On orbits of antichains of positive roots

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Some related articles

- P. CELLINI and P. PAPI. ad-nilpotent ideals of a Borel subalgebra II, *J. Algebra*, **258**(2002), 112–121.
- D. FON-DER-FLAASS. Orbits of antichains in ranked posets, *Europ. J. Combinatorics*, **14**(1993), 17–22.
- P.J. CAMERON and D. FON-DER-FLAASS. Orbits of antichains revisited, *Europ. J. Combinatorics*, **16**(1995), 545–554.
- D. PANYUSHEV. The poset of positive roots and its relatives, *J. Alg. Combinatorics*, **23**(2006), 79–101.
- D. PANYUSHEV. On orbits of antichains of positive roots, `arXiv: math 0711.3353`, 12 pp., to appear in *Europ. J. Combinatorics*.
- E. SOMMERS. *B*-stable ideals in the nilradical of a Borel subalgebra, *Canad. Math. Bull.* **48**(2005), 460–472.

Main definitions

$(\mathcal{P}, \preccurlyeq)$ is an arbitrary finite poset. For any $\mathcal{S} \subset \mathcal{P}$, let $\mathcal{S}_{min}$ and $\mathcal{S}_{max}$ denote the set of minimal and maximal elements of $\mathcal{S}$, respectively.

Definitions

- An antichain in $\mathcal{P}$ is a subset of mutually incomparable elements.
- An upper ideal (or filter) is a subset $\mathcal{I} \subset \mathcal{P}$ such that if $\gamma \in \mathcal{I}$ and $\gamma \preccurlyeq \beta$, then $\beta \in \mathcal{I}$.

The set of all antichains in $\mathcal{P}$ is denoted by $\mathfrak{An}(\mathcal{P})$.

- $\Gamma \in \mathfrak{An}(\mathcal{P})$ if and only if $\Gamma = \Gamma_{min}$ (or $\Gamma = \Gamma_{max}$).
- If $\Gamma \in \mathfrak{An}(\mathcal{P})$, then $\mathcal{I}(\Gamma)$ denotes the upper ideal of $\mathcal{P}$ generated by Γ. That is, $\mathcal{I}(\Gamma) = \{\varepsilon \in \mathcal{P} \mid \exists\gamma \in \Gamma \text{ such that } \gamma \preccurlyeq \varepsilon\}$.
- If $\mathcal{I}$ is an upper ideal of $\mathcal{P}$, then $\mathcal{I}_{min} \in \mathfrak{An}(\mathcal{P})$.

This yields a natural bijection between the upper ideals and antichains of $\mathcal{P}$.

Letting $\Gamma' \lessdot \Gamma$ if $\mathcal{I}(\Gamma') \subset \mathcal{I}(\Gamma)$, we make $\mathfrak{An}(\mathcal{P})$ a poset.

Example

$\Gamma = \varnothing$ is an antichain and $\mathcal{I}(\varnothing)$ is the empty upper ideal.

For $\Gamma \in \mathfrak{An}(\mathcal{P})$, we set $\mathfrak{X}(\Gamma) := (\mathcal{P} \setminus \mathcal{I}(\Gamma))_{max}$. This defines the map $\mathfrak{X} = \mathfrak{X}_{\mathcal{P}} : \mathfrak{An}(\mathcal{P}) \to \mathfrak{An}(\mathcal{P})$. Clearly, $\mathfrak{X}$ is one-to-one, i.e., it is a permutation of the finite set $\mathfrak{An}(\mathcal{P})$.

We say that $\mathfrak{X}$ is the reverse operator for $\mathcal{P}$.

If $\#\mathfrak{An}(\mathcal{P}) = m$, then $\mathfrak{X}$ is an element of the symmetric group Σ_m. The order of $\mathfrak{X}$, $\mathrm{ord}(\mathfrak{X})$, is the order of the group generated by $\mathfrak{X}$.

Main problem: Study connections between combinatorial properties of $\mathcal{P}$ and algebraic properties of $\mathfrak{X}$.

Definition

$\mathcal{P}$ is graded (of level r) if there is a function $d : \mathcal{P} \to \{1, 2, \dots, r\}$ such that both $d^{-1}(1)$ and $d^{-1}(r)$ are non-empty, and $d(y) = d(x) + 1$ whenever y covers x. Then $d^{-1}(1) \subset \mathcal{P}_{min}$ and $d^{-1}(r) \subset \mathcal{P}_{max}$.

