{\mathfrak{q}},\mathfrak{q}) \ ,$$

where $\operatorname{ind}(\tilde{\mathfrak{q}},\mathfrak{q})$ is the index of $\mathfrak{q}$ as $\tilde{\mathfrak{q}}$-module. In particular, $\mathfrak{p}^u$ being the nilpotent radical of a parabolic subalgebra $\mathfrak{p}\subset\mathfrak{g}$, we obtain $\operatorname{ind}\mathfrak{p}+\operatorname{ind}\mathfrak{p}^u\le\dim(\mathfrak{p}/\mathfrak{p}^u)$.

Section 2 begins with generalities on $\mathfrak{sl}_2$-triples and centralisers of nilpotent elements. Some properties of $\mathfrak{d}_\mathfrak{g}(x)$ are discussed afterwards. Let $\{x,h,y\}$ be an $\mathfrak{sl}_2$-triple. As is known, h determines a $\mathbb{Z}$-grading of $\mathfrak{g}$, which is nonnegative on $\mathfrak{z}_\mathfrak{g}(x)$ and hence on $\mathfrak{d}_\mathfrak{g}(x)$. We give a simple proof for the result of R. Brylinski and Kostant [**BK2**] that $\mathfrak{d}_\mathfrak{g}(x)(2)=\Bbbk x$, if $\mathfrak{g}$ is simple. Another result is that the Killing form is non-degenerate on $\mathfrak{d}_\mathfrak{g}(x)\oplus\mathfrak{d}_\mathfrak{g}(y)$. This is equivalent to the fact that $\mathfrak{d}_\mathfrak{g}(y)\oplus[\mathfrak{g},\mathfrak{z}_\mathfrak{g}(x)]=\mathfrak{g}$.

In Section 3, we discuss a conjecture of Elashvili to the effect that $\operatorname{ind}\mathfrak{z}_\mathfrak{g}(x)=\operatorname{rk}\mathfrak{g}$ for all $x\in\mathfrak{g}$. The Jordan decomposition immediately reduces this conjecture to nilpotent elements. We show that the conjecture holds for 'large' and 'small' nilpotent G-orbits. The claim for the regular and subregular orbits follows from the fact that $\mathfrak{z}_\mathfrak{g}(x)$ is Abelian if and only if x regular. The case-by-case proof of the latter was known, but we give a short conceptual proof, see Theorem 3·3. This solves Problem 15 posed by Steinberg at the Moscow I.C.M., see [**St**]. Our proof exploits work of Springer [**Sp**] and the above-mentioned result of Brylinski and Kostant. At the other extreme, we prove that Elashvili's conjecture holds for x such that $(\operatorname{ad}x)^3=0$.

In Section 4, we consider $\mathfrak{n}_\mathfrak{g}(x)$ and its natural representation on $\mathfrak{d}_\mathfrak{g}(x)$. It is proved that $\operatorname{ind}\mathfrak{n}_\mathfrak{g}(x)=\operatorname{ind}\mathfrak{z}_\mathfrak{g}(x)-\dim\mathfrak{d}_\mathfrak{g}(x)$ whenever there exists a $\xi\in\mathfrak{d}_\mathfrak{g}(x)^*$ such that $\mathfrak{n}_\mathfrak{g}(x){\cdot}\xi=\mathfrak{d}_\mathfrak{g}(x)^*$. We also show that the last condition (and even a stronger one) is satisfied for all x if $\mathfrak{g}\in\{\mathfrak{sl}_n,\mathfrak{sp}_{2n},\mathfrak{so}_{2n+1}\}$ and for x in $\mathfrak{g}=\mathfrak{so}_{2n}$ such that the corresponding partition has at least three parts, see Theorem 4·7.

In Section 5, we prove that if x is regular, then $\mathfrak{n}_\mathfrak{g}(x)$ is a *Frobenius* Lie algebra, i.e., it is of index 0. It is worth mentioning that each Frobenius subalgebra of $\mathfrak{g}$ determines a solution of the classical Yang-Baxter equation (CYBE), see [**BD**, § 8]. Hence our result means that the regular nilpotent orbit yields a solution of CYBE.

We conjecture that there is a simple general formula for the index of $\mathfrak{n}_\mathfrak{g}(x)$, namely $\operatorname{ind}\mathfrak{n}_\mathfrak{g}(x)=\operatorname{rk}\mathfrak{g}-\dim\mathfrak{d}_\mathfrak{g}(x)$. Our results show that it is true if either x is regular or $(\operatorname{ad}x)^3=0$. This can be thought of as more general version of Elashvili's conjecture, as the conjecture for $\operatorname{ind}\mathfrak{n}_\mathfrak{g}(x)$ implies that for $\operatorname{ind}\mathfrak{z}_\mathfrak{g}(x)$.

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1 The index of a Lie algebra

The index of a Lie algebra $\mathfrak{q}$, as defined in the Introduction, is an integer attached to the adjoint representation of $\mathfrak{q}$. A similar procedure applied to an arbitrary representation of $\mathfrak{q}$ yields the notion of the *index of a representation*. This is quite useful, even though one is mainly interested in the index of Lie algebras. Let $\rho : \mathfrak{q} \to \mathfrak{gl}(V)$ be a finite-dimensional representation of $\mathfrak{q}$, i.e. V is a $\mathfrak{q}$-module. Abusing notation, we write $s{\cdot}v$ in place of $\rho(s)v$, if $s \in \mathfrak{q}$ and $v \in V$. An element $v \in V$ is called *regular* or $\mathfrak{q}$-*regular* whenever its stabiliser $\mathfrak{q}_v = \{s \in \mathfrak{q} \mid s{\cdot}v = 0\}$ has minimal dimension. Because the function $v \mapsto \dim \mathfrak{q}_v$ $(v \in V)$ is upper semicontinuous, the set of all $\mathfrak{q}$-regular elements is open and dense in V. Consider also the dual $\mathfrak{q}$-module V^*.

Definition. The nonnegative integer $\dim V - \max\limits_{\xi \in V^*}(\dim \mathfrak{q}{\cdot}\xi)$ is called the *index* of (the $\mathfrak{q}$-module) V. It will be denoted by $\mathrm{ind}\,(\rho, V)$ or $\mathrm{ind}\,(\mathfrak{q}, V)$.

Notice that in order to define the index of V we used the regular elements in V^*! It is clear that $\mathrm{ind}\,(\mathrm{ad}\,, \mathfrak{q}) = \mathrm{ind}\,\mathfrak{q}$ is the index of $\mathfrak{q}$ in the sense of Introduction.

Consider the bilinear form with values in V

$$\mathcal{K} = \mathcal{K}(\mathfrak{q}, V) : \mathfrak{q} \times V \to V \ ; \quad (s, v) \mapsto s{\cdot}v$$

Evaluating this form against an arbitrary element $\xi \in V^*$ gives a form with values in $\Bbbk$:

$$\mathcal{K}_\xi : \mathfrak{q} \times V \to V \xrightarrow{\xi(\cdot)} \Bbbk \ .$$

We may consider $\mathcal{K}_\xi$ as an element of $\mathrm{Hom}(\mathfrak{q}, V^*)$. The following is obvious.

1·1 Lemma. $\mathrm{Ker}\,(\mathcal{K}_\xi) = \mathfrak{q}_\xi$ *and* $\mathrm{Im}\,(\mathcal{K}_\xi) = \mathfrak{q}{\cdot}\xi$.

It follows that $\mathrm{ind}\,(\mathfrak{q}, V) = \dim V - \max\limits_{\xi \in V^*}(\mathrm{rank}\,\mathcal{K}_\xi)$. Let $n = \dim \mathfrak{q}$ and $m = \dim V$. Having chosen bases for $\mathfrak{q}$ and V, we may regard $\mathcal{K}$ as $n \times m$-matrix with entries in V, where V is identified with the component of grade 1 in the symmetric algebra $\mathcal{S}^\bullet(V)$. Then $\mathrm{rank}\,\mathcal{K} = \max_{\xi \in V^*}(\mathrm{rank}\,\mathcal{K}_\xi)$. Thus,

$$\mathrm{ind}\,(\mathfrak{q}, V) = \dim V - \mathrm{rank}\,\mathcal{K}(\mathfrak{q}, V) \ . \tag{1·2}$$

In case $V = \mathfrak{q}$, we see that $\mathcal{K} = \mathcal{K}(\mathfrak{q})$ is nothing but the Lie bracket, i.e., $\mathcal{K}(s', s'') = [s', s'']$, $s', s'' \in \mathfrak{q}$. Notice also that

$$\mathcal{K}(\mathfrak{q})_\xi : \mathfrak{q} \times \mathfrak{q} \to \mathfrak{q} \xrightarrow{\xi(\cdot)} \Bbbk \ .$$

is the *Kirillov form* associated with $\xi \in \mathfrak{q}^*$. Since $\operatorname{rank} \mathcal{K}(\mathfrak{q})_\xi$ is even, we deduce that $\dim \mathfrak{q} - \operatorname{ind} \mathfrak{q}$ is even for any Lie algebra.
A Lie algebra $\mathfrak{q}$ is called *Frobenius*, if $\operatorname{ind} \mathfrak{q} = 0$, i.e., $\mathcal{K}(\mathfrak{q})_\xi$ is nonsingular for some $\xi \in \mathfrak{q}^*$.

In case $\mathfrak{q}$ is an algebraic Lie algebra, a more geometric description is available. Denote by Q an algebraic group with Lie algebra $\mathfrak{q}$. Then $\operatorname{ind} \mathfrak{q} = \dim \mathfrak{q} - \max_{\xi \in \mathfrak{q}^*} \dim Q{\cdot}\xi$. In particular, $\mathfrak{q}$ is Frobenius if and only if Q has an open orbit in $\mathfrak{q}^*$.

A useful tool for computing index is a theorem of M. Raïs. It also shows that the index of a representation is helpful in computing the index of Lie algebras. Let $\rho : \mathfrak{q} \to \mathfrak{gl}(V)$ be any representation of $\mathfrak{q}$. The linear space $V \times \mathfrak{q}$ has a natural structure of Lie algebra, with bracket $[\ ,\ ]^{\sim}$ defined by the equality

$$[(v_1, s_1), (v_2, s_2)]^{\sim} = (\rho(s_1)v_2 - \rho(s_2)v_1, [s_1, s_2]) \ .$$

Following Raïs, the resulting Lie algebra is denoted by $V \times_\rho \mathfrak{q}$. It is a semi-direct product of $\mathfrak{q}$ and an Abelian ideal V. We identify the dual space $(V \times_\rho \mathfrak{q})^*$ with $V^* \oplus \mathfrak{q}^*$.

1·3 Theorem (Raïs [**Ra**]). *Let $\xi \in V^*$ be a $\mathfrak{q}$-regular element such that $\xi + \eta$ is a regular element in $(V \times_\rho \mathfrak{q})^*$ for some $\eta \in \mathfrak{q}^*$. Then* $\operatorname{ind}(V \times_\rho \mathfrak{q}) = \operatorname{ind}(\mathfrak{q}, V) + \operatorname{ind} \mathfrak{q}_\xi$.

It is not assumed that the stabilisers of all (or almost all) regular elements are conjugated. The following result, which relates the indices of a Lie algebra and an ideal in it, appears to be helpful in various applications.

1·4 Theorem.
Let $\mathfrak{q}$ be an ideal in a Lie algebra $\tilde{\mathfrak{q}}$. Then $\operatorname{ind} \mathfrak{q} + \operatorname{ind} \tilde{\mathfrak{q}} \le \dim(\tilde{\mathfrak{q}}/\mathfrak{q}) + 2 \operatorname{ind}(\tilde{\mathfrak{q}}, \mathfrak{q})$.

