Periodic contractions of semisimple Lie algebras and their invariants

D. Panyushev

Independent University of Moscow, Russia
(z.Z. MPIM)

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Chevalley, Kostant, Vinberg

$\Bbbk = \bar{\Bbbk}$, $\mathfrak{g}$ semisimple, G adjoint, $r = \mathrm{rk}(\mathfrak{g})$, $\pi_G : \mathfrak{g} \to \mathfrak{g}/\!\!/G = \mathrm{Spec}\,\Bbbk[\mathfrak{g}]^G$ – the quotient morphism; $\mathfrak{g}_{reg} := \{x \in \mathfrak{g} \mid \dim G{\cdot}x = \dim \mathfrak{g} - r\}$;

- $\Bbbk[\mathfrak{g}]^G = \Bbbk[f_1, \dots, f_r]$ is a polynomial algebra;
- $\pi_G^{-1}(0) = \mathfrak{N}$ – the set of nilpotent elements of $\mathfrak{g}$;
- π_G is equidimensional, $\dim \pi_G^{-1}(\xi) = \dim \mathfrak{g} - r$;
- each fibre contains a dense G-orbit.

Kostant-Rallis (1971), $\mathbb{Z}_2$-gradings:
$\sigma \in \mathrm{Inv}(\mathfrak{g}) \ \Rightarrow\ \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $G \supset G_0$; $\quad G_0 \to SO(\mathfrak{g}_1)$.

Vinberg (1975), periodic gradings:
$\theta \in \mathrm{Aut}(\mathfrak{g})$, $\theta^m = id \ \Rightarrow\ \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_{m-1}$; $\quad G_0 \to GL(\mathfrak{g}_1)$

There are similar results for the G_0-action on $\mathfrak{g}_1$ and $\pi : \mathfrak{g}_1 \to \mathfrak{g}_1/\!\!/G_0$, e.g. $\pi^{-1}(0) = \mathfrak{N} \cap \mathfrak{g}_1$ and $\Bbbk[\mathfrak{g}_1]^{G_0}$ is polynomial.

Contractions of Lie algebras

Let $\mathfrak{q}$ be a Lie algebra of dimension n and $c : \Bbbk^\times \to GL_n$ a polynomial mapping. Define the new Lie algebra structure on the vector space $\mathfrak{q}$ by

$$[x, y]_{(t)} := c_t^{-1}[c_t(x), c_t(y)], \quad t \in \Bbbk^\times.$$

Then $\mathfrak{q}_{(1)} = \mathfrak{q}$ and all algebras $\mathfrak{q}_{(t)}$ are isomorphic. If $\lim_{t\to 0}[x, y]_{(t)}$ exists for all $x, y \in \mathfrak{q}$, then we obtain a new Lie algebra, say $\mathfrak{s}$, which is called a contraction of $\mathfrak{q}$. We write $\lim_{t\to 0} \mathfrak{q}_{(t)} = \mathfrak{s}$ or $\mathfrak{q} \rightsquigarrow \mathfrak{s}$. Conversely, $\mathfrak{q}$ is a deformation of $\mathfrak{s}$.

Examples

1. $\mathfrak{q} \supset \mathfrak{h}$ are Lie algebras. Take any decomposition $\mathfrak{q} = \mathfrak{h} \oplus \mathfrak{p}$, and set $c_t(h + p) = h + tp$, $h \in \mathfrak{h}$, $p \in \mathfrak{p}$. Then $\lim_{t\to 0} \mathfrak{q}_{(t)} = \mathfrak{h} \ltimes \mathfrak{q}/\mathfrak{h}$.
   If $\mathfrak{p}$ is an $\mathfrak{h}$-module, then $\mathfrak{q} \rightsquigarrow \mathfrak{h} \ltimes \mathfrak{p}$.
2. For any $\theta \in \mathrm{Aut}(\mathfrak{q})$ of finite order m, one can construct a contraction, which is called periodic or a $\mathbb{Z}_m$-contraction.

Q is a connected algebraic group, $\mathfrak{q} = \mathrm{Lie}\,(Q)$.