Lemma

Suppose $\mathcal{P}$ is graded of level r, $d^{-1}(1) = \mathcal{P}_{min}$ and $d^{-1}(r) = \mathcal{P}_{max}$. Then $\mathfrak{X}$ has an orbit of cardinality $r + 1$.

- $\mathcal{P}(i) := d^{-1}(i)$ is an antichain for any i.
- By our hypotheses, $\mathfrak{X}(\mathcal{P}(i)) = \mathcal{P}(i-1)$ for $i = 2, \dots, r$, $\mathfrak{X}(\mathcal{P}(1)) = \varnothing$, and $\mathfrak{X}(\varnothing) = \mathcal{P}(r)$.
- Thus, $\{\varnothing, \mathcal{P}(r), \dots, \mathcal{P}(1)\}$ is an $\mathfrak{X}$-orbit.

Such an orbit of $\mathfrak{X}$ is said to be standard.

Notation for root systems

- Δ is a reduced irreducible root system in V ($\dim V = n$).
- Δ^+ is a set of positive roots, with the corresponding simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$.
- $W \subset GL(V)$ is the Weyl group of Δ; $w_0 \in W$ is the longest element.

Definition

The root order in Δ^+ is given by letting $x \preccurlyeq y$ if $y - x$ is a non-negative integral combination of positive roots. In particular, y covers x if $y - x$ is a simple root.

- $\theta \in \Delta^+$ is the highest root. It is the maximal element of $(\Delta^+, \preccurlyeq)$.
- If $\gamma = \sum_{i=1}^{n} a_i\alpha_i \in \Delta^+$, then $\mathrm{ht}(\gamma) := \sum a_i$ is the height of γ.
- $h = h(\Delta)$ is the Coxeter number of Δ.

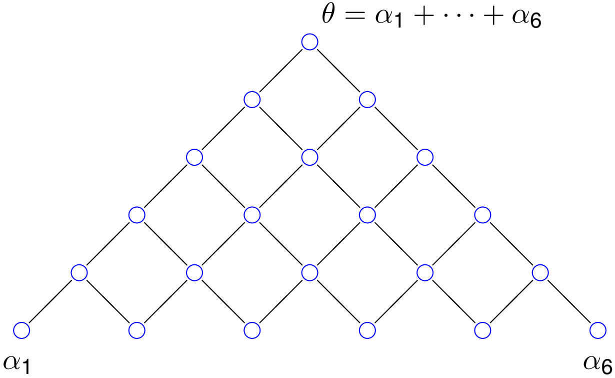


Figure: The poset $\Delta^+(\mathfrak{sl}_7)$

Some properties of Δ^+ and $\mathfrak{An}(\Delta^+)$

- The function $\alpha \mapsto \mathrm{ht}(\alpha)$ makes Δ^+ the graded poset of level $h{-}1$.
- If $e_1, \dots, e_n$ are the exponents of Δ, then
$$\#(\mathfrak{An}(\Delta^+)) = \prod_{i=1}^{n} \frac{h + e_i + 1}{e_i + 1} \text{ (Cellini-Papi, 2002)).}$$
- $\#\Gamma$ equals the number of elements of $\mathfrak{An}(\Delta^+)$ covered by Γ.

(For, Γ covers Γ' with respect to the order '$\lessdot$' described above if and only if $\Gamma' = (\mathcal{I}(\Gamma) \setminus \{\gamma_i\})_{min}$ for some $\gamma_i \in \Gamma$.) Hence $\displaystyle\sum_{\Gamma \in \mathfrak{An}(\Delta^+)} \#\Gamma$ equals the total number of edges in the Hasse diagram of $(\mathfrak{An}(\Delta^+), \lessdot)$.