Proof. Let $m = \dim \mathfrak{q}$ and $n = \dim \tilde{\mathfrak{q}}$, $m \le n$. Choose a basis $\{e_1, \dots, e_n\}$ for $\tilde{\mathfrak{q}}$ so that $\{e_1, \dots, e_m\}$ is a basis for $\mathfrak{q}$. The $n \times n$-matrix $\mathcal{K}(\tilde{\mathfrak{q}}) = ([e_i, e_j])_{i,j=1}^n$ has the following block structure in this basis:

$$\mathcal{K}(\tilde{\mathfrak{q}}) = \begin{pmatrix} C & B \\ -B^t & D \end{pmatrix} \ .$$

Here $C = \mathcal{K}(\mathfrak{q})$ is a skew-symmetric matrix of order m and B is a rectangular $m \times (n - m)$-matrix. Consider the block matrix of order $m + n$ that depends on a parameter $\mu \in \Bbbk$:

$$M(\mu) = \begin{pmatrix} C & B & C \\ -B^t & \mu D & -\mu B^t \\ C & \mu B & \mu C \end{pmatrix} \ .$$

We look at the rank of $M(0)$ in two different ways, which gives us the required inequality.

1) Since $(B\ \ C)^t = \begin{pmatrix} B^t \\ -C \end{pmatrix}$, we have $\operatorname{rank} M(0) \geq 2\operatorname{rank}(B\ \ C)$. Also, up to a permutation of columns, the rectangular matrix $(B\ \ C)$ is nothing but $\mathcal{K}(\tilde{\mathfrak{q}},\mathfrak{q})$. Thus, $\operatorname{rank} M(0) \geq 2\max\limits_{\xi\in\mathfrak{q}^*}\dim(\tilde{\mathfrak{q}}{\cdot}\xi) = 2(\dim\mathfrak{q} - \operatorname{ind}(\tilde{\mathfrak{q}},\mathfrak{q}))$.

2) If $\mu\neq 0$, then a chain of elementary transformations brings $M(\mu)$ in the following form:

$$M(\mu)\sim\begin{pmatrix} C & B & (1-\mu)C \\ -B^t & \mu D & 0 \\ (1-\mu)C & 0 & (\mu-1)\mu C \end{pmatrix}\sim\begin{pmatrix} \frac{1}{\mu}C & B & 0 \\ -B^t & \mu D & 0 \\ 0 & 0 & (\mu-1)\mu C \end{pmatrix}.$$

Therefore if $\mu\neq 0,1$, then

$$\operatorname{rank} M(\mu) = \operatorname{rank} C + \operatorname{rank}\begin{pmatrix} \frac{1}{\mu}C & B \\ -B^t & \mu D \end{pmatrix} = \operatorname{rank} C + \operatorname{rank}\begin{pmatrix} C & B \\ -B^t & D \end{pmatrix}.$$

Thus,

$$\operatorname{rank} M(\mu) = \operatorname{rank}\mathcal{K}(\mathfrak{q}) + \operatorname{rank}\mathcal{K}(\tilde{\mathfrak{q}}) = \dim\mathfrak{q} - \operatorname{ind}\mathfrak{q} + \dim\tilde{\mathfrak{q}} - \operatorname{ind}\tilde{\mathfrak{q}}\ .$$

From the semi-continuity of the rank, it follows that $\operatorname{rank} M(\mu)\geq\operatorname{rank} M(0)$ for $\mu\neq 0,1$, which yields the required inequality. q.e.d.

Let us apply this theorem to parabolic subalgebras. Let $\mathfrak{p}$ be a parabolic subalgebra of a semisimple Lie algebra $\mathfrak{g}$, with the nilpotent radical $\mathfrak{p}^u$ and a Levi subalgebra $\mathfrak{l}$. Let $\mathfrak{b}\subset\mathfrak{p}$ be any Borel subalgebra of $\mathfrak{g}$. Obviously, $\mathfrak{b}\supset\mathfrak{p}^u$ and $\mathfrak{p}^u$ is a $\mathfrak{b}$-module. It is known that the Borel subgroup $B\subset G$ has a dense orbit in $(\mathfrak{p}^u)^*$, i.e., $\operatorname{ind}(\mathfrak{b},\mathfrak{p}^u)=0$, see [**Jo**, 2.4]. This clearly implies that $\operatorname{ind}(\mathfrak{p},\mathfrak{p}^u)=0$.

1·5 Corollary. *For any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ we have*

(i) $\operatorname{ind}\mathfrak{p} + \operatorname{ind}\mathfrak{p}^u \leq \dim\mathfrak{l}$. *In particular,* $\operatorname{ind}\mathfrak{b} + \operatorname{ind}\mathfrak{b}^u \leq \operatorname{rk}\mathfrak{g}$.

(ii) $\operatorname{ind}\mathfrak{b} + \operatorname{ind}\mathfrak{p}^u \leq \dim(\mathfrak{b}/\mathfrak{p}^u)$.

Proof. Take $\mathfrak{q}=\mathfrak{p}^u$ and then $\tilde{\mathfrak{q}}=\mathfrak{p}$ or $\mathfrak{b}$. q.e.d.

1·5·1 Remarks. Note that $\dim(\mathfrak{b}/\mathfrak{p}^u)$ is dimension of a Borel subalgebra of $\mathfrak{l}$. It can conceptually be proved that $\operatorname{ind}\mathfrak{b}^u = \operatorname{rk}\mathfrak{g}^\theta$ and $\operatorname{ind}\mathfrak{b} = \operatorname{rk}\mathfrak{g} - \operatorname{rk}\mathfrak{g}^\theta$, where θ is an involutory automorphism of $\mathfrak{g}$ of maximal rank. In particular, $\operatorname{ind}\mathfrak{b} + \operatorname{ind}\mathfrak{b}^u = \operatorname{rk}\mathfrak{g}$.

The next Proposition was communicated to me by A.G. Elashvili at the beginning of 90's. He attributed the result to E.B. Vinberg. Being rather simple, this result is extremely helpful. As Vinberg's proof is not published, we give a proof below.
Let $\rho:\mathfrak{q}\to\mathfrak{gl}(V)$ be a representation of $\mathfrak{q}$. Given $w\in V$, consider the natural representation of $\mathfrak{q}_w$ on $\overline{V} = V/\mathfrak{q}{\cdot}w$.

1·6 Proposition (Vinberg). *For any $w\in V$, we have*

$$\max_{v\in V}\dim(\mathfrak{q}{\cdot}v) \ge \max_{\eta\in\overline{V}}\dim(\mathfrak{q}_w{\cdot}\eta) + \dim(\mathfrak{q}{\cdot}w) \ .$$

Proof. Fix a vector space direct sum $\mathfrak{q} = \mathfrak{q}_w \oplus \tilde{\mathfrak{q}}$. Choose $\eta = v + \mathfrak{q}{\cdot}w \in \overline{V}$ so that $\dim(\mathfrak{q}_w{\cdot}\eta)$ is maximal. For any $t \in \Bbbk \setminus \{0\}$, one has $\mathfrak{q}{\cdot}(tv + w) = \mathfrak{q}_w{\cdot}v + \tilde{\mathfrak{q}}{\cdot}(tv + w)$. If $t = 0$, then dimension of the RHS is equal to $\dim(\mathfrak{q}_w{\cdot}\eta) + \dim(\mathfrak{q}{\cdot}w)$. Consequently, for all but finitely many t's, we obtain $\dim\mathfrak{q}{\cdot}(tv + w) \ge \dim(\mathfrak{q}_w{\cdot}\eta) + \dim(\mathfrak{q}{\cdot}w)$. q.e.d.

Remark. If $\mathfrak{q}_w$ is reductive, then one can identify $\overline{V}$ with a $\mathfrak{q}_w$-stable complement to $\mathfrak{q}{\cdot}w$. In this case, one can actually prove that equality holds. It is an intersting open problem to find out weaker sufficient conditions for the equality in 1·6 to hold. See e.g. Conjecture 3·2 below.

1·7 Corollary. *Let $\mathfrak{q}$ be a Lie algebra. Then* $\operatorname{ind}\mathfrak{q}_\xi \ge \operatorname{ind}\mathfrak{q}$ *for any $\xi \in \mathfrak{q}^*$.*

Proof. Apply the previous Proposition to $V = \mathfrak{q}^*$. Here $\mathfrak{q}^*/\mathfrak{q}{\cdot}w \simeq (\mathfrak{q}_w)^*$. q.e.d.

1·8 Corollary (Duflo–Vergne, 1969)**.** *If $w \in \mathfrak{q}^*$ is regular, then $\mathfrak{q}_w$ is Abelian.*

Proof. Since w is regular, $\operatorname{ind}\mathfrak{q} = \dim\mathfrak{q}_w$. Hence $\dim\mathfrak{q}_w \ge \operatorname{ind}\mathfrak{q}_w \ge \operatorname{ind}\mathfrak{q} = \dim\mathfrak{q}_w$. q.e.d.

2 Some properties of the double centraliser of a nilpotent element

Let $\mathfrak{g}$ be a semisimple Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{u}_-\oplus\mathfrak{t}\oplus\mathfrak{u}_+$, Δ the corresponding root system, and $\Pi = \{\alpha_1, \dots, \alpha_p\}$ the set of simple roots. The Killing form on $\mathfrak{g}$ is denoted by Φ. For any subset $M \subset \mathfrak{g}$, let $Z_G(M)$ and $\mathfrak{z}_\mathfrak{g}(M)$ denote its centraliser in G and $\mathfrak{g}$, respectively; $M^\perp$ is the orthogonal complement to M with respect to the Killing form. Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone. By the Morozov-Jacobson theorem, each nonzero element $e \in \mathcal{N}$ can be included in an $\mathfrak{sl}_2$-triple $\{e, h, f\}$ (i.e., $[e, f] = h,\ [h, e] = 2e,\ [h, f] = -2f$). The semisimple element h determines a $\mathbb{Z}$-grading in $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i) \ ,$$

where $\mathfrak{g}(i) = \{\, x \in \mathfrak{g} \mid [h, x] = ix \,\}$. Since all $\mathfrak{sl}_2$-triples containing e are $Z_G(e)$-conjugate, the properties of this $\mathbb{Z}$-grading does not depend on a particular choice of h.

The adjoint orbit $G{\cdot}h$ contains a unique element h_+ such that $h_+ \in \mathfrak{t}$ and $\alpha(h_+) \ge 0$ for all $\alpha \in \Pi$. The Dynkin diagram of $\mathfrak{g}$ equipped with the numerical labels $\alpha_i(h_+)$, $\alpha_i \in \Pi$, at the corresponding nodes is called the *weighted Dynkin diagram* of e. After Dynkin, it is known (see [**Dy2**, 8.1, 8.3]) that

(a) $\alpha_i(h_+) \in \{0, 1, 2\}$;

(b) $\mathfrak{sl}_2$-triples $\{e, h, f\}$ and $\{e', h', f'\}$ are G-conjugate *if and only if* h and h' are G-conjugate *if and only if* their weighted Dynkin diagrams coincide.

We also need the following standard results on the structure of $Z_G(e) \subset G$ and $\mathfrak{z}_{\mathfrak{g}}(e) \subset \mathfrak{g}$ (see e.g. [**SS**, ch. III] or [**CM**]).

2·1 Proposition.

(i) *the Lie algebra* $\mathfrak{z}_{\mathfrak{g}}(e)$ *(resp.* $\mathfrak{z}_{\mathfrak{g}}(f)$*) is positively (resp. negatively) graded;* $\mathfrak{z}_{\mathfrak{g}}(e) = \bigoplus_{i\ge 0} \mathfrak{z}_{\mathfrak{g}}(e)(i)$, *where* $\mathfrak{z}_{\mathfrak{g}}(e)(i) = \mathfrak{z}_{\mathfrak{g}}(e) \cap \mathfrak{g}(i)$, *and likewise for* $\mathfrak{z}_{\mathfrak{g}}(f)$*;*

(ii) *let* $G(0)$ *be the connected subgroup of* G *with Lie algebra* $\mathfrak{g}(0)$ *and* $K := Z_{G(0)}(e)$ *(*K *can be disconnected !). Then* $K = Z_G(e) \cap Z_G(f)$ *and it is a maximal reductive subgroup in both* $Z_G(e)$ *and* $Z_G(f)$*;*

(iii) *for any* i*, there are* K*-stable decompositions:*

$$\mathfrak{g}(i) = \mathfrak{z}_{\mathfrak{g}}(e)(i) \oplus [f, \mathfrak{g}(i+2)], \qquad \mathfrak{g}(i) = \mathfrak{z}_{\mathfrak{g}}(f)(i) \oplus [e, \mathfrak{g}(i-2)].$$

In particular, $\mathrm{ad}\, e : \mathfrak{g}(i-2) \to \mathfrak{g}(i)$ *is injective for* $i \le 1$ *and surjective for* $i \ge 1$*;*

An element $e \in \mathcal{N}$ is said to be *even*, if the eigenvalues of $\mathrm{ad}\, h$ are even; $e \in \mathcal{N}$ is called *distinguished*, if $\mathfrak{z}_{\mathfrak{g}}(e)(0) = \{0\}$. By a well-known result of Bala-Carter and Vinberg, "distinguished" implies "even". The integer $\max\{i \in \mathbb{N} \mid \mathfrak{g}(i) \ne 0\}$ is called the *height* of e or of the orbit $G{\cdot}e$ and is denoted by $\mathrm{ht}\,(e)$. It is easily seen that $\mathrm{ht}\,(e) = m$ if and only if $(\mathrm{ad}\, e)^m \ne 0$ and $(\mathrm{ad}\, e)^{m+1} = 0$. See [**Pa2**, Sect. 2] for some results concerning the height.

