Let X be a projective variety defined over an algebraically closed field k. For a polynomial $f(x) = \sum_{i=1}^{n} a_i x^i$, where a_i are nonnegative integers, and a vector bundle E on X, define the vector bundle

$$f(E) := \bigoplus_{i=1}^{n} (E^{\otimes i})^{\oplus a_i} .$$

A vector bundle E is called finite if there are two such distinct polynomials f_1 , f_2 such that $f_1(E)$ and $f_2(E)$ are isomorphic.

Nori proved the following:

A vector bundle E on X is finite if and only if there is a finite étale cover

$$\phi : Y \longrightarrow X$$

such that $\phi^* E$ is trivial.

Let us explain the two conditions "E is finite" and "$\phi^* E$ is trivial".

A vector bundle V on X is called decomposable if $V = V_1 \oplus V_2$, where both V_1 and V_2 are of positive ranks. A vector bundle V is called indecomposable if it is not decomposable.

If a vector bundle is not indecomposable, we may keep breaking it up into direct sums of successively smaller ranks, and finally it is expressed as a direct sum of indecomposable bundles. A well-known, and very useful, theorem of Atiyah says:

If a vector bundle V is expressed as

$$\oplus_{i=1}^{m} E_i = V = \oplus_{j=1}^{n} F_j ,$$

where every E_i and F_j are indecomposable, then $m = n$, and there is a permutation σ of $\{1, \cdots, m\}$ such that E_i is isomorphic to $F_{\sigma(i)}$ for all $1 \leq i \leq m$. The vector bundles E_i, $1 \leq i \leq m$, are called the indecomposable components of V. The theorem of Atiyah makes very precise the notion of being an indecomposable component of V.

Nori showed that a vector bundle V on X is finite if and only if there are finitely many indecomposable vector bundles $V_1, \cdots, V_\ell$ such that for every integer $b \geq 1$, the indecomposable components of $V^{\otimes b}$ are contained in $\{V_1, \cdots, V_\ell\}$, equivalently

$$V^{\otimes b} = \bigoplus_{i=1}^{\ell} V_i^{\oplus n_{b,i}}$$

for all $b \geq 1$, where $n_{b,i}$ are nonnegative integers.

Let us take $k = \mathbb{C}$, and take X to be a smooth projective variety. Let D be a flat holomorphic connection on V whose monodromy homomorphism

$$\rho_D : \pi_1(X, x_0) \longrightarrow \mathrm{GL}(V_{x_0})$$

has finite image. Let $\phi : Y \longrightarrow X$ be the Galois étale covering corresponding to kernel$(\rho_D) \subset \pi_1(X, x_0)$. Then the flat holomorphic connection ϕ^*D on ϕ^*V has trivial monodromy. Hence ϕ^*V is a trivial bundle. Conversely, let F be a vector bundle on X and $\phi : Y \longrightarrow X$ an étale Galois covering such that ϕ^*F is trivial. Then the trivial connection on ϕ^*F is preserved by the action of Gal(ϕ) on ϕ^*F; note that the trivial connection on ϕ^*F does not depend on the choice of trivialization of ϕ^*F. Therefore, the trivial connection on ϕ^*F descends to a connection on F. The flat holomorphic connection on F thus obtained has finite monodromy.

Therefore, given a vector bundle V on X, there is a finite étale cover

$$\phi : Y \longrightarrow X$$

such that ϕ^*V is trivial if and only if V admits a flat holomorphic connection with finite monodromy.

Back to the general algebraically closed field.

Comments on the strategy of the proof of Nori: If there is a finite étale cover $\phi : Y \longrightarrow X$ such that ϕ^*E is trivial, then it is relatively easy to prove that E is finite.

From the Tannakian category theory developed by Saavedra Rivano and Nori, to prove the converse, it suffices to show that there are finitely many vector bundles $\{V_1, \cdots, V_\ell\}$ satisfying the following conditions:

- $E^{\otimes a} \otimes (E^{\otimes b})^*$ is a direct sum of copies of $\{V_1, \cdots, V_\ell\}$ for all $a, b \geq 0$.
- $V_i^{\otimes a} \otimes (V_j^{\otimes b})^*$ is a direct sum of copies of $\{V_1, \cdots, V_\ell\}$ for all $a, b \geq 0$ and $1 \leq i, j \leq \ell$.

  For any $\mathcal{O}_X$–linear homomorphism

$$\gamma : \bigoplus_{i=1}^{\ell} V_i^{\oplus m_i} \longrightarrow \bigoplus_{i=1}^{\ell} V_i^{\oplus n_i},$$

both kernel(γ) and cokernel(γ) are direct sums of copies of $\{V_1, \cdots, V_\ell\}$.

By successively cutting X by hyperplanes, the question is reduced to curves. For curves, Nori's theorem is far simpler. This is mostly because of the fact that the semistable vector bundles of degree zero on a curve form an abelian category.

Atiyah proved the earlier mentioned theorem also for holomorphic vector bundles on compact complex manifolds. Note that Atiyah's theorem is not valid in the C^∞ category. Consider the unit sphere S^2 embedded in $\mathbb{R}^3$. We have

$$S^2 \times \mathbb{R}^3 = TS^2 \oplus (S^2 \times \mathbb{R})\,.$$

Since TS^2 is indecomposable, this example shows that Atiyah's theorem fails in the C^∞ category.