Notation: $\mathrm{Inv}(\mathfrak{q}, \mathrm{ad}\,) := \Bbbk[\mathfrak{q}]^Q$, $\mathrm{Inv}(\mathfrak{q}, \mathrm{ad}^*) := \Bbbk[\mathfrak{q}^*]^Q = \mathcal{S}(\mathfrak{q})^Q$

Suppose $\mathfrak{q}$ is a contraction of a semisimple Lie algebra $\mathfrak{g}$.

Problem: When is $\Bbbk[\mathfrak{q}]^Q$ or $\Bbbk[\mathfrak{q}^*]^Q$ polynomial?

Simplest contractions lead to semi-direct products; therefore we begin with considering invariants of arbitrary semi-direct products.

R is a connected algebraic group, $\phi : R \to GL(V) \Rightarrow \mathfrak{r} \to \mathfrak{gl}(V)$.
Let $\mathfrak{q} = \mathfrak{r} \ltimes V$ be the semi-direct product of $\mathfrak{r}$ and V. Then

- $Q = R \ltimes V$ is the corresponding connected algebraic group;
- $1 \ltimes V$ is a commutative unipotent normal subgroup of Q (and $0 \ltimes V$ is a nilpotent ideal of $\mathfrak{q}$).

The adjoint representation of Q is given by:

$$\mathrm{ad}\,(z,u){\cdot}(x,v) = ((\mathrm{ad}\,z)x, z{\cdot}v - x{\cdot}u), \quad u,v \in V,\ z \in R, x \in \mathfrak{r}.$$

Examples (interesting cases)

- $\mathfrak{r} \dotplus \mathfrak{r} \rightsquigarrow \mathfrak{r} \ltimes \mathfrak{r}$ – Takiff Lie algebra;
- $\mathfrak{r} \ltimes \mathfrak{r}^*$ – quadratic Lie algebra;
- $\mathfrak{so}_{n+1} \rightsquigarrow \mathfrak{so}_n \ltimes \Bbbk^n$, the Lie algebra of the affine orthogonal group;
- $\mathfrak{so}(1,3) \ltimes \mathbb{R}^4$ is the Lie algebra of the Poincaré group.

Ingredients for $\mathrm{Inv}(\mathfrak{r} \ltimes V, \mathrm{ad}\,)$:

- $\mathfrak{r} \ltimes V \to \mathfrak{r} \ltimes V/(0 \ltimes V) \simeq \mathfrak{r}$ is Q-equivariant $\Rightarrow$ $\Bbbk[\mathfrak{r}]^R \hookrightarrow \mathrm{Inv}(\mathfrak{r} \ltimes V, \mathrm{ad}\,)$.
- $F \in \mathrm{Mor}_R(\mathfrak{r}, V^*)$ – covariant. Define $\hat{F} \in \Bbbk[\mathfrak{r} \ltimes V]$ by $\hat{F}(x, v) := \langle F(x), v\rangle$ $(x \in \mathfrak{r}, v \in V)$.

Lemma

$\hat{F} \in \mathrm{Inv}(\mathfrak{r} \ltimes V, \mathrm{ad}\,)$

Proof reduces to assertion that $x{\cdot}F(x) = 0$ for all $x \in \mathfrak{r}$. $\square$

- $\mathrm{Mor}_R(\mathfrak{r}, V^*) \simeq (\Bbbk[\mathfrak{r}] \otimes V^*)^R$ and it is a graded $\Bbbk[\mathfrak{r}]^R$-module. (It is a finitely generated module whenever $\mathfrak{r}$ is reductive.)

Ingredients for $\mathrm{Inv}(\mathfrak{r} \ltimes V, \mathrm{ad}^*)$:

- $(\mathfrak{r} \ltimes V)^* \to V^*$ is Q-equivariant $\Rightarrow \Bbbk[V^*]^R \hookrightarrow \mathrm{Inv}(\mathfrak{r} \ltimes V, \mathrm{ad}^*)$.
- for $F \in \mathrm{Mor}_R(V^*, \mathfrak{r})$, define $\check{F} \in \Bbbk[(\mathfrak{r} \ltimes V)^*]$ by $\check{F}(\eta, \xi) := \langle F(\xi), \eta\rangle$ $(\eta \in \mathfrak{r}^*, \xi \in V^*)$