The notation related to the $\mathbb{Z}$-grading associated with a nilpotent orbit will be used throughout the paper. Whenever we discuss in the sequel a $\mathbb{Z}$-grading associated with $e \in \mathcal{N}$, this means that e is regarded as member of an $\mathfrak{sl}_2$-triple and the grading is determined by h.

Remark. In the context of the adjoint representation (of $\mathfrak{g}$) we write $\mathfrak{z}_{\mathfrak{g}}(e)$ in place of $\mathfrak{g}_e$. This algebra is also referred to as *centraliser* of e, and not stabiliser.

Set $\mathfrak{d}_{\mathfrak{g}}(e) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{z}_{\mathfrak{g}}(e))$. Sometimes, this space is called the *double centraliser* of e. It is also easily seen that $\mathfrak{d}_{\mathfrak{g}}(e)$ is the centre of $\mathfrak{z}_{\mathfrak{g}}(e)$. The subalgebra $\mathfrak{d}_{\mathfrak{g}}(e)$ inherits the (nonnegative) graded structure from $\mathfrak{z}_{\mathfrak{g}}(e)$. R. Brylinski and B. Kostant have obtained several results on the graded structure of $\mathfrak{d}_{\mathfrak{g}}(e)$. For instance, they proved that

($\ast$) $\mathfrak{d}_{\mathfrak{g}}(e)(i) = \{0\}$ for $i \le 1$;

($\diamond$) $\dim \mathfrak{d}_{\mathfrak{g}}(e)(2)$ is equal to the number of those simple ideals of $\mathfrak{g}$ to which e has nonzero projection.

But whereas relation ($\ast$) is proved rather elementary (see [**BK1**, Prop. 17]), the proof of ($\diamond$) is very involved. In fact, the proof of Theorem 7.1 in [**BK2**] consists of a list of reference to previous results in two papers, which requires, in particular, understanding some properties of a Poisson structure on the algebra of regular functions on $G{\cdot}e$.

We are going to give a simple self-contained proof of ($\diamond$). First, recall a required result of [**BK1**]. Obviously, the spaces $\mathfrak{d}_{\mathfrak{g}}(e)(2)$, $\mathfrak{d}_{\mathfrak{g}}(f)(-2)$, and $[f, \mathfrak{d}_{\mathfrak{g}}(e)(2)]$ have the same dimension, say d. Set $\mathfrak{r} := \mathfrak{d}_{\mathfrak{g}}(f)(-2) \oplus [f, \mathfrak{d}_{\mathfrak{g}}(e)(2)] \oplus \mathfrak{d}_{\mathfrak{g}}(e)(2)$.

2·2 Proposition [**BK1**, Prop. 20]**.**
The vector space $\mathfrak{r}$ is a subalgebra of $\mathfrak{g}$. It is isomorphic to $(\mathfrak{sl}_2)^d$.

Set $\mathfrak{a} = \Bbbk e + \Bbbk h + \Bbbk f$. It is a subalgebra of $\mathfrak{r}$. Let $\mathfrak{a}_1, \dots, \mathfrak{a}_d$ be the simple ideals in $\mathfrak{r}$. Choose a basis $\{e_i, h_i, f_i\}$ of $\mathfrak{a}_i$ such that it is an $\mathfrak{sl}_2$-triple, $e_i \in \mathfrak{d}_{\mathfrak{g}}(e)(2)$, and $f_i \in \mathfrak{d}_{\mathfrak{g}}(f)(-2)$. It immediately follows from the definition of $\mathfrak{r}$ and (2·2) that $\mathfrak{a}$ projects isomorphically to each $\mathfrak{a}_i$. We may therefore assume that $e = \sum_{i=1}^d e_i$. Let φ_i (resp. φ) denote the fundamental weight of $\mathfrak{a}_i$ (resp. $\mathfrak{a}$) relative to the Borel subalgebra $\Bbbk e_i + \Bbbk h_i$ (resp. $\Bbbk e + \Bbbk h$). We shall use dominant weights in multiplicative notation to denote irreducible $\mathfrak{r}$- and $\mathfrak{a}$-modules. For instance, $\varphi_1^{k_1} \dots \varphi_d^{k_d}$ stands for the irreducible $\mathfrak{r}$-module with highest weight $k_1\varphi_1 + \dots + k_d\varphi_d$.

2·3 Theorem (R. Brylinski & Kostant)**.**
Let e be a nilpotent element in a semisimple Lie algebra $\mathfrak{g}$. Then $d = \dim \mathfrak{d}_{\mathfrak{g}}(e)(2)$ equals the number of those simple ideals of $\mathfrak{g}$ to which e has nonzero projection.

Proof. 1. Let $\mathfrak{g}_1, \dots, \mathfrak{g}_m$ be the simple ideals in $\mathfrak{g}$ and $e = e^{(1)} + \dots + e^{(s)}$, where $0 \neq e^{(j)} \in \mathfrak{g}_j$ and $s \le m$. Since $\mathfrak{d}_{\mathfrak{g}}(e)(2) = \mathfrak{d}_{\mathfrak{g}_1}(e^{(1)})(2) \oplus \dots \oplus \mathfrak{d}_{\mathfrak{g}_s}(e^{(s)})(2)$, it suffices to prove that $d = 1$ whenever $\mathfrak{g}$ is simple.

2. Consider $\mathfrak{g}$ as $\mathfrak{r}$-module:

(2·3·1)
$$\mathfrak{g}\,|_{\mathfrak{r}} = \bigoplus_{k_1,\dots,k_d} m_{k_1\dots k_d} \varphi_1^{k_1} \dots \varphi_d^{k_d}.$$

Here $m_{k_1\dots k_d}$ is the multiplicity in $\mathfrak{g}$ of the respective irreducible $\mathfrak{r}$-module. Restricting further to $\mathfrak{a}$, we obtain:

$$\mathfrak{g}\,|_{\mathfrak{a}} = \bigoplus_{k_1,\dots,k_d} m_{k_1\dots k_d} \varphi^{k_1} \otimes \dots \otimes \varphi^{k_d} \ .$$

A highest weight vector in $\mathfrak{g}$ relative to $\mathfrak{a}$ is nothing but a homogeneous element of $\mathfrak{z}_{\mathfrak{g}}(e)$. Since each e_i lies in $\mathfrak{d}_{\mathfrak{g}}(e)$, any highest weight vector in $\mathfrak{g}$ relative to $\mathfrak{a}$ is also a highest weight vector relative to $\mathfrak{a}_i$ and hence relative to $\mathfrak{r}$. This means that the above two decompositions have "the same" irreducible constituents. In other words, $\varphi^{k_1} \otimes \dots \otimes \varphi^{k_d}$ is an irreducible $\mathfrak{a}$-module for any d-tuple $(k_1, \dots, k_d)$ occurring in the (2·3·1). This clearly implies that any such d-tuple contains *at most one* nonzero coordinate.

Let $\mathfrak{L}_j$ ($j = 1, \dots, d$) denote the sum of all nontrivial $\mathfrak{a}_j$-submodules, i.e., the sum of all summands in (2·3·1), where $k_j \neq 0$. Since each d-tuple contains *at most* one nonzero coordinate, $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{r}) \oplus \mathfrak{L}_1 \oplus \dots \oplus \mathfrak{L}_d$, the sum of vector spaces. Note that $[\mathfrak{L}_i, \mathfrak{L}_j] = 0$ $(i \neq j)$, because any nonzero element would generate a $\mathfrak{r}$-submodule with a nontrivial action of *both* $\mathfrak{a}_i$ and $\mathfrak{a}_j$. Denoting $\mathfrak{r}^{(j)} = \oplus_{i \neq j} \mathfrak{a}_i$, we see that $\mathfrak{L}_j \oplus \mathfrak{z}_{\mathfrak{g}}(\mathfrak{r}) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{r}^{(j)})$ is a reductive Lie subalgebra.

If $\mathfrak{h}$ is a reductive subalgebra of a simple Lie algebra $\mathfrak{g}$, then the subalgebra $\mathfrak{s}$ generated by $\mathfrak{h}^{\perp}$ equals $\mathfrak{g}$. Indeed, $\mathfrak{s}$ is stable relative to both $\mathrm{ad}\,(\mathfrak{h}^{\perp})$ and $\mathrm{ad}\,\mathfrak{h}$, so that it is an ideal in $\mathfrak{g}$, see [**Ka**, 4.1]. Apply this to $\mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{r}^{(j)})$, for arbitrary j. Here $\mathfrak{h}^\perp = \bigoplus_{i\ne j} \mathfrak{L}_i$, i.e., it is nonzero, if $d > 1$. Then the relations of the previous paragraph show that the subalgebra generated by $\mathfrak{h}^\perp$ lies in $\mathfrak{h}^\perp \oplus \mathfrak{z}_{\mathfrak{g}}(\mathfrak{r})$, i.e., it does not contain $\mathfrak{L}_j$. Thus, $\mathfrak{g}$ cannot be simple, if $d > 1$. q.e.d.

It follows from $\mathfrak{sl}_2$-theory that $\mathfrak{z}_{\mathfrak{g}}(e) \oplus [\mathfrak{g}, f] = \mathfrak{g}$, see Proposition 2·1(iii). It is equivalent to the fact that the restriction of the Killing form to $\mathfrak{z}_{\mathfrak{g}}(e) + \mathfrak{z}_{\mathfrak{g}}(f)$ is nondegenerate. It is then rather natural to ask about a good complementary subspace to $\mathfrak{d}_{\mathfrak{g}}(e)$. This question together with a conjectural answer was communicated to me by P. Tauvel in 1996. Here we give a proof for Tauvel's conjecture.

2·4 Theorem. *For any $\mathfrak{sl}_2$-triple we have $\mathfrak{d}_{\mathfrak{g}}(e) \oplus [\mathfrak{g}, \mathfrak{z}_{\mathfrak{g}}(f)] = \mathfrak{g}$. Equivalently, the Killing form is nondegenerate on $\mathfrak{d}_{\mathfrak{g}}(e) \oplus \mathfrak{d}_{\mathfrak{g}}(f) \subset \mathfrak{g}$.*

Before giving a proof, we state an auxiliary result.
Assume for a while that $\Bbbk = \mathbb{C}$. Let θ be an involutory automorphism of $\mathfrak{g}$. It is a standard fact (see e.g. [**VGO**, 4.3.1]) that there exists an antilinear involution σ such that $\theta\sigma = \sigma\theta$ and $\mathfrak{g}^{\theta\sigma}$ is a compact real form of $\mathfrak{g}$. We shall say that σ is a *(complex) conjugation* associated with θ. In other words, θ is a Cartan involution for the real form $\mathfrak{g}^\sigma$.

2·5 Proposition. *Let $\theta \in \mathrm{Aut}\,\mathfrak{g}$ be an involutory automorphism of $\mathfrak{g}$ and σ an associated complex conjugation. Suppose an $\mathfrak{sl}_2$-triple $\{e', h', f'\}$ satisfies the condition that $G{\cdot}h'$ has a nonempty intersection with the (-1)-eigenspace of θ. Then $\{e', h', f'\}$ is G-conjugate to a triple $\{e, h, f\}$ such that $\{e, h, f\} \subset \mathfrak{g}^\sigma$, $\theta(h) = -h$, and $\theta(e) = -f$.*

Proof. This is proved, although not stated explicitly, in [**An**, Theorem 1]. q.e.d.