Let M be a compact connected complex manifold. As before, a holomorphic vector bundle E on M is called finite if there are two distinct polynomials $f_1\,, f_2$, whose coefficients are nonnegative integers, such that $f_1(E)$ and $f_2(E)$ are isomorphic.

We proved the following (Adv. Math., Vol. 369):

A vector bundle E on X is finite if and only if E admits a flat holomorphic connection with finite monodromy.

When M is Kähler, this was proved with Y. Holla and G. Schumacher. When M is a Gauduchon astheno-Kähler, this was proved with V. Pingali. The key step in these two works: A finite bundle is numerically flat.

The notion of numerically flat bundles was introduced by Demailly, Peternell and Schneider. A holomorphic line bundle L on a compact complex manifold Y is called nef if for a fixed Hermitian structure g on Y and every $\epsilon > 0$, there is a Hermitian structure $h(\epsilon)$ on L whose curvature $\Theta_{h(\epsilon)}$ satisfies the inequality

$$\frac{1}{2\pi\sqrt{-1}}\Theta_{h(\epsilon)} \;>\; -\epsilon\omega_g\,,$$

where ω_g is the (1, 1)-form on Y associated to g. This condition does not depend on the choice of the Hermitian structure g. A holomorphic vector bundle F on Y is called nef if the tautological line bundle $\mathcal{O}_{\mathbb{P}(F)}(1)$ is nef. A holomorphic vector bundle F on Y is called numerically flat if both E and E^* are nef.

Demailly–Peternell–Schneider proved that a holomorphic vector bundle F on a compact Kähler manifold Y is numerically flat if and only if F

admits a filtration of holomorphic subbundles such that each successive quotient admits a unitary flat connection. This theorem was extended to bundles on Gauduchon astheno-Kähler manifolds in the work with Pingali. But this theorem is not known for bundles on compact complex manifolds. For stable vector bundles with vanishing Chern classes on a Gauduchon astheno-Kähler manifold, the corresponding Hermitian–Einstein connection is flat. This is also not known for bundles on compact complex manifolds. For these reasons, the proofs in the works with Holla–Schumacher and Pingali do not extend to compact complex manifolds.

On the proof of the theorem: We go back to the original work of Nori and prove the following: Let E be a finite bundle on a compact complex manifold M. Then there are finitely many holomorphic vector bundles $\{V_1, \cdots, V_\ell\}$ satisfying the following conditions:

- $E^{\otimes a} \otimes (E^{\otimes b})^*$ is a direct sum of copies of $\{V_1, \cdots, V_\ell\}$ for all $a, b \geq 0$.
- $V_i^{\otimes a} \otimes (V_j^{\otimes b})^*$ is a direct sum of copies of $\{V_1, \cdots, V_\ell\}$ for all $a, b \geq 0$ and $1 \leq i, j \leq \ell$.

  For any $\mathcal{O}_X$–linear homomorphism

$$\gamma : \bigoplus_{i=1}^{\ell} V_i^{\oplus m_i} \longrightarrow \bigoplus_{i=1}^{\ell} V_i^{\oplus n_i},$$

  both kernel(γ) and cokernel(γ) are direct sums of copies of $\{V_1, \cdots, V_\ell\}$.

We construct $\{V_1, \cdots, V_\ell\}$ without using hyperplanes.

We won't go into the details of the construction. We will just explain the starting point of our approach, which is a very simple lemma.

Lemma: Let E be a finite bundle on M and $s \in H^0(M, E)$. If $s(x) = 0$ for some point $x \in M$, then $s = 0$.

Proof: Since E is finite, there are vector bundles $\{W_1, \cdots, W_\ell\}$, such that

$$E^{\otimes b} = \bigoplus_{i=1}^{\ell} W_i^{\oplus n_{b,i}}$$

for all $b \geq 1$. Suppose that $s(x_0) = 0$ but $s \neq 0$. Then $s^{\otimes b} \in H^0(M, E^{\otimes b})$ vanishes at x of order at least b. Therefore, some W_i has a nonzero holomorphic section which vanishes at x of order at least b.

On the other hand for fixed i, the set of integers

$$\{k \mid W_i \text{ has a nonzero holomorphic section vanishing at } x \text{ of order } \geq k\}$$

is finite. This is because $\dim H^0(M, W_i) < \infty$ and $H^0(M, W_i)$ is filtered by subspaces $H^0(M, W_i)_k$ consisting of sections that vanish at x of order at least k.

Therefore, we get a contradiction, and hence $s = 0$ if $s(x_0) = 0$.

Steps in the proof of the theorem:

(1) If $f : E \longrightarrow F$ is a homomorphism of finite bundles, then both kernel(f) and cokernel(f) are locally free.
(2) Fix a Gauduchon metric on M. Any finite bundle E is semistable.
(3) If E is finite, all the successive quotients of the Jordan-Hölder filtration of E are locally free.
(4) A finite stable bundle admits a flat holomorphic connection with finite monodromy.
(5) If a finite bundle E admits a filtration of holomorphic subbundles such that each successive quotient is a trivial bundle, then E is trivial. (This step is already there in the work with Pingali.)