- $\check{F}$ is R-invariant, but is not always Q-invariant!
- $\check{F}$ is Q-invariant $\Leftrightarrow$ $F(\xi) \in \mathfrak{r}_\xi \ \ \forall \xi \in V^*$
- Then, by R-equivariance of F, we obtain $[F(\xi), \mathfrak{r}_\xi] = 0$, i.e., $F(\xi)$ belongs to the centre of $\mathfrak{r}_\xi$.
  Hence if such a covariant F exists, then for generic $\xi \in V^*$, $\mathfrak{r}_\xi$ has a non-trivial centre.

Lemma

Both $\mathrm{Inv}(\mathfrak{r} \ltimes V, \mathrm{ad})$ and $\mathrm{Inv}(\mathfrak{r} \ltimes V, \mathrm{ad}^*)$ are bi-graded.

The "reductive" adjoint case: $(\mathfrak{q} = \mathfrak{g} \ltimes V, \mathrm{ad})$.

Theorem (A)

For any G-module V, $\mathrm{Inv}(\mathfrak{g} \ltimes V, \mathrm{ad})$ *is polynomial, of Krull dimension* $\mathrm{rk}(\mathfrak{g}) + \dim V^T$. *It is generated by* $\Bbbk[\mathfrak{g}]^G$ *and "covariants".*

- let $f_1, \dots, f_r$ be the free generators of $\Bbbk[\mathfrak{g}]^G$;
- let $F_1, \dots, F_m$ be a basis of the free $\Bbbk[\mathfrak{g}]^G$-module $\mathrm{Mor}_G(\mathfrak{g}, V^*)$;
- $\varphi = (f_1, \dots, f_r, \hat{F}_1, \dots, \hat{F}_m) : \mathfrak{g} \ltimes V \to \mathbb{A}^{m+r}$ is surjective, and generic fibre of φ contains a dense Q-orbit.
- This implies that $\mathrm{Inv}(\mathfrak{g} \ltimes V, \mathrm{ad}) \simeq \Bbbk[f_1, \dots, f_r, \hat{F}_1, \dots, \hat{F}_m]$. □

[One needs: $x \in \mathfrak{g}_{reg} \Rightarrow F_1(x), \dots, F_m(x) \in V^*$ are linearly independent.]

A stronger result:

$\Bbbk[\mathfrak{g} \ltimes V]^{Q^u} = \Bbbk[\mathfrak{g}][\hat{F}_1, \dots, \hat{F}_m]$ is also a polynomial algebra! ($Q^u = 1 \ltimes V$.)

Hence $\Bbbk[\mathfrak{g} \ltimes V]^Q = \Bbbk[\mathfrak{g}]^G[\hat{F}_1, \dots, \hat{F}_m]$.

The “reductive” coadjoint case: $(\mathfrak{q} = \mathfrak{g} \ltimes V,\ \mathrm{ad}^*)$.

Bad:
- The covariant method doesn’t work properly;
- $\mathrm{Inv}(\mathfrak{g} \ltimes V, \mathrm{ad}^*)$ is free $\Rightarrow \Bbbk[V^*]^G$ is free;

Good:
- $\mathcal{S}(\mathfrak{q})$ is a Poisson algebra;
- If generic stabiliser for $(G : V)$ is finite, then $\mathrm{Inv}(\mathfrak{g} \ltimes V, \mathrm{ad}^*) = \Bbbk[V^*]^G$.
- $\mathcal{S}(\mathfrak{q})^Q$ is isomorphic to the centre of $\mathcal{U}(\mathfrak{q})$.

$\operatorname{trdeg} \Bbbk(\mathfrak{q}^*)^Q =: \operatorname{ind}(\mathfrak{q})$ is the index of $\mathfrak{q}$.

$$\mathfrak{q}^*_{reg} := \{\xi \in \mathfrak{q}^* \mid \dim Q{\cdot}\xi \geqslant \dim Q{\cdot}\eta \text{ for all } \eta \in \mathfrak{q}^*\} = \{\xi \in \mathfrak{q}^* \mid \dim \mathfrak{q}_\xi = \operatorname{ind}(\mathfrak{q})\}$$

Def $(\mathfrak{g}, \operatorname{ad}^*)$ has *codim–2* property if $\operatorname{codim}(\mathfrak{q}^* \setminus \mathfrak{q}^*_{reg}) \geqslant 2$.