Proof of (2·4). 1. First, we prove that the sum $\mathfrak{d}_{\mathfrak{g}}(e) + \mathfrak{d}_{\mathfrak{g}}(f)$ is direct. Indeed, if $z \in \mathfrak{d}_{\mathfrak{g}}(e) \cap \mathfrak{d}_{\mathfrak{g}}(f)$, then it commutes with f and $\mathfrak{z}_{\mathfrak{g}}(e)$. Since the subalgebra generated by f and $\mathfrak{z}_{\mathfrak{g}}(e)$ is the whole $\mathfrak{g}$, we see that $z = 0$.

2. Let θ be an involutory automorphism of *maximal rank*, i.e., its (-1)-eigenspace contains a Cartan subalgebra of $\mathfrak{g}$. Then the hypothesis of Proposition 2·5 is satisfied for all $\mathfrak{sl}_2$-triples. We may therefore assume that $\{e, h, f\}$ are chosen as prescribed in (2·5). Then $\theta(\mathfrak{z}_{\mathfrak{g}}(e)) = \mathfrak{z}_{\mathfrak{g}}(f)$ and hence $\theta(\mathfrak{d}_{\mathfrak{g}}(e)) = \mathfrak{d}_{\mathfrak{g}}(f)$, while σ leaves all these spaces intact. It follows that $\mathfrak{d}_{\mathfrak{g}}(e) \oplus \mathfrak{d}_{\mathfrak{g}}(f)$ is a θ- and σ-stable subspace of $\mathfrak{g}$. Therefore it is the complexification of a subspace in the compact real form $\mathfrak{g}^{\theta\sigma}$. This shows that Φ is nondegenerate on $\mathfrak{d}_{\mathfrak{g}}(e) \oplus \mathfrak{d}_{\mathfrak{g}}(f)$. More explicitly, given $x \in \mathfrak{d}_{\mathfrak{g}}(e)$, we have $\sigma\theta(x) \in \mathfrak{d}_{\mathfrak{g}}(f)$ and $\Phi(x, \sigma\theta(x)) \ne 0$.

3. Using the invariance of the Killing form, one readily verifies that $[\mathfrak{g}, \mathfrak{z}_{\mathfrak{g}}(f)]^\perp = \mathfrak{d}_{\mathfrak{g}}(f)$. This yields the equivalence stated in the Theorem. $\square$

We shall need a stronger result than $(*)$. It was proved by Tauvel in [**Ta2**][1]. Here we give a shorter proof.

2·6 Proposition. *For any $e \in \mathcal{N}$, we have $\mathfrak{d}_{\mathfrak{g}}(e) = \oplus_{i\ge 1}\mathfrak{d}_{\mathfrak{g}}(e)(2i)$.*

Proof. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{z}_{\mathfrak{g}}(e)(0)$. Consider the reductive Levi subalgebra $\mathfrak{l} := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$. It is clear that $\mathfrak{h}$ is the centre of $\mathfrak{l}$, i.e., $\mathfrak{l} = [\mathfrak{l}, \mathfrak{l}] \oplus \mathfrak{h}$. By the construction of $\mathfrak{l}$, we have $\mathfrak{d}_{\mathfrak{g}}(e) \subset \mathfrak{z}_{\mathfrak{l}}(e)$. Furthermore, $\mathfrak{d}_{\mathfrak{g}}(e) \subset \mathfrak{z}_{[\mathfrak{l},\mathfrak{l}]}(e)$, since $\mathfrak{d}_{\mathfrak{g}}(e)(0) = 0$. Finally, the very definition of $\mathfrak{l}$ implies that e is distinguished in $[\mathfrak{l}, \mathfrak{l}]$ and therefore $\mathfrak{z}_{[\mathfrak{l},\mathfrak{l}]}(e)$ is concentrated in positive even degrees. q.e.d.

Remark. The previous proof shows that the double centraliser of any $e \in \mathcal{N}$ is isomorphic, as graded space, to the double centraliser of a distinguished element in a smaller semisimple subalgebra. For this reason, all questions about $\mathfrak{d}_{\mathfrak{g}}(e)$ can be reduced to the case, where e is distinguished.

3 On a conjecture of A.G. Elashvili

By Corollary 1·7, we have $\operatorname{ind}\mathfrak{q}_\xi \ge \operatorname{ind}\mathfrak{q}$ for any Lie algebra $\mathfrak{q}$ and any $\xi \in \mathfrak{q}^*$. It is however easily seen that this inequality can be strict.

3·1 Example. Let $\mathfrak{q}$ be a Borel subalgebra of $\mathfrak{sp}_4$. It is not hard to verify that $\operatorname{ind}\mathfrak{q} = 0$ and there exists $\xi \in \mathfrak{q}^*$ such that $\mathfrak{q}_\xi$ is two-dimensional and Abelian, i.e., $\operatorname{ind}\mathfrak{q}_\xi = 2$.

The following remarkable conjecture claimes that such a phenomenon cannot occur in reductive Lie algebras. For a reductive $\mathfrak{g}$, we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ and consider centralisers $\mathfrak{z}_{\mathfrak{g}}(x)$ in place of stabilisers $\mathfrak{g}_\xi$ ($\xi \in \mathfrak{g}^*$).

3·2 Conjecture (A.G. Elashvili)**.** *If $\mathfrak{g}$ is a reductive Lie algebra, then $\operatorname{ind}\mathfrak{z}_{\mathfrak{g}}(x) = \operatorname{rk}\mathfrak{g}$ for any $x \in \mathfrak{g}$.*

Let us record several helpful (and easy) observations related to this conjecture.

- It suffices to prove the conjecture for the simple Lie algebras;
- The conjecture is true, if x is semisimple.
- The Jordan decomposition in $\mathfrak{g}$ shows that it is sufficient to prove the conjecture only for the nilpotent elements. Indeed, let $x = x_s + x_n$ be the Jordan decomposition, where x_s is semisimple and x_n is nilpotent. Here $\mathfrak{l} = \mathfrak{z}_{\mathfrak{g}}(x_s)$ is reductive, $\operatorname{rk}\mathfrak{g} = \operatorname{rk}\mathfrak{l}$, x_n is a nilpotent element of $\mathfrak{l}$, and the equality $\mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{z}_{\mathfrak{g}}(x_s) \cap \mathfrak{z}_{\mathfrak{g}}(x_n)$ means that $\mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{z}_{\mathfrak{l}}(x_n)$.

[1] Practically, this book seems to be a virtual object because of the bankruptcy of the publisher.

For this reason, we shall only consider in the sequel the centralisers of *nilpotent* elements. It is worth noting that Problem 15 stated by R.Steinberg in his address to the I.C.M. in Moscow [**St**, p. 280] (see also [**SS**, III.1.18]) can be regarded as a restricted version of this conjecture. A solution to this problem is given by Kurtzke [**Ku2**] and Tauvel [**Ta1**], independently. However, both proofs exploit the classification of nilpotent elements and/or computer computations. Here we give a short conceptual proof.

3·3 Theorem. *Let $e \in \mathfrak{g}$ be a nilpotent element. Suppose $\mathfrak{z}_{\mathfrak{g}}(e)$ is Abelian. Then e is regular, i.e.,* $\dim \mathfrak{z}_{\mathfrak{g}}(e) = \mathrm{rk}\,\mathfrak{g}$.

Proof. Without loss of generality, we assume that $\mathfrak{g}$ is simple. Let $\{e, h, f\}$ be an $\mathfrak{sl}_2$-triple. Recall from Section 2 the non-negative grading on $\mathfrak{z}_{\mathfrak{g}}(e)$ determined by h:

$$\mathfrak{z}_{\mathfrak{g}}(e) = \bigoplus_{i \geq 0} \mathfrak{z}_{\mathfrak{g}}(e)(i) \ .$$

Take any $z \in \mathfrak{z}_{\mathfrak{g}}(e)(0)$. By Proposition 2·1(ii), z commutes with f and, by the assumption, with $\mathfrak{z}_{\mathfrak{g}}(e)$. From the $\mathfrak{sl}_2$-theory it follows that the Lie subalgebra generated by f and $\mathfrak{z}_{\mathfrak{g}}(e)$ is $\mathfrak{g}$. Hence z lies in the centre of $\mathfrak{g}$, i.e., $z = 0$. This means that e is distinguished and therefore is even. By Proposition 2·1(iii), we have

$$\dim \mathfrak{g}(2i) - \dim \mathfrak{g}(2i+2) = \dim \mathfrak{z}_{\mathfrak{g}}(e)(2i) = \dim \mathfrak{d}_{\mathfrak{g}}(e)(2i)$$

for all $i \geq 0$. Then invoking Theorem 2·3, we conclude that

(3·3·1) $$\dim \mathfrak{g}(2) - \dim \mathfrak{g}(4) = 1.$$

This condition for a distinguished nilpotent element x was considered by T.A. Springer in [**Sp**]. Let $m = \mathrm{ht}\,(e)$ and let q be a nonzero element in $\mathfrak{g}(-m)$. Springer proved that $c := q+e$ is a regular semisimple element in $\mathfrak{g}$ whenever (3·3·1) is satisfied, see Lemma 9.6 in [loc. cit.]

3·3·2 Claim. *For every $x \in \mathfrak{z}_{\mathfrak{g}}(e)$ there exists a unique $z \in \bigoplus_{i<0} \mathfrak{g}(i)$ such that $x+z \in \mathfrak{z}_{\mathfrak{g}}(c)$.*

As e is distinguished, x is nilpotent and therefore $x \notin \mathfrak{z}_{\mathfrak{g}}(c)$. Therefore $z \neq 0$, if $x \neq 0$. The uniqueness of z follows from the fact that $\mathfrak{z}_{\mathfrak{g}}(c)$ contains no nonzero nilpotent elements. So, it remains to prove that such a z exists. Consider the equation on z

$$0 = [x+z, e+q] = [z, e] + [x, q].$$

(Note that $[q, z] = 0$ in view of our assumption on z and q.) Since $\mathfrak{z}_{\mathfrak{g}}(e)$ is Abelian, $[x, q]$ is orthogonal to $\mathfrak{z}_{\mathfrak{g}}(e)$ with respect to the Killing form on $\mathfrak{g}$. Therefore $[x, q] \in \mathfrak{z}_{\mathfrak{g}}(e)^{\perp} = [\mathfrak{g}, e]$, i.e. the above equation has a solution. Because $[x, q] \in \oplus_{i \leq 0} \mathfrak{g}(i)$ and $e \in \mathfrak{g}(2)$, we also deduce that there exists a solution z lying in $\oplus_{i<0} \mathfrak{g}(i)$, as required.

It follows from the Claim that $\dim \mathfrak{z}_{\mathfrak{g}}(e) \leq \dim \mathfrak{z}_{\mathfrak{g}}(c) = \mathrm{rk}\,\mathfrak{g}$, which completes the proof of Theorem 3·3. q.e.d.

Remarks. 1. In [**Sp**], Springer gave a classification of all nilpotent orbits in simple Lie algebras satisfying (3·3·1). It turns that all such orbits in the classical Lie algebras are necessarily regular, and there are a few nonregular nilpotent orbits in the exceptional Lie algebras satisfying (3·3·1), see Table 11 on p.196 in [loc. cit.]. However, the subregular orbit in type $\mathbb{G}_2$ is erroneously placed in that list. Indeed, in this case $\dim \mathfrak{g}(2) = 4$ and $\dim \mathfrak{g}(4) = 1$.

2. As was explained in [**SS**, III.4], all results related to $\mathfrak{sl}_2$-triples that can be proved in characteristic zero remain valid if $\mathrm{char}(\Bbbk) \ge 4m+3$, where m is the height of the highest root. This means that our proof of Theorem 3·3 goes through under this constraint as well.