- $$\sum_{\Gamma \in \mathfrak{An}(\Delta^+)} \frac{\#\Gamma}{\#\mathfrak{An}(\Delta^+)} = \frac{\#\Delta^+}{h} \qquad \text{(Panyushev, 2006)}$$

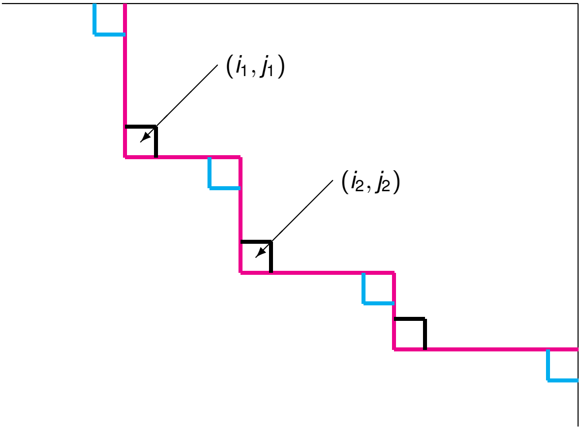


Figure: Antichains Γ and $\mathfrak{X}(\Gamma)$ for $\Delta^+(\mathfrak{sl}_{n+1})$

Some orbits of $\mathfrak{X} = \mathfrak{X}_{\Delta^+}$

Set $\Delta(i) = \{\alpha \in \Delta^+ \mid \mathrm{ht}(\alpha) = i\}$.

Then $\Delta(1) = \Pi = \Delta^+_{min}$ and $\Delta(h{-}1) = \{\theta\} = \Delta^+_{max}$.

Example

There are two specific orbits of $\mathfrak{X} = \mathfrak{X}_{\Delta^+}$:

- By Lemma, there is an orbit of cardinality h. Namely, $\{\varnothing, \Delta(h{-}1), \dots, \Delta(2), \Delta(1)\}$ is the standard $\mathfrak{X}$-orbit in $\mathfrak{An}(\Delta^+)$.
- There is an orbit of cardinality 2. Let $\mathcal{A} \subset \Pi$ a set of mutually orthogonal roots such that $\Pi \setminus \mathcal{A}$ also has that property. The partition $\{\mathcal{A}, \Pi \setminus \mathcal{A}\}$ is uniquely determined, since the Dynkin diagram of Δ is a tree. Then $\mathfrak{X}(\mathcal{A}) = \Pi \setminus \mathcal{A}$ and $\mathfrak{X}(\Pi \setminus \mathcal{A}) = \mathcal{A}$.

Remark

If Δ is of rank 2, then these two orbits exhaust $\mathfrak{An}(\Delta^+)$.

Conjecture 1 (for $\mathfrak{X} = \mathfrak{X}_{\Delta^+}$)

(i) If $w_0 = -1$, then $\mathrm{ord}(\mathfrak{X}) = h$;

(ii) If $w_0 \neq -1$, then $\mathfrak{X}^h$ is the involution of $\mathfrak{An}(\Delta^+)$ induced by $-w_0$ and $\mathrm{ord}(\mathfrak{X}) = 2h$;

(iii) Let $\mathcal{O}$ be an arbitrary $\mathfrak{X}$-orbit in $\mathfrak{An}(\Delta^+)$. Then
$$\frac{1}{\#\mathcal{O}} \sum_{\Gamma \in \mathcal{O}} \#\Gamma = \frac{\#\Delta^+}{h} = \frac{n}{2}.$$

- $w_0 \neq -1$ if and only if Δ is of type $\mathbf{A}_n$ ($n \geqslant 2$), $\mathbf{D}_{2n+1}$, $\mathbf{E}_6$.
- Conjecture 1 has been verified for $\mathbf{A}_n$ ($n \leqslant 5$), $\mathbf{C}_n$ ($n \leqslant 4$), $\mathbf{D}_4$, $\mathbf{F}_4$.
- $\Delta^+(\mathbf{B}_n) \simeq \Delta^+(\mathbf{C}_n)$.
- Part (iii) is a refinement of the formula for the number of edges in the Hasse diagram.

Example (for $\Delta^+(\mathbf{A}_n)$)

Usual notation: $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1, 2, \dots, n$, and $\theta = \varepsilon_1 - \varepsilon_{n+1}$. Suppose $\Gamma = \{\alpha_1\}$ and $n \geqslant 3$. Then $\mathfrak{X}(\{\alpha_1\}) = \alpha_2 + \ldots + \alpha_n$ and

$$\mathfrak{X}^k(\{\alpha_1\}) = \{\gamma \in \Delta(\alpha_1, \dots, \alpha_{n-1}) \mid \mathrm{ht}(\gamma) = n+1-k\} \sqcup \{\alpha_{k+1} + \ldots + \alpha_n\}$$

for $1 \leqslant k \leqslant n$. In particular, $\mathfrak{X}^n(\{\alpha_1\}) = \{\alpha_1, \dots, \alpha_{n-1}\}$ and hence $\mathfrak{X}^{n+1}(\{\alpha_1\}) = \{\alpha_n\}$. Therefore the $\mathfrak{X}$-orbit of $\{\alpha_1\}$ is of cardinality $2h = 2n + 2$.