Theorem (B)

Suppose $\operatorname{Quot}(\Bbbk[\mathfrak{q}^*]^Q) = \Bbbk(\mathfrak{q}^*)^Q$ *and* $(\mathfrak{q}, \operatorname{ad}^*)$ *has codim–2 property.*

(i) *If* $f_1, \dots, f_r \in \Bbbk[\mathfrak{q}^*]^Q$ *are homogeneous algebraically independent* ($r = \operatorname{ind}\mathfrak{q}$), *then* $\sum_{i=1}^r \deg f_i \geqslant (\dim \mathfrak{q} + \operatorname{ind}\mathfrak{q})/2$.

(ii) *if the equality holds, then* $\Bbbk[\mathfrak{q}^*]^Q = \Bbbk[f_1, \dots, f_r]$ *is polynomial and "Kostant's criterion" is satisfied, i.e.,* $\xi \in \mathfrak{q}^*_{reg}$ *if and only if* $(df_1)_\xi, \dots, (df_r)_\xi$ *are linearly independent.*

$\mathbb{Z}_2$-contractions

$\sigma \in \mathrm{Inv}(\mathfrak{g}) \ \Rightarrow\ \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Let $\mathfrak{q} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$. It is a $\mathbb{Z}_2$-contraction of $\mathfrak{g}$.

By Theorem (A), $\mathrm{Inv}(\mathfrak{q}, \mathrm{ad}\,)$ is a polynomial algebra of Krull dimension $\mathrm{rk}(\mathfrak{g}_0) + \dim(\mathfrak{g}_1)^{T_0} = \mathrm{rk}(\mathfrak{g})$.

What happens for the coadjoint representation?

Since $\mathfrak{g} \rightsquigarrow \mathfrak{q}$, we have $\mathrm{ind}(\mathfrak{q}) \geqslant \mathrm{ind}(\mathfrak{g})$.

Lemma

1. $\mathrm{ind}(\mathfrak{q}) = \mathrm{ind}(\mathfrak{g}) = \mathrm{rk}(\mathfrak{g})$;
2. $(\mathfrak{q}, \mathrm{ad}^*)$ has *codim–2* property;
3. The algebra $\mathrm{Inv}(\mathfrak{q}, \mathrm{ad}^*)$ is bi-graded.

The reason for (1) is that G/G_0 is a spherical homogeneous space. To prove (2), we argue by induction on $\mathrm{rk}(\mathfrak{g})$; (3) holds for any semi-direct product.

To apply Theorem (B), we need a method for producing invariants of $(\mathfrak{q}, \mathrm{ad}^*)$ from invariants of $(\mathfrak{g}, \mathrm{ad}\, = \mathrm{ad}^*)$.

Contractions of G-invariants

We may identify $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \mathfrak{q}, \mathfrak{q}^*$ as vectors spaces and $\mathfrak{g}_0$-modules.

$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \ \Rightarrow\ \mathcal{S} = \mathcal{S}(\mathfrak{g})$ is bi-graded, $\mathcal{S} = \bigoplus\limits_{a,b} \mathcal{S}_{(a,b)}$.

$f \in \mathcal{S}_n \ \Rightarrow\ f = \sum f_i$, where $f_i \in \mathcal{S}_{(n-i,i)}$

Set $\begin{cases} f^\bullet = f_i \neq 0 \text{ with maximal } i, \\ f_\bullet = f_j \neq 0 \text{ with minimal } j. \end{cases}$ Notice that $\deg f = \deg f^\bullet = \deg f_\bullet$.

Lemma

$f \in \mathcal{S}(\mathfrak{g})^G \simeq \Bbbk[\mathfrak{g}]^G \ \Rightarrow\ f^\bullet \in \mathcal{S}(\mathfrak{q})^Q$ and $f_\bullet \in \Bbbk[\mathfrak{q}]^Q$.