3·4 Corollary. *Suppose* $e \in \mathcal{N}$ *is subregular (i.e.,* $\dim \mathfrak{z}_{\mathfrak{g}}(e) = \mathrm{rk}\,\mathfrak{g} + 2$*). Then* $\mathrm{ind}\,\mathfrak{z}_{\mathfrak{g}}(e) = \mathrm{rk}\,\mathfrak{g}$.

Proof. From Corollary 1·7 and Theorem 3·3, we conclude that $\mathrm{rk}\,\mathfrak{g} \le \mathrm{ind}\,\mathfrak{z}_{\mathfrak{g}}(e) < \mathrm{rk}\,\mathfrak{g} + 2$. Since $\dim \mathfrak{z}_{\mathfrak{g}}(e) - \mathrm{ind}\,\mathfrak{z}_{\mathfrak{g}}(e)$ is even, we are done. q.e.d.

Our next goal is to show that Elashvili's Conjecture is true for 'small' nilpotent orbits. Note that $\mathrm{ht}\,(e) \ge 2$ for all $x \in \mathcal{N} \setminus \{0\}$.

3·5 Theorem. *Suppose* $\mathrm{ht}\,(e) = 2$. *Then* $\mathrm{ind}\,\mathfrak{z}_{\mathfrak{g}}(e) = \mathrm{rk}\,\mathfrak{g}$.

Proof. Consider the $\mathbb{Z}$-grading of $\mathfrak{g}$ associated with an $\mathfrak{sl}_2$-triple $\{e, h, f\}$. We have $\mathfrak{g} = \bigoplus_{i=-2}^{2} \mathfrak{g}(i)$ and $\mathfrak{z}_{\mathfrak{g}}(e) = \mathfrak{k} \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$, where $\mathfrak{k}$ is a reductive subalgebra of $\mathfrak{g}(0)$. Consider f as element of $\mathfrak{z}_{\mathfrak{g}}(e)^*$, using the Killing form, and compute its stabiliser $\mathfrak{z}_{\mathfrak{g}}(e)_f$. It readily follows from Proposition 2·1 that $\mathfrak{z}_{\mathfrak{g}}(e)_f = \mathfrak{k} \oplus \mathfrak{g}(2)$. Combining Corollary 1·7 and Theorem 1·3 yields

$$\mathrm{rk}\,\mathfrak{g} \le \mathrm{ind}\,\mathfrak{z}_{\mathfrak{g}}(e) \le \mathrm{ind}\,(\mathfrak{k} \oplus \mathfrak{g}(2)) = \mathrm{ind}\,(\mathfrak{k} : \mathfrak{g}(2)) + \mathrm{ind}\,\mathfrak{k}_\xi \ ,$$

where ξ is a regular element in the $\mathfrak{k}$-module $\mathfrak{g}(2)^*$. Thus, it suffices to prove that the last expression is equal to $\mathrm{rk}\,\mathfrak{g}$. Note that $\mathfrak{g}(2)$ is the isotropy representation for the *affine* homogeneous space $G(0)/K$. This implies that $\mathfrak{g}(2) \simeq \mathfrak{g}(2)^*$ as $\mathfrak{k}$-modules and there exists an open subset $\Omega \in \mathfrak{g}(2)$ such that the stabilisers $\mathfrak{k}_\xi$ $(\xi \in \Omega)$ are reductive and conjugate in $\mathfrak{k}$, see [**Lu**]. Therefore we may assume that $\mathfrak{k}_\xi$ is such a reductive generic stabiliser. By [**Pa1**, 3.3], $\mathfrak{k}$ is a 'symmetric' subalgebra of $\mathfrak{g}(0)$ whenever $\mathrm{ht}\,(e) \le 3$. In particular, $G(0)/K$ is a spherical homogeneous space. In this case, one knows that

$$\dim \mathfrak{g}(2)/\!\!/K = \mathrm{rk}\,G(0) - \mathrm{rk}\,\mathfrak{k}_\xi \quad (\xi \in \Omega)$$

(see [**Pa3**, ch.II]). Here $\mathfrak{g}(2)/\!\!/K = \mathrm{Spec}\,(\Bbbk[\mathfrak{g}(2)]^K)$. Since $\mathfrak{g}(2)$ is an orthogonal K-module, the field of invariants $\Bbbk(\mathfrak{g}(2))^K$ is the quotient field of $\Bbbk[\mathfrak{g}(2)]^K$. Therefore $\mathrm{ind}\,(\mathfrak{k} : \mathfrak{g}(2)) = \dim \mathfrak{g}(2)/\!\!/K$ and, since $\mathrm{ind}\,\mathfrak{k}_\xi = \mathrm{rk}\,\mathfrak{k}_\xi$, we conclude that $\mathrm{ind}\,(\mathfrak{k} : \mathfrak{g}(2)) + \mathrm{ind}\,\mathfrak{k}_\xi = \mathrm{rk}\,G(0) = \mathrm{rk}\,\mathfrak{g}$. q.e.d.

Using some structure theory for the nilpotent orbits of height 3, one can also show that Conjecture 3·2 holds for such orbits. This will be published elsewhere.

(3·6) Current status of Elashvili's conjecture. The preceding exposition shows that the Conjecture is true for 'large' and 'small' nilpotent orbits. The area in between seems to be a "terra incognita". However, explicit computations, performed by Elashvili for all exceptional Lie algebras but $\mathbb{E}_8$, confirm the conjecture. He also succeeded in proving the conjecture for $\mathfrak{sl}_n$ (personal communication). I hope that the details of his work will be published in the near future.

4 The normaliser of the centraliser of a nilpotent element and its index

In this section, $\mathfrak g$ is a simple Lie algebra, and we study the index of the Lie algebra $\mathfrak n_{\mathfrak g}(e) := \{s \in \mathfrak g \mid [s, \mathfrak z_{\mathfrak g}(e)] \subset \mathfrak z_{\mathfrak g}(e)\}$, $e \in \mathcal N$. The main structure results about $\mathfrak n_{\mathfrak g}(e)$ are obtained Brylinski and Kostant in [**BK1**]. It has to be mentioned that a nice exposition of these results, as well as some complements, is given in Tauvel's book, see [**Ta2**, ch.17]. The following assertion contains, in a condensed form, all what we need from that theory.

4·1 Theorem. *Let $\{e, h, f\}$ be an $\mathfrak{sl}_2$-triple. Then*

(i) *[**BK1**, Theorem 18]* $\mathfrak n_{\mathfrak g}(e) = \mathfrak z_{\mathfrak g}(e) \oplus [f, \mathfrak d_{\mathfrak g}(e)]$*; in particular,* $\dim \mathfrak n_{\mathfrak g}(e) = \dim \mathfrak z_{\mathfrak g}(e) + \dim \mathfrak d_{\mathfrak g}(e)$.

(ii) *[**BK1**, Theorem 23] the Lie algebra $\mathfrak q := \mathfrak n_{\mathfrak g}(e)/\mathfrak z_{\mathfrak g}(e)$ is solvable; the image of the 1-dimensional subspace $\Bbbk h \subset [f, \mathfrak d_{\mathfrak g}(e)]$ is a maximal reductive subalgebra in it.*

(iii) *[**Ta2**, ch.17]* $\mathfrak n_{\mathfrak g}(e) = \{s \in \mathfrak g \mid [s, e] \in \mathfrak d_{\mathfrak g}(e)\}$.

Let $N_{\mathfrak g}(e)$ denote the connected algebraic subgroup of G with Lie algebra $\mathfrak n_{\mathfrak g}(e)$. An immediate consequence of Theorem 4·1(iii) is the claim that

(4·2) $N_{\mathfrak g}(e)$ has an open orbit in $\mathfrak d_{\mathfrak g}(e)$; namely, $N_{\mathfrak g}(e){\cdot}e$ is open in $\mathfrak d_{\mathfrak g}(e)$.

We will be interested in the property that $N_{\mathfrak g}(e)$ has an open orbit in $\mathfrak d_{\mathfrak g}(e)^*$. Recall that if an algebraic group $A \subset GL(V)$ has an open orbit in V, then this does not imply in general that A has an open orbit in V^*. However, the following is true, see [**Py**] :

A has finitely many orbits in V if and only if it has finitely many orbits in V^*.

From this we derive the following sufficient condition:

(4·3) If $N_{\mathfrak g}(e)$ has finitely many orbits in $\mathfrak d_{\mathfrak g}(e)$, then it has an open orbit in $\mathfrak d_{\mathfrak g}(e)^*$.

4·4 Theorem.

(i) *For any $e \in \mathcal N$, we have*

a) $\operatorname{ind} \mathfrak n_{\mathfrak g}(e) \ge \operatorname{ind} \mathfrak z_{\mathfrak g}(e) - \dim \mathfrak d_{\mathfrak g}(e) \ge \operatorname{rk} \mathfrak g - \dim \mathfrak d_{\mathfrak g}(e)$;

b) $\operatorname{ind} (\mathfrak n_{\mathfrak g}(e), \mathfrak z_{\mathfrak g}(e) \ge \operatorname{ind} \mathfrak z_{\mathfrak g}(e) - \dim \mathfrak d_{\mathfrak g}(e)$;

(ii) *suppose $N_{\mathfrak g}(e)$ has an open orbit in $\mathfrak d_{\mathfrak g}(e)^*$. Then*

$$\operatorname{ind} (\mathfrak n_{\mathfrak g}(e), \mathfrak z_{\mathfrak g}(e)) = \operatorname{ind} \mathfrak n_{\mathfrak g}(e) = \operatorname{ind} \mathfrak z_{\mathfrak g}(e) - \dim \mathfrak d_{\mathfrak g}(e).$$

Proof. Choose a basis $(e_1,\dots,e_n)$ for $\mathfrak{z}_{\mathfrak g}(e)$ so that $(e_1,\dots,e_m)$ is a basis for $\mathfrak{d}_{\mathfrak g}(e)$, $m\le n$. Then $(e_1,\dots,e_n,[f,e_1],\dots,[f,e_m])$ is a basis for $\mathfrak{n}_{\mathfrak g}(e)$. Write the matrix $\mathcal{K}(\mathfrak{n}_{\mathfrak g}(e))$ in this basis. It has the following block structure: $\begin{pmatrix} 0 & 0 & \mathfrak{D} \\ 0 & \mathfrak{C} & * \\ -\mathfrak{D}^t & * & * \end{pmatrix}$, where $\mathfrak{D}$ (resp. $\mathfrak{C}$) is a square matrix of order m (resp. $n-m$). Notice that some fragments of $\mathcal{K}(\mathfrak{n}_{\mathfrak g}(e))$ are $\mathcal{K}$-matrices in its own right. Indeed, $\mathcal{K}(\mathfrak{z}_{\mathfrak g}(e)) = \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{C} \end{pmatrix}$ and $\mathcal{K}(\mathfrak{n}_{\mathfrak g}(e),\mathfrak{d}_{\mathfrak g}(e)) = (0\ 0\ \mathfrak{D})$. It follows that $\operatorname{rank}\mathfrak{C} = \dim\mathfrak{z}_{\mathfrak g}(e) - \operatorname{ind}\mathfrak{z}_{\mathfrak g}(e)$ and $\operatorname{rank}\mathfrak{D} = \dim\mathfrak{d}_{\mathfrak g}(e) - \operatorname{ind}(\mathfrak{n}_{\mathfrak g}(e),\mathfrak{d}_{\mathfrak g}(e))$, in view of (1·2).

(i)a The block structure of $\mathcal{K}(\mathfrak{n}_{\mathfrak g}(e))$ shows that

$$\dim\mathfrak{n}_{\mathfrak g}(e) - \operatorname{ind}\mathfrak{n}_{\mathfrak g}(e) = \operatorname{rank}\mathcal{K}(\mathfrak{n}_{\mathfrak g}(e)) \le \operatorname{rank}\mathfrak{C} + 2\,\dim\mathfrak{d}_{\mathfrak g}(e)\ ,$$

which yields the first inequality. The second inequality stems from Corollary 1·7.

(i)b Applying Theorem 1·4 to $\tilde{\mathfrak q} = \mathfrak{n}_{\mathfrak g}(e)$ and $\mathfrak q = \mathfrak{z}_{\mathfrak g}(e)$, we obtain

(4·4·1) $$\operatorname{ind}\mathfrak{z}_{\mathfrak g}(e) + \operatorname{ind}\mathfrak{n}_{\mathfrak g}(e) \le \dim\mathfrak{d}_{\mathfrak g}(e) + 2\operatorname{ind}(\mathfrak{n}_{\mathfrak g}(e),\mathfrak{z}_{\mathfrak g}(e)).$$

Combining this with (a) yields the desired inequality.