- For this orbit, we have $\dfrac{1}{\#\mathcal{O}} \displaystyle\sum_{\Gamma \in \mathcal{O}} \#\Gamma = n/2$, as required.

Challenging problem: construct “invariants” of $\mathfrak{X}$, i.e., functions on $\mathfrak{An}(\Delta^+)$ that are constant on the $\mathfrak{X}$-orbits. Ideally, one could ask for a family of invariants that separates the orbits. Below, we describe one invariant in the case of type $\mathbf{A}_n$.

- $\Delta^+ \setminus \Pi = \Delta(\geqslant 2)$ is a subposet of Δ^+.
- $\Delta^+ \setminus \Pi$ the graded poset of level $h-2$. (Use $\alpha \mapsto \mathrm{ht}(\alpha)-1$.)
- The theory of antichains in $\Delta^+ \setminus \Pi$ resembles that for Δ^+. In particular, $\#(\mathfrak{An}(\Delta^+ \setminus \Pi)) = \displaystyle\prod_{i=1}^{n} \frac{h+e_i-1}{e_i+1}$ (Sommers, 2005).
- $\mathfrak{X}_{\Delta^+ \setminus \Pi}$ has the standard orbit of cardinality $h-1$.

Conjecture 2 (for $\mathfrak{X}_0 = \mathfrak{X}_{\Delta^+ \setminus \Pi}$)

(i) If $w_0 = -1$, then $\mathrm{ord}(\mathfrak{X}_0) = h-1$;

(ii) If $w_0 \neq -1$, then $\mathfrak{X}_0^{h-1}$ is the involution of $\mathfrak{An}(\Delta^+ \setminus \Pi)$ induced by $-w_0$ and $\mathrm{ord}(\mathfrak{X}_0) = 2h-2$;

(iii) For any $\mathfrak{X}_0$-orbit $\mathcal{O} \subset \mathfrak{An}(\Delta^+ \setminus \Pi)$, we have
$$\frac{1}{\#\mathcal{O}} \sum_{\Gamma \in \mathcal{O}} \#\Gamma = \frac{\#(\Delta^+ \setminus \Pi)}{h-1} = \frac{n}{2} \cdot \frac{h-2}{h-1} .$$

Again, part (iii) is a refinement of a formula for the number of edges in the Hasse diagram of $\mathfrak{An}(\Delta^+ \setminus \Pi)$.

Empirical evidences supporting Conjecture 2:

$\Delta^+ \setminus \Pi(\mathbf{A}_{n+1}) \simeq \Delta^+(\mathbf{A}_n)$. Therefore Conjecture 2 holds for $\mathbf{A}_n$ ($n \leqslant 6$). It has also been verified for $\mathbf{C}_n$ ($n \leqslant 5$), $\mathbf{D}_n$ ($n \leqslant 5$), and $\mathbf{F}_4$.

Warning

One might have thought that posets $\Delta(\geqslant j)$ enjoy similar good properties for any j. However, this is not the case!

Example

For $\mathbf{F}_4$ and $\Delta(\geqslant 3)$, the reverse operator has orbits of cardinality 10 and 8. Hence its order equals 40, while $h - 2 = 10$. Furthermore, the mean value of the size of antichains along the orbits is not constant.

- If Δ has two root lengths, then $\Delta_s^+ = \{\alpha \in \Delta^+ \mid \alpha \text{ is short}\}$.
- Δ_s^+ is regarded as subposet of Δ^+.
- θ_s is the only maximal element of Δ_s^+ and $(\Delta_s^+)_{min} = \Pi \cap \Delta_s^+ = \Pi_s$.

Let $h^*(\Delta)$ be the dual Coxeter number of Δ. If $\Delta^\vee = \{\frac{2\alpha}{(\alpha,\alpha)} \mid \alpha \in \Delta\}$ is the dual root system, then $h^*(\Delta^\vee) - 1 = \mathrm{ht}(\theta_s)$.