Practical conclusion:

Suppose $f_1, \dots, f_r \in \Bbbk[\mathfrak{g}]^G$ are alg. indep. homogeneous generators such that $f_1^\bullet, \dots, f_r^\bullet$ are algebraically independent. Then $\mathcal{S}(\mathfrak{q})^Q = \Bbbk[f_1^\bullet, \dots, f_r^\bullet]$ and Kostant's criterion holds for $\mathfrak{q}$.

Definition

$f_1, \dots, f_r \in \Bbbk[\mathfrak{g}]^G$ is a good generating system (g.g.s.) for σ, if $f_1^\bullet, \dots, f_r^\bullet$ are algebraically independent.

Example (symmetric pairs $\mathfrak{g} \supset \mathfrak{g}_0$ with g.g.s.)

1. $SL_{n+m} \supset SL_n \times SL_m \times \Bbbk^\times$;
2. $SO_{n+m} \supset SO_n \times SO_m$;
3. $\mathbf{F}_4 \supset \mathbf{B}_4 = \mathfrak{so}_9$.

In the first two cases, the coefficients of the characteristic polynomial of a matrix form a g.g.s.

A necessary condition for the existence of a g.g.s.:

$\Bbbk[\mathfrak{g}]^G \to \Bbbk[\mathfrak{g}_1]^{G_0}$ is onto.

There are four bad cases! E.g. $\mathbf{E}_6 \supset \mathbf{F}_4$, $\mathbf{E}_8 \supset \mathbf{E}_7 \times \mathbf{A}_1$.

$\mathbb{Z}_2$-grading is called $\mathfrak{N}$-regular, if $\mathfrak{g}_1$ contains a regular nilpotent element.

Theorem (C)

If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is $\mathfrak{N}$-regular, then

- *there is a g.g.s.*
- *$\mathcal{S}(\mathfrak{q})^Q$ is generated by $\mathcal{S}(\mathfrak{g}_1)^{G_0}$ and certain "covariants" $F_i : \mathfrak{g}_1^* \to \mathfrak{g}_0$.*
- *$\pi_Q : \mathfrak{q}^* \to \mathfrak{q}^* /\!\!/ Q \simeq \mathbb{A}^r$ is equidimensional (flat).*

Conclusion:

- g.g.s cannot provide a description of $\mathrm{Inv}(\mathfrak{g}_0 \ltimes \mathfrak{g}_1, \mathrm{ad}^*)$ for all σ.
- There are some cases, where we expect the presence of g.g.s., but it is not proved yet. (It is difficult to check by hand whether a generating system is good.)
- Recently, O.Yakimova proved by another method that $\mathrm{Inv}(\mathfrak{g}_0 \ltimes \mathfrak{g}_1, \mathrm{ad}^*)$ is polynomial for all σ.

Arbitrary periodic contractions

$\theta \in \mathrm{Aut}(\mathfrak{g})$, $\theta^m = id$, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{m-1}$;

Periodic (or $\mathbb{Z}_m$-) contraction: $\mathfrak{g} \rightsquigarrow \mathfrak{q}$.

$c_t(x_0 + x_1 + \ldots + x_{m-1}) = x_0 + tx_1 + \ldots + t^{m-1}x_{m-1}$.

- Bracket in $\mathfrak{g}$: $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ $(i, j \in \mathbb{Z}_m)$.
- Bracket in $\mathfrak{q}$: $[\mathfrak{g}_i, \mathfrak{g}_j]_{(\mathfrak{q})} \begin{cases} \subset \mathfrak{g}_{i+j}, & i+j \leqslant m-1, \\ = 0 & i+j \geqslant m \, . \end{cases}$

$\mathfrak{q}$ is $\mathbb{N}$-graded Lie algebra; notation: $\mathfrak{q} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1 \ltimes \ldots \ltimes \mathfrak{g}_{m-1}$.

The algebra $\mathcal{S} = \mathcal{S}(\mathfrak{q})$ is $\mathbb{N}^m$-graded: $\mathcal{S} = \bigoplus_{i_0, i_1, \ldots, i_{m-1}} \mathcal{S}_{i_0, i_1, \ldots, i_{m-1}}$

and $f = \sum f_{i_0, i_1, \ldots, i_{m-1}} \in \mathcal{S}$.