(ii) The hypothesis means that $\operatorname{ind}(\mathfrak{n}_{\mathfrak g}(e),\mathfrak{d}_{\mathfrak g}(e)) = 0$, i.e., $\mathfrak{D}$ is non-singular. It also implies that $\operatorname{rank}\mathcal{K}(\mathfrak{n}_{\mathfrak g}(e)) = 2\operatorname{rank}\mathfrak{D} + \operatorname{rank}\mathfrak{C}$. Hence

$$\dim\mathfrak{n}_{\mathfrak g}(e) - \operatorname{ind}\mathfrak{n}_{\mathfrak g}(e) = 2\,\dim\mathfrak{d}_{\mathfrak g}(e) + \dim\mathfrak{z}_{\mathfrak g}(e) - \operatorname{ind}\mathfrak{z}_{\mathfrak g}(e)\ ,$$

which together with Theorem 4·1(1) gives the second equality. To obtain the first equality, notice that $\mathcal{K}(\mathfrak{n}_{\mathfrak g}(e),\mathfrak{z}_{\mathfrak g}(e)) = \begin{pmatrix} 0 & 0 & \mathfrak{D} \\ 0 & \mathfrak{C} & * \end{pmatrix}$ and therefore $\operatorname{rank}\mathcal{K}(\mathfrak{n}_{\mathfrak g}(e),\mathfrak{z}_{\mathfrak g}(e)) = \dim\mathfrak{d}_{\mathfrak g}(e) + \operatorname{rank}\mathfrak{C}$. Thus,

$$\dim\mathfrak{z}_{\mathfrak g}(e) - \operatorname{ind}(\mathfrak{n}_{\mathfrak g}(e),\mathfrak{d}_{\mathfrak g}(e)) = \dim\mathfrak{d}_{\mathfrak g}(e) + \dim\mathfrak{z}_{\mathfrak g}(e) - \operatorname{ind}\mathfrak{z}_{\mathfrak g}(e)\ ,$$

which completes the proof. q.e.d.

Remark. The validity of Conjecture 3·2 would imply that $\operatorname{ind}\mathfrak{n}_{\mathfrak g}(e) = \operatorname{rk}\mathfrak g - \dim\mathfrak{d}_{\mathfrak g}(e)$ under the hypothesis of Theorem 4·4(ii).

It is difficult to directly verify whether $N_{\mathfrak g}(e)$ has an open orbit in $\mathfrak{d}_{\mathfrak g}(e)^*$. However, it appears to be easy to describe a wide class of examples, where $N_{\mathfrak g}(e)$ has finitely many orbits in $\mathfrak{d}_{\mathfrak g}(e)$.
Because h has even eigenvalues on $\mathfrak{d}_{\mathfrak g}(e)$, it is convenient to replace it by $\tilde h = \frac12 h$. Let $1 = m_1 < m_2 \le m_3 \le \dots \le m_l$ be the $\tilde h$-eigenvalues on $\mathfrak{d}_{\mathfrak g}(e)$, counted with multiplicities, i.e., $l = \dim\mathfrak{d}_{\mathfrak g}(e)$. (That $m_2 > 1$ follows from Theorem 2·3.) Let $(e_1,\dots,e_l)$ be a respective basis of $\tilde h$-eigenvectors. Notice that $e_1 = e$.

4·5 Proposition. *The group $N_{\mathfrak g}(e)$ has finitely many orbits in $\mathfrak{d}_{\mathfrak g}(e)$ if and only if*

$$(\heartsuit_1) \qquad [[f,e_j],e_i]=\alpha e_{i+j-1} \textit{ for all } i,j \textit{ such that } i+j-1\le l$$

and for some $\alpha=\alpha_{ij}\in \Bbbk\setminus\{0\}$. *In particular,*

$$(\heartsuit_2) \qquad m_i+m_j-1=m_{i+j-1}$$

and $\tilde h$ *has a simple spectrum on* $\mathfrak d_{\mathfrak g}(e)$.

Proof. Set $Z_G(e)^o = Z_G(e)\cap N_{\mathfrak g}(e)$. It is the uneffectivity kernel for the $N_{\mathfrak g}(e)$-action on $\mathfrak d_{\mathfrak g}(e)$. Letting $Q=N_{\mathfrak g}(e)/Z_G(e)^o$, we conclude from Theorem 4·1(ii) that $Q\simeq T_1\rightthreetimes Q^u$, where T_1 is a 1-dimensional torus and Q^u is the unipotent radical of Q. Of course, we choose T_1 so that all e_i's are eigenvectors of it. Considering the Q-action on $\mathfrak d_{\mathfrak g}(e)$, we may identify $\mathfrak q$, the Lie algebra of Q, with the space $[f,\mathfrak d_{\mathfrak g}(e)]$. Letting $L_i:=\oplus_{j\ge i}\Bbbk e_j$, we obtain a Q-stable complete flag $\mathfrak d_{\mathfrak g}(e)=L_1\supset L_2\supset\cdots\supset L_l$.

a) Suppose conditions $(\heartsuit_1)$ are satisfied. Obviously, these conditions mean that $[\mathfrak q,e_i]=L_i$ for all i. In other words, $Q{\cdot}e_i$ is dense in L_i. More precisely, we have $Q^u{\cdot}e_i=e_i+L_{i+1}$ and $T_1{\cdot}e_i=(\Bbbk\setminus\{0\})e_i$. Thus, $L_i=Q{\cdot}e_i\cup L_{i+1}$, and finiteness follows.

b) Conversely, suppose Q has finitely many orbits in $\mathfrak d_{\mathfrak g}(e)$. Then each L_i must contain a dense orbit. This already implies that all m_i's to be different. Indeed, assume that $m_i=m_{i+1}$ for some i. Let $x=\sum_{j\ge i}x_je_j$ be a generic element of L_i. Then $x\mapsto x_i/x_{i+1}$ is a Q-invariant rational function on L_i, and therefore a dense orbit cannot exist. Thus $m_i<m_{i+1}$ for all i. Next, let $x\in L_i$ be such that $[\mathfrak q,x]=L_i$. Clearly $[[f,e_j],x]=0$ whenever $j>l+1-i$. Writing the condition that the vectors $[[f,e_j],x]$ $(j\le l+1-i)$ are linearly independent, we obtain exactly conditions $(\heartsuit_1)$ with fixed i. q.e.d.

4·6 Corollary (of the proof). *If* $N_{\mathfrak g}(e)$ *has finitely many orbits in* $\mathfrak d_{\mathfrak g}(e)$, *then the number of orbits is* $\dim\mathfrak d_{\mathfrak g}(e)+1$ *and each orbit closure is an affine space.* □

Note that conditions $(\heartsuit_2)$ are vacuous for $i=1$, whereas for $i=2$, we obtain that $\{m_i\}$ is an arithmetic progression with first term 1. For $i>2$, condition $(\heartsuit_2)$ yields no new constraints on $\{m_i\}$.

4·7 Theorem.
Condition 4·5$(\heartsuit_1)$ *is satisfied in the following cases:*

(i) *for every nilpotent element, if* $\mathfrak g\in\{\mathfrak{sl}_n,\mathfrak{sp}_{2n},\mathfrak{so}_{2n+1}\}$;

(ii) *for any nilpotent element whose partition contains at least three parts, if* $\mathfrak g=\mathfrak{so}_{2n}$.

(iii) *if* $\dim\mathfrak d_{\mathfrak g}(e)\le 2$.

In particular, if $e\in\mathcal N$ *falls in either of these cases, then*

a) $N_{\mathfrak g}(e)$ *has finitely many orbits in* $\mathfrak d_{\mathfrak g}(e)$;

b) $\operatorname{ind}\mathfrak n_{\mathfrak g}(e)=\operatorname{ind}(\mathfrak n_{\mathfrak g}(e),\mathfrak z_{\mathfrak g}(e))=\operatorname{ind}\mathfrak z_{\mathfrak g}(e)-\dim\mathfrak d_{\mathfrak g}(e)$.

Proof. Let $e \in \mathfrak{sl}_n$ be a nilpotent matrix and let $\{e,h,f\}$ be an $\mathfrak{sl}_2$-triple. Then, denoting by e^i the usual i^{th} matrix power, one easily proves by induction that

$$[e^i,f] = e^i f - f e^i = \sum_{\alpha+\beta=i-1} e^\alpha h e^\beta \quad \text{and} \quad [h,e^j] = 2j{\cdot}e^j \ .$$

Then

$$[[e^i,f],e^j] = \sum_{\alpha+\beta=i-1} e^\alpha h e^{\beta+j} - \sum_{\alpha+\beta=i-1} e^{\alpha+j} h e^\beta = \sum_{\alpha+\beta=i-1} e^\alpha [h,e^j] e^\beta = 2ij{\cdot}e^{i+j-1} \ .$$

This equality is an incarnation of 4·5($\heartsuit_1$) for classical Lie algebras. Indeed, it is well known (see e.g. [**Ku1**]) that, for $e \in \mathfrak{sl}_n$, $\mathfrak{d}_{\mathfrak g}(e)$ is the span of all nonzero powers e^j. For $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n+1}$, the same holds with the *odd* powers of e. This proves (i). For $\mathfrak{so}_{2n}$, it is known that $\mathfrak{d}_{\mathfrak g}(e)$ is the span of all odd powers if and only if the partition of e has at least three parts, whence (ii). Part (iii) is obvious.
The conclusions a) and b) follows by 4·4(ii) and 4·5. q.e.d.

Example. Take a subregular nilpotent element $e \in \mathfrak{so}_8$. Here $\dim \mathfrak{z}_{\mathfrak g}(e) = 6$, $\dim \mathfrak{d}_{\mathfrak g}(e) = 3$, and the corresponding partition is $(5,3)$, i.e., it has two parts. It is easy to find that here $m_1 = 1, m_2 = m_3 = 3$. Thus, condition 4·5($\heartsuit_2$) is not satisfied for $i = j = 2$. Therefore $N_{\mathfrak g}(e)$ (or Q) has infinitely many orbits in $\mathfrak{d}_{\mathfrak g}(e)$. This shows that assumptions 4·7(ii),(iii) cannot be weakened in general. The fact that $m_2 = m_3$ also implies that $N_{\mathfrak g}(e)$ does not have an open orbit in $\mathfrak{d}_{\mathfrak g}(e)^*$.

5 The normaliser of the centraliser of a regular nilpotent element

As above, $\mathfrak g$ is simple. In this section, $e \in \mathcal N$ is a regular (= principal) nilpotent element. In this case, an $\mathfrak{sl}_2$-triple $\{e,h,f\}$ is said to be *principal*. It is worth noting that principal $\mathfrak{sl}_2$-triples were first considered by E.B. Dynkin in [**Dy1**]. Our aim in this section is to prove that $\mathfrak{n}_{\mathfrak g}(e)$ is a Frobenius Lie algebra. Note that, for $\mathfrak{sl}_n, \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$, and $\mathbb{G}_2$, this already follows from Theorem 4·7. But the proof to be given below is absolutely general and does not appeal to classification. Along the way we prove several properties of principal $\mathfrak{sl}_2$-triples, which seem to be new.
We keep the notation of Section 2. Recall that $p = \operatorname{rk}\mathfrak g$ and $\{\alpha_1,\dots,\alpha_p\}$ are the simple roots. Without loss of generality, we assume that $h = h_+$ (see Section 2). Then $\mathfrak g(0) = \mathfrak t$ and $\mathfrak g(2i) = \oplus_{\mathrm{ht}\,(\alpha)=i}\mathfrak g_\alpha$, where $\mathrm{ht}\,(\alpha)$ is the usual height of a root α. In particular, $\mathfrak g(2) = \oplus_{\alpha\in\Pi}\mathfrak g_\alpha$. Recall some standard invariant theoretic features of this situation, see [**Ko1**], [**Ko2**].