- Δ_s^+ is a graded poset of level $h^*(\Delta^\vee) - 1$.
- $\mathfrak{X}_s$ has the standard orbit of cardinality $h^*(\Delta^\vee)$.

Conjecture 3 (for $\mathfrak{X}_s = \mathfrak{X}_{\Delta_s^+}$)

(i) $\mathrm{ord}(\mathfrak{X}_s) = h^*(\Delta^\vee)$;

(ii) Let $\mathcal{O}$ be an $\mathfrak{X}_s$-orbit in $\mathfrak{An}(\Delta_s^+)$. Then $\dfrac{1}{\#\mathcal{O}} \sum_{\Gamma \in \mathcal{O}} \#\Gamma = \dfrac{\#(\Delta_s^+)}{h^*(\Delta^\vee)}$.

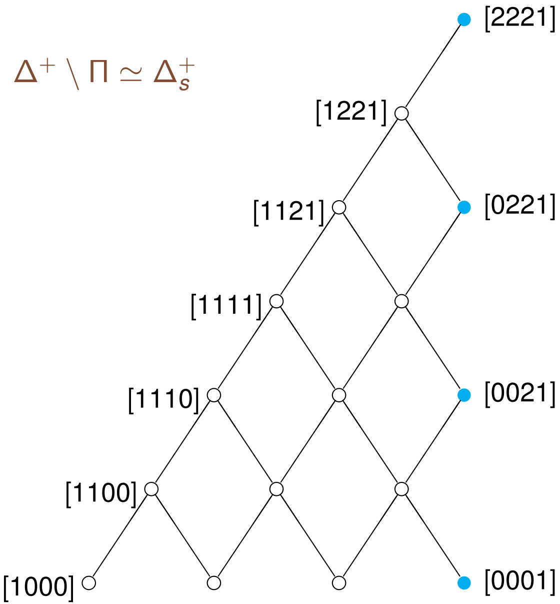
- Conjecture 3 is true for $\mathbf{B}_n$, $\mathbf{F}_4$, and $\mathbf{G}_2$, where the number of $\mathfrak{X}_s$-orbits equals $1, 3$, and 1, respectively.
- It is also verified for $\mathbf{C}_n$, $n \leqslant 5$.
- $\Delta^+ \setminus \Pi(\mathbf{C}_n) \simeq \Delta_s^+(\mathbf{C}_n)$ (hence $\mathfrak{An}(\Delta^+ \setminus \Pi)$ and $\mathfrak{An}(\Delta_s^+)$ are also isomorphic). There is a more precise conjecture in this case:

Conjecture 4

For $\Delta_s^+(\mathbf{C}_n)$, every $\mathfrak{X}_s$-orbit is of cardinality $2n-1 = h^*(\mathbf{B}_n)$. Each $\mathfrak{X}_s$-orbit contains a unique antichain lying in $\Delta^+(\alpha_1, \dots, \alpha_{n-2}) \simeq \Delta^+(\mathbf{A}_{n-2})$.

Since $\#(\mathfrak{An}(\Delta_s^+)) = \binom{2n-1}{n}$ for $\mathbf{C}_n$ (Panyushev, 2004), Conjecture 4 would imply that the number of $\mathfrak{X}_s$-orbits equals $\frac{1}{2n-1}\binom{2n-1}{n}$, the $(n-1)$-th Catalan number. This conjecture also provides a canonical representative in each $\mathfrak{X}_s$-orbit in $\mathfrak{An}(\Delta_s^+(\mathbf{C}_n))$.

Fact: $\Delta^+ \setminus \Pi \simeq \Delta_s^+$



More possibilities

- Similar conjecture can be formulated for $\Delta_s^+ \setminus \Pi_s$:
  - Everything is easy for $\mathbf{B}_n$, $\mathbf{F}_4$, $\mathbf{G}_2$.
  - We also have $\Delta_s^+ \setminus \Pi_s(\mathbf{C}_n) \simeq \Delta^+(\mathbf{C}_{n-1})$;
- There is a unique non-reduced irreducible root system $\mathbf{BC}_n$, where $\Delta^+(\mathbf{BC}_n) \simeq \Delta^+ \setminus \Pi(\mathbf{C}_{n+1})$.