$\mathbb{N}$-specialisation: $\deg_\theta(f_{i_0, i_1, \ldots, i_{m-1}}) = i_1 + 2i_2 + \cdots + (m-1)i_{m-1}$.

One can define $f_\bullet$ and $f^\bullet$ w.r.t. $\deg_\theta$. Then

$$f \in \mathcal{S}(\mathfrak{g})^G \simeq \Bbbk[\mathfrak{g}]^G \ \Rightarrow \ f^\bullet \in \mathcal{S}(\mathfrak{q})^Q \text{ and } f_\bullet \in \Bbbk[\mathfrak{q}]^Q.$$

Difficulties (coadjoint case)

- It is not a priori clear that $\mathrm{ind}(\mathfrak{q}) = \mathrm{rk}(\mathfrak{g})$,
- $f_1^\bullet, \dots, f_r^\bullet$ can be algebraically dependent.

In general, we have inclusions:

- $\mathcal{S}(\mathfrak{q})^Q \supset \mathcal{L}^\bullet(\mathcal{S}(\mathfrak{g})^G) := \{f^\bullet \mid f \in \mathcal{S}(\mathfrak{g})^G\}$,
- $\Bbbk[\mathfrak{q}]^Q \supset \mathcal{L}_\bullet(\Bbbk[\mathfrak{g}]^G) := \{f_\bullet \mid f \in \Bbbk[\mathfrak{g}]^G\}$.

A more fancy contraction (quasi-graded contraction):

$\mathfrak{g} \dot{+} \mathfrak{g}_0 \rightsquigarrow \mathfrak{g}_0 \ltimes \mathfrak{g}_1 \ltimes \dots \ltimes \mathfrak{g}_{m-1} \ltimes \mathfrak{g}_0 =: \mathfrak{r}\langle \mathfrak{g} \dot{+} \mathfrak{g}_0 \rangle$.

Much more fancy:

$$\mathfrak{g}^{\oplus n} = n\mathfrak{g} \rightsquigarrow \underbrace{\mathfrak{g}_0 \ltimes \mathfrak{g}_1 \ltimes \dots \ltimes \mathfrak{g}_{m-1} \ltimes \mathfrak{g}_0 \ltimes \dots \ltimes \mathfrak{g}_{m-1}}_{nm} =: \mathfrak{r}\langle n\mathfrak{g} \rangle.$$

$$n\mathfrak{g} \dot{+} \mathfrak{g}_0 \rightsquigarrow \underbrace{\mathfrak{g}_0 \ltimes \mathfrak{g}_1 \ltimes \dots \ltimes \mathfrak{g}_{m-1}}_{nm} \ltimes \mathfrak{g}_0 =: \mathfrak{r}\langle n\mathfrak{g} \dot{+} \mathfrak{g}_0 \rangle.$$

Let $\mathcal{O}^{reg}$ be the regular nilpotent orbit in $\mathfrak{g}$.

Theorem (good cases for adjoint representations)

Suppose θ has the property that $\mathfrak{g}_0 \cap \mathcal{O}^{reg} \neq \varnothing$. Then

(i) $\mathcal{L}_\bullet(\Bbbk[n\mathfrak{g}]^{nG}) = \mathrm{Inv}(\mathfrak{r}\langle n\mathfrak{g}\rangle, \mathrm{ad}\,)$, *and it is a polynomial algebra of Krull dimension $n{\cdot}\mathrm{rk}(\mathfrak{g})$. In particular, for $n = 1$, $\mathcal{L}_\bullet(\Bbbk[\mathfrak{g}]^G) = \Bbbk[\mathfrak{q}]^Q$.*

(ii) $\mathcal{L}_\bullet(\Bbbk[n\mathfrak{g} \dotplus \mathfrak{g}_0]^{n\mathfrak{g}\dotplus\mathfrak{g}_0}) = \mathrm{Inv}(\mathfrak{r}\langle n\mathfrak{g} \dotplus \mathfrak{g}_0\rangle, \mathrm{ad}\,)$, *and it is a polynomial algebra of Krull dimension $\mathrm{rk}(\mathfrak{g}_0) + n{\cdot}\mathrm{rk}(\mathfrak{g})$.*

Remark: $\mathfrak{r}\langle n\mathfrak{g} \dotplus \mathfrak{g}_0\rangle$ is quadratic for any n, hence $\mathrm{ad}\, = \mathrm{ad}^*$.