- The eigenvalues of h on $\mathfrak{z}_{\mathfrak g}(e)$ are $\{2m_1,\dots,2m_p\}$, where $\{m_1,\dots,m_p\}$ are the *exponents* of (the Weyl group of) $\mathfrak g$. We order the exponents so that $1 = m_1 < m_2 \le \dots \le m_p$;

- $\Bbbk[\mathfrak{g}]^G$ is a graded polynomial algebra. The degrees of free generators are $\{m_i+1\}_{i=1,\dots,p}$. The quotient map $\pi_G : \mathfrak{g} \to \mathfrak{g}/\!\!/G$ is flat and $\pi_G^{-1}(\pi_G(0)) = \mathcal{N}$;
- Let M be any $\mathrm{ad}\, h$-stable complement of $[\mathfrak{g}, e]$ in $\mathfrak{g}$. Then the eigenvalues of h on M are $\{-2m_1, \dots, -2m_p\}$. The affine subspace $e+M$ is transversal to $G{\cdot}e$ at e. The restriction homomorphism $\Bbbk[\mathfrak{g}] \to \Bbbk[e+M]$ induces an isomorphism $\Bbbk[\mathfrak{g}]^G \xrightarrow{\sim} \Bbbk[e+M]$. This means, in particular, that $(e+M)\cap\mathcal{N} = \{e\}$. Such an affine subspace is said to be a *special transversal slice* (at e).

The usual choice for M is $\mathfrak{z}_\mathfrak{g}(f)$; however we shall see that there are some other natural possibilities.
For each $\alpha \in \Delta$, define $h_\alpha \in \mathfrak{t}$ by the formula $\Phi(h_\alpha, x) = \alpha(x)$ for all $x \in \mathfrak{t}$. Let e_α be a nonzero element in $\mathfrak{g}_\alpha$. We normalise them so that $\Phi(e_\alpha, e_{-\alpha}) = 1$. Then $[e_\alpha, e_{-\alpha}] = h_\alpha$. We have $\mathrm{ht}\,(e) = 2m_p$ and $\mathfrak{g}(2m_p) = \mathfrak{g}_\lambda$, where λ is the highest root in Δ_+. Since $\dim\mathfrak{g}(2) - \dim\mathfrak{g}(4) = 1$, Springer's result [**Sp**, 9.6] says that $c = e + e_{-\lambda}$ is regular semisimple, cf. the proof of Theorem 3·3. Applying Claim 3·3·2 in this situation, we see that, for $e_\lambda \in \mathfrak{z}_\mathfrak{g}(e)$, there exists a unique $\hat f \in \mathfrak{u}_-$ such that $e_\lambda + \hat f \in \mathfrak{z}_\mathfrak{g}(c)$.

5·1 Proposition. (i) *$\hat f$ lies in $\mathfrak{g}(-2)$; it is a regular nilpotent element.*
(ii) $\mathfrak{z}_\mathfrak{g}(c) \subset \mathfrak{z}_\mathfrak{g}(e) \oplus \mathfrak{z}_\mathfrak{g}(\hat f)$.

Proof. (i) The element $\hat f \in \mathfrak{u}_-$ is being determined by the relation

$$[e, \hat f] = [e_\lambda, e_{-\lambda}] = h_\lambda \in \mathfrak{g}(0)\ .$$

This already shows that $\hat f \in \mathfrak{g}(-2)$. Letting $e = \sum_{i=1}^p X_i e_{\alpha_i}$ and $\hat f = \sum_{i=1}^p Y_i e_{-\alpha_i}$, we obtain $[e, \hat f] = \sum_{i=1}^p (X_iY_i) h_{\alpha_i}$. On the other hand, $h_\lambda = \sum_{i=1}^p n_i h_{\alpha_i}$, where $\{n_i\}$ are defined by the equality $\lambda = \sum_{i=1}^p n_i\alpha_i$. Hence all $n_i \ne 0$, Since e is regular, we have all $X_i \ne 0$. It follows that all $Y_i \ne 0$, too. Thus, $\hat f$ is regular as well.
(ii) Claim 3·3·2 applied in our situation gives the following: For any $x \in \mathfrak{z}_\mathfrak{g}(e)$ there exists a unique $z \in \mathfrak{u}_-$ such that $x + z \in \mathfrak{z}_\mathfrak{g}(c)$. Since $\dim\mathfrak{z}_\mathfrak{g}(e) = \dim\mathfrak{z}_\mathfrak{g}(c)$, this construction gives the whole centraliser of c. Thus, each $y \in \mathfrak{z}_\mathfrak{g}(c)$ has only 'positive' and 'negative' parts; $y = y_+ + y_-$, where $y_+ \in \mathfrak{u}_+$ and $y_- \in \mathfrak{u}_-$. The totality of all positive parts forms $\mathfrak{z}_\mathfrak{g}(e)$. Because $\mathfrak{z}_\mathfrak{g}(c)$ is Abelian, the totality of all negative parts forms an Abelian subalgebra of dimension p. Since $\hat f$ is one of such negative parts and it is regular, the totality of all negative parts is precisely $\mathfrak{z}_\mathfrak{g}(\hat f)$. q.e.d.

5·1·1 Remark. Because $\dim[\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}] = 1$, the line $\Bbbk\hat f$ and the centraliser $\mathfrak{z}_\mathfrak{g}(\hat f)$ do not depend on the particular choice of e_λ and $e_{-\lambda}$. It is easily seen that $\hat f$ is proportional to f if and only if h_λ is proportional to h if and only if $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{sl}_3$. That is, in general, these two elements are completely different. Nevertheless, since $\hat f$ is regular and $\hat f \in \mathfrak{g}(-2)$, the space $\mathfrak{z}_\mathfrak{g}(\hat f)$ is $\mathrm{ad}\, h$-stable and the eigenvalues of h on $\mathfrak{z}_\mathfrak{g}(\hat f)$ are the same as on $\mathfrak{z}_\mathfrak{g}(f)$.

Until Theorem 5·4 we will work with $\Bbbk = \mathbb{C}$. Let θ be an involution of $\mathfrak{g}$ of maximal rank that acts as $-id$ on $\mathfrak{t} = \mathfrak{g}(0)$. Let σ be an associated complex conjugation. Then $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ and $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_\alpha$ for each $\alpha \in \Delta$. Note that $\mathfrak{g}^\sigma$ is a split real form of $\mathfrak{g}$ and all h_α ($\alpha \in \Delta$) lie in $\mathfrak{g}^\sigma$. In view of Proposition 2·5, we may assume that $e, h, f \in \mathfrak{g}^\sigma$ and $\theta(e) = -f$. Set $\mathfrak{t}_\mathbb{R} = \mathfrak{t} \cap \mathfrak{g}^\sigma$. Then $\mathfrak{t} = \mathfrak{t}_\mathbb{R} \oplus \sqrt{-1}\,\mathfrak{t}_\mathbb{R}$ and $\sqrt{-1}\,\mathfrak{t}_\mathbb{R} \subset \mathfrak{g}^{\sigma\theta}$. Set $T_\mathbb{R} := exp(\mathfrak{t}_\mathbb{R})$ and $T_c = exp(\sqrt{-1}\,\mathfrak{t}_\mathbb{R})$. Then $T = T_\mathbb{R}{\cdot}T_c$ is a maximal torus in G.

5·2 Lemma. *Suppose $t \in T_\mathbb{R}$. Then Φ is nondegenerate on $\mathfrak{z}_\mathfrak{g}(e) \oplus t{\cdot}\mathfrak{z}_\mathfrak{g}(f)$.*

Proof. Take any $s \in T$ such that $s^2 = t$. Writing $s = s_\mathbb{R} s_c$ (in the obvious notation), we obtain $(s_c)^2 = 1 \in T$. Set $V = \mathfrak{z}_\mathfrak{g}(e) \oplus t{\cdot}\mathfrak{z}_\mathfrak{g}(f)$. Then $s^{-1}{\cdot}V = s^{-1}{\cdot}\mathfrak{z}_\mathfrak{g}(e) \oplus s{\cdot}\mathfrak{z}_\mathfrak{g}(f)$. Since $\sigma\theta(s^{-1}{\cdot}\mathfrak{z}_\mathfrak{g}(e)) = (s_c^{-2}s){\cdot}\mathfrak{z}_\mathfrak{g}(f) = s{\cdot}\mathfrak{z}_\mathfrak{g}(f)$, the space $s^{-1}{\cdot}V$ is the complexification of a space in the compact real form $\mathfrak{g}^{\sigma\theta}$. Thus, Φ is nondegenerate on $s^{-1}{\cdot}V$ and hence on V. q.e.d.

The following result says that $\hat{f}$ can be used in place of f for constructing a special transversal slice to $G{\cdot}e$ at e.

5·3 Proposition. $\mathfrak{z}_\mathfrak{g}(\hat{f}) \oplus [\mathfrak{g}, e] = \mathfrak{g}$, *or equivalently, Φ is nondegenerate on $\mathfrak{z}_\mathfrak{g}(e) \oplus \mathfrak{z}_\mathfrak{g}(\hat{f})$.*

Proof. Since both f and $\hat{f}$ are regular nilpotent elements in $\mathfrak{g}(-2)$, there exists a unique $t \in T$ such that $t{\cdot}f = \hat{f}$. To order to define $\hat{f}$, we used the relation $[e, \hat{f}] = h_\lambda$. Because $\mathfrak{h}_\lambda \in \mathfrak{g}^\sigma$, we have $\hat{f} \in \mathfrak{g}^\sigma$ as well, and therefore $t \in T_\mathbb{R}$. Since $\mathfrak{z}_\mathfrak{g}(\hat{f}) = t{\cdot}\mathfrak{z}_\mathfrak{g}(f)$, we conclude by Lemma 5·2. q.e.d.

Yet another possibility for a special transversal slice is provided by the next theorem.

5·4 Theorem. $[e_{-\lambda}, [f, \mathfrak{z}_\mathfrak{g}(e)]] \oplus [\mathfrak{g}, e] = \mathfrak{g}$.

Proof. Set $\mathcal{A} = [e_{-\lambda}, [f, \mathfrak{z}_\mathfrak{g}(e)]]$ and $\mathcal{B} = [e_{-\lambda}, \mathfrak{z}_\mathfrak{g}(e)]$. We begin with proving that $\mathcal{A}$ has the required dimension, i.e., p. Since $c = e_{-\lambda} + e$ is regular semisimple, $\mathfrak{z}_\mathfrak{g}(e_{-\lambda}) \cap \mathfrak{z}_\mathfrak{g}(e) = \{0\}$. Hence $\dim \mathcal{B} = p$. Because $\mathfrak{z}_\mathfrak{g}(e)$ is Abelian, we see that $\mathcal{B}$ is orthogonal to $\mathfrak{z}_\mathfrak{g}(e)$ with respect to the Killing form. In other words, $\mathcal{B} \subset [\mathfrak{g}, e]$ and hence $\mathcal{B} \cap \mathfrak{z}_\mathfrak{g}(f) = \{0\}$. Consequently, $\dim \mathcal{A} = p$, as $\mathcal{A} = [f, \mathcal{B}]$.
Using the fact the Cartan subalgebra $\mathfrak{z}_\mathfrak{g}(c)$ is "diagonally" embedded in $\mathfrak{z}_\mathfrak{g}(e) \oplus \mathfrak{z}_\mathfrak{g}(\hat{f})$, we obtain

$$\mathcal{B} = [e_{-\lambda} + e, \mathfrak{z}_\mathfrak{g}(e)] = [e_{-\lambda} + e, \mathfrak{z}_\mathfrak{g}(\hat{f})] = [e, \mathfrak{z}_\mathfrak{g}(\hat{f})] \ .$$

It follows that $\mathcal{A} = [f, [e, \mathfrak{z}_\mathfrak{g}(\hat{f})]]$. Given $x \in \mathfrak{z}_\mathfrak{g}(\hat{f})$, we have $[f, [e, x]] = -[h, x] + [e, [f, x]]$. Since $\mathfrak{z}_\mathfrak{g}(h) \cap \mathfrak{z}_\mathfrak{g}(\hat{f}) = \{0\}$, each nonzero element of $\mathcal{A}$ is a sum of a nonzero element in $\mathfrak{z}_\mathfrak{g}(\hat{f})$ and an element in $[\mathfrak{g}, e]$. Thus, the validity of Theorem 5·4 follows from that of Proposition 5·3. q.e.d.