The OY-number

Here $\Delta = \Delta(\mathbf{A}_n) = \Delta(\mathfrak{sl}_{n+1})$. We describe an $\mathfrak{X}$-invariant function $\mathcal{Y} : \mathfrak{An}(\Delta^+) \to \mathbb{N}$, which is found by Oksana Yakimova.
Let $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ be an arbitrary antichain in Δ^+ and $\mathcal{I} = \mathcal{I}(\Gamma)$ the corresponding upper ideal, so that $\Gamma = \mathcal{I}_{min}$. To each γ_s, we attach certain integer as follows. Clearly, $\mathcal{I} \setminus \{\gamma_s\}$ is again an upper ideal. Set

$$r_\Gamma(\gamma_s) := \#(\mathcal{I} \setminus \{\gamma_s\})_{min} - \#\mathcal{I}_{min} + 1 \ .$$

For $\mathfrak{sl}_{n+1}$, the difference between the numbers of minimal elements of $\mathcal{I}$ and $\mathcal{I} \setminus \{\gamma_s\}$ always belongs to $\{-1, 0, 1\}$. Therefore $r_\Gamma(\gamma_s) \in \{0, 1, 2\}$. The OY-number of Γ is defined by

$$\mathcal{Y}(\Gamma) := \sum_{s=1}^{k} r_\Gamma(\gamma_s).$$

We specially set $\mathcal{Y}(\varnothing) = 0$.

Example

- For $\Gamma = \Pi = \{\alpha_1, \dots, \alpha_n\}$, we have $\mathfrak{Y}(\Pi) = 0$. More generally, the same is true for $\Gamma = \Delta(i)$.
- For $\Gamma = \{\alpha_1, \alpha_3, \dots\}$ (all simple roots with odd numbers) or $\Gamma = \{\alpha_2, \alpha_4, \dots\}$ (all simple roots with even numbers), we have $\mathfrak{Y}(\Gamma) = n-1$.

Theorem (O. Yakimova)

The OY-number is $\mathfrak{X}$-invariant, i.e., $\mathfrak{Y}(\Gamma) = \mathfrak{Y}(\mathfrak{X}(\Gamma))$ for all $\Gamma \in \mathfrak{An}(\Delta^+)$.

- The minimal (resp. maximal) value of $\mathfrak{Y}$ is 0 (resp. $n-1$).
- Each of them is attained on a unique $\mathfrak{X}$-orbit.

The above definition of $\mathfrak{Y}(\Gamma)$ can be repeated verbatim for any other root system. However, such a function will not be $\mathfrak{X}$-invariant.

A duality for $\mathfrak{An}(\Delta^+)$

- For $\Delta(\mathbf{A}_n)$, there is an involutory map ("duality") $\ast : \mathfrak{An}(\Delta^+) \to \mathfrak{An}(\Delta^+)$ (Panyushev, 2004).
- Γ^* is called the dual antichain for Γ.

For $i \leqslant j$, the root $\alpha_i + \ldots + \alpha_j$ is denoted by (i, j). If $\Gamma = \{(i_1, j_1), \ldots, (i_k, j_k)\}$ with $i_1 < \cdots < i_k$, then it is represented as an array:

$$\Gamma = \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \end{pmatrix}.$$

Set $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$. That is, $\Gamma = (I, J)$ is determined by two strictly increasing sequences of equal cardinalities lying in $[n] := \{1, \ldots, n\}$ such that $I \leqslant J$ (componentwise). Then $\Gamma^* = (I^*, J^*)$ is defined by

$$I^* := [n] \setminus J \ \text{ and } \ J^* := [n] \setminus I\,.$$

$\mathfrak{X}$-orbits, OY-invariant, and duality

The duality has the following properties:

1. $\#\Gamma + \#(\Gamma^*) = n$;
2. If $\Gamma \subset \Pi$, then $\Gamma^* = \Pi \setminus \Gamma$;
3. $\Delta(i)^* = \Delta(n+2-i)$.

Theorem

- *For any* $\Gamma \in \mathfrak{An}(\Delta^+)$, *we have* $\mathfrak{X}(\Gamma)^* = \mathfrak{X}^{-1}(\Gamma^*)$.
- $\mathfrak{Y}(\Gamma) = \mathfrak{Y}(\Gamma^*)$.