Examples (suitable automorphisms θ)

$m = 2$: (SL_{2n}, Sp_{2n}), (SL_{2n+1}, SO_{2n+1}), (SO_{2n}, SO_{2n-1}), $(\mathbf{E}_6, \mathbf{F}_4)$.

$m = 3$: $(\mathbf{D}_4, \mathbf{G}_2)$

The corresponding contractions occur in real life as centralisers of nilpotent elements.

For a $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, set
$\mathfrak{E}_M(\mathfrak{g}_0, \mathfrak{g}_1) := \underbrace{\mathfrak{g}_0 \ltimes \mathfrak{g}_1 \ltimes \mathfrak{g}_0 \ltimes \mathfrak{g}_1 \ltimes \dots}_{M}$.

1^o. Let $e \in \tilde{\mathfrak{g}} = \mathfrak{sp}_{2ns}$ be a nilpotent element with partition $((2s)^n)$. Then $\tilde{\mathfrak{g}}^e \simeq \mathfrak{E}_{2s}(\mathfrak{so}_n, \mathsf{R}(2\varpi_1) \oplus \mathbb{I}) \simeq \mathfrak{E}_{2s}(\mathfrak{so}_n, \mathsf{R}(2\varpi_1)) \dotplus \mathfrak{z}_s$. Here $\mathfrak{so}_n \oplus (\mathsf{R}(2\varpi_1) \oplus \mathbb{I}) = \mathfrak{gl}_n$ and $\tilde{\mathfrak{g}}^e$ is a contraction of ${\mathfrak{gl}_n}^{\oplus s}$.

2^o. Let $e \in \tilde{\mathfrak{g}} = \mathfrak{so}_{n(2s+1)}$ be a nilpotent element with partition $((2s+1)^n)$. Then $\tilde{\mathfrak{g}}^e \simeq \mathfrak{E}_{2s+1}(\mathfrak{so}_n, \mathsf{R}(2\varpi_1) \oplus \mathbb{I}) \simeq \mathfrak{E}_{2s+1}(\mathfrak{so}_n, \mathsf{R}(2\varpi_1)) \dotplus \mathfrak{z}_s$. Here ${\mathfrak{gl}_n}^{\oplus s} \dotplus \mathfrak{so}_n \rightsquigarrow \tilde{\mathfrak{g}}^e$.

3^o. Let $e \in \tilde{\mathfrak{g}} = \mathfrak{so}_{4ns}$ be a nilpotent element with partition $((2s)^{2n})$. Then $\tilde{\mathfrak{g}}^e \simeq \mathfrak{E}_{2s}(\mathfrak{sp}_{2n}, \mathsf{R}(\varpi_2) \oplus \mathbb{I}) \simeq \mathfrak{E}_{2s}(\mathfrak{sp}_{2n}, \mathsf{R}(\varpi_2)) \dotplus \mathfrak{z}_s$. Here $\mathfrak{sp}_{2n} \oplus (\mathsf{R}(\varpi_2) \oplus \mathbb{I}) = \mathfrak{gl}_{2n}$ and $\tilde{\mathfrak{g}}^e$ is a contraction of ${\mathfrak{gl}_{2n}}^{\oplus s}$.

4^o. Let $e \in \tilde{\mathfrak{g}} = \mathfrak{sp}_{2n(2s+1)}$ be a nilpotent element with partition $((2s+1)^{2n})$. Then $\tilde{\mathfrak{g}}^e \simeq \mathfrak{E}_{2s+1}(\mathfrak{sp}_{2n}, \mathsf{R}(\varpi_2) \oplus \mathbb{I}) \simeq \mathfrak{E}_{2s+1}(\mathfrak{sp}_{2n}, \mathsf{R}(\varpi_2)) \dotplus \mathfrak{z}_s$. Here ${\mathfrak{gl}_{2n}}^{\oplus s} \dotplus \mathfrak{sp}_{2n} \rightsquigarrow \tilde{\mathfrak{g}}^e$.