The following is our main result in this section.

5·5 Theorem. *$\mathfrak{n}_{\mathfrak{g}}(e)$ is a Frobenius Lie algebra; more precisely, $e_{-\lambda}$, considered as element of $\mathfrak{n}_{\mathfrak{g}}(e)^*$, is a representative of the open $N_{\mathfrak{g}}(e)$-orbit.*

Proof. Using the Killing form, each $x \in \mathfrak{g}$ can be considered as element of the dual space $\mathfrak{n}_{\mathfrak{g}}(e)^*$.
Let us describe a convenient model for $\mathfrak{n}_{\mathfrak{g}}(e)^*$. In our situation, $\mathfrak{n}_{\mathfrak{g}}(e) = \mathfrak{z}_{\mathfrak{g}}(e) \oplus [f, \mathfrak{z}_{\mathfrak{g}}(e)] = \mathrm{Ker}\,(\mathrm{ad}\, e)^2$. It follows that $\mathfrak{n}_{\mathfrak{g}}(e)^{\perp} = \mathrm{Im}\,(\mathrm{ad}\, e)^2$. Since $\mathfrak{g} = \mathrm{Im}\,(\mathrm{ad}\, e)^2 \oplus \mathrm{Ker}\,(\mathrm{ad}\, f)^2$ (which actually holds for any distinguished element), we may identify $\mathfrak{n}_{\mathfrak{g}}(e)^*$ with $\mathrm{Ker}\,(\mathrm{ad}\, f)^2$. The $\mathfrak{n}_{\mathfrak{g}}(e)$-module structure on $\mathrm{Ker}\,(\mathrm{ad}\, f)^2$ is given by the usual Lie bracket in $\mathfrak{g}$ coupled with the subsequent projection to $\mathrm{Ker}\,(\mathrm{ad}\, f)^2$, with kernel $\mathrm{Im}\,(\mathrm{ad}\, e)^2$.

Using the notation of the proof of Theorem 5·4, we have $[e_{-\lambda}, \mathfrak{n}_{\mathfrak{g}}(e)] = \mathcal{A} + \mathcal{B}$. It is already proved therein that $\dim \mathcal{A} = \dim \mathcal{B} = p$, $\mathcal{B} \subset [\mathfrak{g}, e]$, and $\mathcal{A} \cap [\mathfrak{g}, e] = \{0\}$. Hence $\dim(\mathcal{A}+\mathcal{B}) = 2p$. Thus, we only need to show that $(\mathcal{A}+\mathcal{B}) \cap \mathrm{Im}\,(\mathrm{ad}\, e)^2 = \{0\}$. As was noted in the proof of Theorem 5·4, a stronger condition holds for$\mathcal{A}$; namely, $\mathcal{A} \cap \mathrm{Im}\,(\mathrm{ad}\, e) = \{0\}$. Next, we can restate Theorem 5·4 as follows: Φ induces a nondegenerate pairing between $\mathcal{A} = [f, \mathcal{B}]$ and $\mathfrak{z}_{\mathfrak{g}}(e) = [\mathfrak{g}, e]^{\perp}$. Using the invariance of Φ, this also yields a nondegenerate pairing between $\mathcal{B}$ and $[f, \mathfrak{z}_{\mathfrak{g}}(e)]$. It follows that $\mathcal{B} \cap [f, \mathfrak{z}_{\mathfrak{g}}(e)]^{\perp} = \{0\}$. Since $[f, \mathfrak{z}_{\mathfrak{g}}(e)] \subset \mathfrak{n}_{\mathfrak{g}}(e)$, we see that $\mathcal{B} \cap \mathfrak{n}_{\mathfrak{g}}(e)^{\perp} = \{0\}$. This means that the image of $[e_{-\lambda}, \mathfrak{n}_{\mathfrak{g}}(e)]$ in $\mathfrak{g}/\mathfrak{n}_{\mathfrak{g}}(e)^{\perp} = \mathfrak{n}_{\mathfrak{g}}(e)^*$ is of dimension $2p$, which completes the proof. q.e.d.

Making use of the matrix approach to the index, we can deduce from Theorem 5·5 that a certain matrix is nonsingular.

5·6 Corollary. *Let $\{e_1, \dots, e_p\}$ be a basis for $\mathfrak{z}_{\mathfrak{g}}(e)$. Then*

(i) *the $p \times p$ matrix $\mathfrak{D} = \big([e_i, [e_j, f]]\big)_{i,j=1}^p$ is nonsingular and symmetric. (The entries of $\mathfrak{D}$, which belong to $\mathfrak{z}_{\mathfrak{g}}(e)$, are regarded as elements of the symmetric algebra $\mathcal{S}(\mathfrak{z}_{\mathfrak{g}}(e))$);*

(ii) *up to a constant multiple, $\det(\mathfrak{D}) = (e_{\lambda})^p$. In particular, the matrix $\mathfrak{D}(e_{-\lambda}) = \big(\Phi([e_i, [e_j, f]], e_{-\lambda})\big)_{i,j=1}^p$ is nonsingular.*

Proof. The symmetricity of $\mathfrak{D}$ is obvious. By Theorem 4·1(i), $(e_1, \dots, e_p, [e_1, f], \dots, [e_p, f])$ is a basis for $\mathfrak{n}_{\mathfrak{g}}(e)$. The matrix $\mathcal{K}(\mathfrak{n}_{\mathfrak{g}}(e))$ in this basis is $\begin{pmatrix} 0 & \mathfrak{D} \\ -\mathfrak{D}^t & * \end{pmatrix}$. This proves (i).
By Theorem 5·5, $\mathcal{K}(\mathfrak{n}_{\mathfrak{g}}(e))$ remains nonsingular after having been evaluated against $e_{-\lambda} \in \mathfrak{n}_{\mathfrak{g}}(e)^*$. Hence $\mathfrak{D}(e_{-\lambda})$ is nonsingular. Furthermore, $\det(\mathfrak{D})$ vanishes precisely on the complement of the open $N_{\mathfrak{g}}(e)$-orbit in $\mathfrak{n}_{\mathfrak{g}}(e)$. In our situation, $e_{-\lambda} \in \mathfrak{n}_{\mathfrak{g}}(e)^*$ is the unique weight vector of minimal weight with respect to h. Let $W \subset \mathfrak{n}_{\mathfrak{g}}(e)^*$ be *the* $\mathrm{ad}\, h$-stable complementary hyperplane. Since $N_{\mathfrak{g}}(e)$ is a semi-direct product of a 1-dimensional torus and a unipotent group, it is easily seen that $\mathfrak{n}_{\mathfrak{g}}(e)^* \setminus N_{\mathfrak{g}}(e){\cdot}e_{-\lambda} = W$ (cf. proof of Theorem 4·4). Because W is the annihilator of e_{λ}, we are done. q.e.d.

More specifically, let us choose a basis for $\mathfrak{z}_{\mathfrak{g}}(e)$ so that $e_i \in \mathfrak{g}(2m_i)$. Then $d_{ij} := [e_i, [e_j, f]] \in \mathfrak{z}_{\mathfrak{g}}(e)(2m_i+2m_j-2)$. Therefore it is 0 whenever $m_i+m_j-1 > m_p$. Recall that the exponents satisfy the property that $m_i+m_{p+1-i} = m_p+1$ for all i (this sum is called the *Coxeter number* of $\mathfrak{g}$). Consequently, if the exponents are different, then $d_{ij} = 0$ for $i + j > p + 1$. That is, all entries below the antidiagonal are equal to 0. Since $\mathfrak{D}$ is nonsingular, the antidiagonal entries must be non-zero. Thus, we derive by 'grading' reasons the following.

5·7 Corollary. *Suppose all the exponents of $\mathfrak{g}$ are different, and let $e_i \in \mathfrak{z}_{\mathfrak{g}}(e)(2m_i)$. Then $[e_i, [e_{p+1-i}, f]] = \mu_i e_\lambda$ with some $\mu_i \in \Bbbk \setminus \{0\}$.*

For $\mathfrak{g} = \mathfrak{so}_{4n}$, there are two equal exponents. However, it is also possible to choose basis vectors in the corresponding 2-dimensional space so that the whole $\mathfrak{D}$ to be triangular.

6 Some open problems

In this section, $x \in \mathfrak{g}$ is an arbitrary nilpotent element. It is proved in Section 4 that, under certain assumptions, $\operatorname{ind}\mathfrak{n}_{\mathfrak{g}}(x) = \operatorname{ind}\mathfrak{z}_{\mathfrak{g}}(x) - \dim\mathfrak{d}_{\mathfrak{g}}(x)$. Since those assumptions are satisfied for 'almost all' classical cases, it is tempting to assume that this phenomenon is completely general. Combining this with Elashvili's conjecture, we arrive at the following nice statement.

6·1 Conjecture. *Let $x \in \mathcal{N} \subset \mathfrak{g}$. Then* $\operatorname{ind}\mathfrak{n}_{\mathfrak{g}}(x) = \operatorname{rk}\mathfrak{g} - \dim\mathfrak{d}_{\mathfrak{g}}(x)$.

If true, this conjecture would give an explanation to the fact observed by Kurtzke [**Ku2**] that $\dim\mathfrak{d}_{\mathfrak{g}}(x) \le \operatorname{rk}\mathfrak{g}$ for all x. It is worth noting that $\dim\mathfrak{d}_{\mathfrak{g}}(x) = \operatorname{rk}\mathfrak{g}$ if and only if either x is regular of $\mathfrak{g} = \mathbb{G}_2$ and x is subregular. So, the conjecture implies that $\mathfrak{n}_{\mathfrak{g}}(x)$ is Frobenius only in these two cases.
Actually, I believe that the following stronger assertion is true.

6·2 Conjecture. $\operatorname{ind}(\mathfrak{n}_{\mathfrak{g}}(x), \mathfrak{z}_{\mathfrak{g}}(x)) = \operatorname{rk}\mathfrak{g} - \dim\mathfrak{d}_{\mathfrak{g}}(x)$.

Indeed, it follows from Theorem 4·4(i) and (4·4·1) that Conjecture 6·2 implies (6·1).

The previous exposition shows that the last conjecture is valid in two extreme cases:

a) x is regular (by Theorem 5·5);

b) $\operatorname{ht}(x) = 2$ (here $\mathfrak{d}_{\mathfrak{g}}(x) = \Bbbk x$, by Theorem 2·3. Then combine Theorem 3·5 and Theorem 4·4(ii)).

A proof of the last conjecture may consists of two steps. The first step is to prove Elashvili's conjecture, and the second is to prove the equality $\operatorname{ind}(\mathfrak{n}_{\mathfrak{g}}(x), \mathfrak{z}_{\mathfrak{g}}(x)) = \operatorname{ind}\mathfrak{z}_{\mathfrak{g}}(x) - \dim\mathfrak{d}_{\mathfrak{g}}(x)$. The latter has a transparent geometric meaning. By dimension reason, we have the inequality

$$\max_{\eta\in\mathfrak{z}_{\mathfrak{g}}(x)^*} \dim\mathfrak{z}_{\mathfrak{g}}(x){\cdot}\eta + \dim\mathfrak{d}_{\mathfrak{g}}(x) \ge \max_{\xi\in\mathfrak{z}_{\mathfrak{g}}(x)^*} \dim\mathfrak{n}_{\mathfrak{g}}(x){\cdot}\xi \ ,$$

and the equality in the second step is the same as equality here.

• It would be interesting to find proofs for Propositions 2·4 and 5·3 that do not appeal to $\mathbb{C}$. Note in this regard that Lemma 5·2 is wrong for arbitrary $t \in T$.

• In conclusion, I state a problem related to parabolic subalgebras. Corollary 1·5 says that $\operatorname{ind}\mathfrak{p}+\operatorname{ind}\mathfrak{p}^u \le \dim\mathfrak{l}$. It is likely that there is also a lower bound, namely, $\operatorname{ind}\mathfrak{p}+\operatorname{ind}\mathfrak{p}^u \ge \operatorname{rk}\mathfrak{g}$.

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