$\mathbf{F}_4$, $\mathfrak{An}(\Delta^+)$

We use the numbering of simple roots from [Vinberg–Onishchik]. The positive root $\beta = \sum_{i=1}^{4} n_i\alpha_i$ is denoted by $(n_1 n_2 n_3 n_4)$. For instance, $\theta = (2432)$ and $\theta_s = (2321)$.

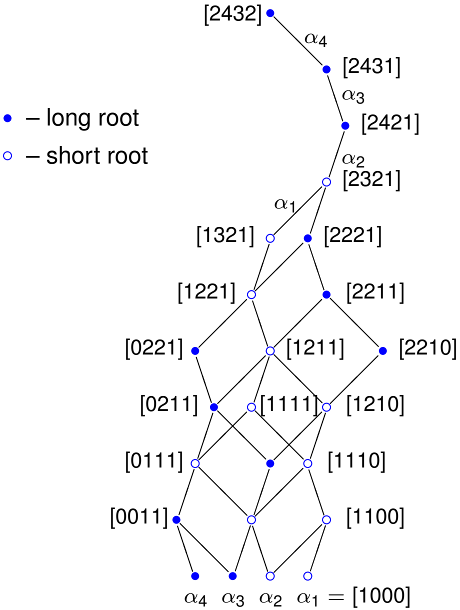
$\#\mathfrak{An}(\Delta^+) = 105$ and $h = 12$. There are eleven $\mathfrak{X}$-orbits: eight orbits of cardinality 12 and orbits of cardinality 2,3, and 4.

We indicate representatives and cardinalities for all $\mathfrak{X}$-orbits:

{1000} – 12; {0100} – 12; {0010} – 12; {0001} – 12; {0011} – 12;

{1100} – 12; {1111} – 12; {2432} – 12 (*the standard orbit*);

{1000, 0010} – 2; {0110} – 3; {0001, 1110} – 4.


• – long root
○ – short root
[2432]
α_4
[2431]
α_3
[2421]
α_2
[2321]
α_1
[1321]
[2221]
[1221]
[2211]
[0221]
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[0111]
[1110]
[0011]
[1100]
α_4 α_3 α_2 α_1 = [1000]

$\mathbf{F}_4$, $\mathfrak{An}(\Delta^+ \setminus \Pi)$

$\#\mathfrak{An}(\Delta^+ \setminus \Pi) = 66$ and $h - 1 = 11$. The notation $\Gamma \rightsquigarrow \Gamma'$ means $\Gamma' = \mathfrak{X}_0(\Gamma)$. The $\mathfrak{X}_0$-orbits are:

1) The standard one:
$\Delta(11) = \{2432\} \rightsquigarrow \{2431\} \rightsquigarrow \cdots \rightsquigarrow \Delta(2) \rightsquigarrow \varnothing \rightsquigarrow \Delta(11)$;

2) $\{1321\} \rightsquigarrow \{2221\} \rightsquigarrow \{1321, 2211\} \rightsquigarrow \{1221, 2210\} \rightsquigarrow \{0221, 1211\} \rightsquigarrow \{0211, 1111, 2210\} \rightsquigarrow \{0111, 1210\} \rightsquigarrow \{0011, 0210, 1110\} \rightsquigarrow \{0110, 1100\} \rightsquigarrow \{0011\} \rightsquigarrow \{2210\} \rightsquigarrow \{1321\}$;

3) $\{1221\} \rightsquigarrow \{0221, 2211\} \rightsquigarrow \{1211, 2210\} \rightsquigarrow \{0221, 1111, 1210\} \rightsquigarrow \{0211, 1110\} \rightsquigarrow \{0111, 0210, 1100\} \rightsquigarrow \{0011, 0110\} \rightsquigarrow \{1100\} \rightsquigarrow \{0221\} \rightsquigarrow \{2211\} \rightsquigarrow \{1321, 2210\} \rightsquigarrow \{1221\}$;

4) $\{1211\} \rightsquigarrow \{0221, 1111, 2210\} \rightsquigarrow \{0211, 1210\} \rightsquigarrow \{1111, 0210\} \rightsquigarrow \{0111, 1110\} \rightsquigarrow \{0011, 0210, 1100\} \rightsquigarrow \{0110\} \rightsquigarrow \{0011, 1100\} \rightsquigarrow \{0210\} \rightsquigarrow \{1111\} \rightsquigarrow \{0221, 2210\} \rightsquigarrow \{1211\}$;