Recall that $\mathfrak{q} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1 \ltimes \ldots \ltimes \mathfrak{g}_{m-1} = \mathfrak{r}\langle\mathfrak{g}\rangle$.

Proposition

If $\mathfrak{g}_1 \cap \mathfrak{g}_{reg} \neq \varnothing$, then $\operatorname{ind}(\mathfrak{q}) = \operatorname{rk}(\mathfrak{g})$.

Theorem (good cases for coadjoint representations)

Suppose that $\mathfrak{g}_1 \cap \mathcal{O}^{reg}$ is dense in $\mathfrak{N} \cap \mathfrak{g}_1$ and $\mathfrak{g}_1$ contains a regular semisimple element of $\mathfrak{g}$. Then $\mathfrak{L}^\bullet(\mathcal{S}(\mathfrak{g})^G) = \mathcal{S}(\mathfrak{q})^Q$ and this is a polynomial algebra of Krull dimension $\operatorname{rk}(\mathfrak{g})$.

It is difficult to verify the assumptions directly!

- There are some sufficient conditions that can be stated in terms of invariant-theoretic properties of the G_0-action on $\mathfrak{g}_1$.
- In case of involutions, this theorem simplifies to Theorem (C).

The Lie algebra $\mathfrak{gl}_{nm}$ has an automorphism of order m such that $\mathfrak{g}_0 = \mathfrak{gl}_n{}^{\oplus m}$ and the assumptions of the previous theorem are satisfied.

Examples

1) Let V_i be an n-dimensional space, $i = 1, \dots, m$, and $\mathfrak{g} = \mathfrak{gl}(V_1 \oplus \dots \oplus V_m)$. Define $A \in GL(V_1 \oplus \dots \oplus V_m)$ by $A|_{V_i} = \zeta^i \cdot \mathrm{id}$, where $\zeta = \sqrt[m]{1}$. Let θ be the inner automorphism of $\mathfrak{g}$ determined by A. Then $\mathfrak{g}_0 = \mathfrak{gl}(V_1) \dotplus \dots \dotplus \mathfrak{gl}(V_m)$ and $\mathfrak{g}_1 = \bigoplus_{i=1}^m \mathrm{Hom}(V_i, V_{i+1})$, where $V_{m+1} = V_1$. Here $\dim \mathfrak{g}_1 = mn^2$.

Using the matrix realisation, one easily verifies that $\mathfrak{g}_1$ contains regular semisimple and nilpotent elements. Here $\mathfrak{N} \cap \mathfrak{g}_1$ has m irreducible components, and each contains regular nilpotent elements of $\mathfrak{g}$.

2) $\mathbf{E}_8$ has an automorphism of order 5 such that $\mathfrak{g}_0 = \mathbf{A}_4 \times \mathbf{A}_4$.

Open problems for periodic contractions

- there is no general formula for the index of $\mathbb{Z}_m$-contactions, $m \geqslant 3$.
- little is known about the flatness of the corresponding quotient morphisms, $\pi_Q : \mathfrak{q}^* \to \mathfrak{q}^*/\!\!/Q$. (Some partial positive results are obtained for $\mathbb{Z}_2$-contractions.)
- In case of $\mathbb{Z}_2$-contractions, it is suggested by O. Yakimova that π_Q is not flat if $\Bbbk[\mathfrak{g}]^G \to \Bbbk[\mathfrak{g}_1]^{G_0}$ is not onto (four "bad" cases!).

Some related articles

- D. PANYUSHEV. Semi-direct products of Lie algebras and their invariants, *Publ. R.I.M.S.*, **43**, no. 4 (2007), 1199–1257.
- D. PANYUSHEV. On the coadjoint representation of $\mathbb{Z}_2$-contractions of reductive Lie algebras, *Advances in Math.*, **213**(2007), 380–404.
- D. PANYUSHEV. Periodic automorphisms of Takiff algebras, contractions, and θ-groups, *Transform. Groups*, (2009), to appear = `arXiv: math 0710.2113`, 21 pp