5) $\{1210\} \rightsquigarrow \{0221, 1111\} \rightsquigarrow \{0211, 2210\} \rightsquigarrow \{1111, 1210\} \rightsquigarrow \{0221, 1110\} \rightsquigarrow \{0211, 1100\} \rightsquigarrow \{0111, 0210\} \rightsquigarrow \{0011, 1110\} \rightsquigarrow \{0210, 1100\} \rightsquigarrow \{0111\} \rightsquigarrow \{0011, 2210\} \rightsquigarrow \{1210\}$;

6) $\{1110\} \rightsquigarrow \{0221, 1100\} \rightsquigarrow \{0211\} \rightsquigarrow \{1111, 2210\} \rightsquigarrow \{0221, 1210\} \rightsquigarrow \{0211, 1111\} \rightsquigarrow \{0111, 2210\} \rightsquigarrow \{0011, 1210\} \rightsquigarrow \{0210, 1110\} \rightsquigarrow \{0111, 1100\} \rightsquigarrow \{0011, 0210\} \rightsquigarrow \{1110\}$.

Each orbit consists of 11 antichains.

$\mathbf{F}_4$, $\mathfrak{An}(\Delta_s^+)$

$\#\mathfrak{An}(\Delta_s^+) = 21$ and $h^* = 9$. The $\mathfrak{X}_s$-orbits are:

1) standard: $\Delta_s(8) = \{2321\} \rightsquigarrow \{1321\} \rightsquigarrow \cdots \rightsquigarrow \Delta_s(1) = \{1000, 0100\} \rightsquigarrow \varnothing \rightsquigarrow \Delta_s(8)$;

2) $\{0100\} \rightsquigarrow \{1000\} \rightsquigarrow \{0111\} \rightsquigarrow \{1210\} \rightsquigarrow \{1111\} \rightsquigarrow \{0111, 1210\} \rightsquigarrow \{1110\} \rightsquigarrow \{0111, 1100\} \rightsquigarrow \{0110, 1000\} \rightsquigarrow \{0100\}$;

3) $\{1100\} \rightsquigarrow \{0111, 1000\} \rightsquigarrow \{0110\} \rightsquigarrow \{1100\}$.

Bonus: $\mathbf{H}_3$

Question

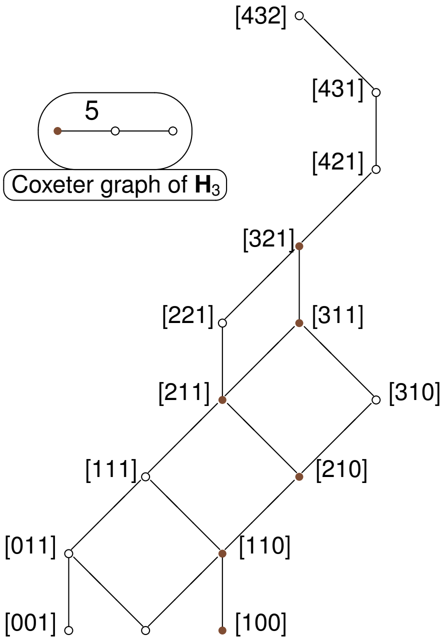
Is there a "poset of positive roots" for $\mathbf{H}_3$ and $\mathbf{H}_4$?

The exponents of $\mathbf{H}_3$ are 1, 5, 9. Therefore one should have

$$\#\mathfrak{An}(\Delta^+) = \frac{12 \cdot 16 \cdot 20}{2 \cdot 6 \cdot 10} = 32 \ \text{ and } \ \#\mathfrak{An}(\Delta^+ \setminus \Pi) = \frac{10 \cdot 14 \cdot 18}{2 \cdot 6 \cdot 10} = 21$$

- The (generalised) Narayana polynomial for $\mathfrak{An}(\Delta^+)$ should be $1 + 15t + 15t^2 + t^3$.
- Analogues of Conjectures 1–3 should hold.

The answer for $\mathbf{H}_3$ is "yes"!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Coxeter graph of $\mathbf{H}_3$
[432]
[431]
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