Null Killing vector fields and structures on complex surfaces

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(Initial stage of a work in progress)

Null vectors and structures on 4-dimensional vector space

Let $(V^4, <,>$ be a 4-dimensional oriented real vector space with metric $<,>$ of signature (2,2). The group $SO_0(2,2)$ is connected component of the isometry group.

Then $SO_0(2,2) \cong SL(2,R) \times SL(2,R)/Z_2$, so $V^4 \cong U \otimes W$ as representation space of $so(2,2)$, where U, W are 2-dimensional real vector spaces with volume 2-forms ϵ_1, ϵ_2, with elements called positive or negative spinors.

Then $< u_1 \otimes v_1, u_2 \otimes v_2) = \epsilon_1(u_1, u_2)\epsilon_2(v_1, v_2)$ If u_1, u_2 is basis of U, v_1, v_2

is a basis of W, then

$$u_1 \otimes v_1 \qquad\qquad u_2 \otimes v_1$$

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form *null-tetrad.*

Observation: If X, Y are null vectors spanning a maximal totally isotropic (null) subspace, then they have the form $X = u_1 \otimes v_1, Y = u_2 \otimes v_1$.So fixing such X and Y, there are the following structures on V^4

1) $I \in End(V^4)$, $I^2 = -Id$, $IX = Y$, $I(u_2 \otimes v_1) = u_2 \otimes v_2$

2) $S \in End(V^4)$, $S^2 = Id$, $S = +Id$ on $U \otimes v_1$ and $S = -Id$ on $U \otimes v_2$

I, S are independent of the choices of u_i, v_j, $IS = -SI$, and I is isometry, S is anti-isometry.

Such $I, S, T = IS$ define an action of *split-quaternions* on V^4.

The algebra of the split quaternions is $\mathbb{H}' = \{a + bi + cs + dt \in \mathbb{R}^4 | i^2 = -1, s^2 = t^2 = 1, is = -si = t\}$. For $p, q \in \mathbb{H}'$, the product is defined by $|p|^2 =< p, p >= a^2 + b^2 - c^2 - d^2$. Then one has:

$$pq + qp = -2 < p, q >$$

which is the definition of the Clifford algebra $\mathcal{C}^{(1,1)}$ determined by this scalar product. Other algebraic relations: $\overline{pq} = \overline{q}\ \overline{p}$ and $|pq|^2 = |p|^2|q|^2$.

Parahypercomplex structures on manifolds

Based on the algebra $\mathbb{H}'$ one defines a para-hypercomplex structure (also called complex product and neutral hypercomplex structure) on M^{2n} - $2n$-dimensional manifold as a triple of endomorphisms I, S, T of TM with $I^2 = -Id, S^2 = T^2 = Id, IS = T = -SI$ satisfying the integrability condition $N_I = N_S = N_T = 0$, where $N_A(X,Y) = A^2[AX, AY] + [X,Y] - A[AX,Y] - A[X, AY]$ is the Nijenhuis tensor associated with $A = I, S, T$.

Moreover there exists a unique torsion-free connection (called the Obata connection) ∇ such that $\nabla I = \nabla S = \nabla T = 0$. The structure is the "split analog" of hypercomplex structure. For a given neutral hypercomplex structure (I, S, T) one can consider the set $K_{(a,b,c)} = aI + bS + cT$. Then $K^2_{(a,b,c)} = (-a^2 + b^2 + c^2)Id$ and $\nabla K_{(a,b,c)} = 0$. So in particular:

i) If $a^2 - b^2 - c^2 = 1$, $K_{(a,b,c)}$ is a complex structure,

ii) If $a^2 - b^2 - c^2 = -1$, $K_{(a,b,c)}$ is a product structure (called also para-complex),

iii) If $a^2 - b^2 - c^2 = 0$, $Ker(K_{(a,b,c)}) = Im(K_{(a,b,c)})$ is involutive middle-dimensional distribution on M.

A pseudo-Riemannian metric g, such that I, S, T are skew-symmetric is called para-hyperhermitian (or neutral hyperhermitian). Such metric has split signature and exists only if $dim(M) = 4n$. When the metric satisfies also $\nabla g = 0$, it is called *parahyperkähler* (also hypersymplectic). Since I, S, T are skewsymmetric they define fundamental 2-forms ω_i as

$$\omega_1(X,Y) = g(IX,Y), \qquad \omega_2(X,Y) = g(SX,Y), \qquad \omega_3(X,Y) = g(TX,Y)$$

A structure is parahyperkähler if the forms ω_i are closed.

Observation: Not every parahypercomplex 4 manifold admits compatible global parahyperhermitian metric. For example a bi-elliptic surface I_b and an Inoue surface of type S^- admit parahyprcomplex structures without such a metric. However every 4-dimensional parahypercomplex manifold admitss such a metric after a double cover.

Observation: The structures $K_{a,b,c}$ above lead to definitions of analogs of *twistor spaces* for hypercomplex 4-manifolds. For example the space $Z = M \times D$, where $D = \{(a,b,c)|a^2 - b^2 - c^2 = 1\}$ has a tautological almost complex structure $J^Z|_{(p,a,b,c)} = (K_{a,b,c}, I_D)$ for $p \in M$ and I_D the structure on the unit disc, obtained after inversion from to D. J^Z is integrable when I, S, T are.

Metrics with two null Killing or parallel vector fields

From before, if (M, g) is a (2,2)-signature pseudo-Riemannian manifold with two nowhere vanishing null and orthogonal vector fields, then M admits an (almost) parahypercomplex structure compatible with g. We are interested in the integrability properties. We start with:

Proposition 1 *Let (M, g) be a 4-manifold with a metric g of signature $(2,2)$. Suppose that M admits two parallel and orthogonal null vector fields K,L, linearly independent at every point of M and let I be the almost complex structure determined by (g, K, L) . Then:*

(i) *The structure (g, J) is (pseudo) Kähler.*

(ii) *The metric g is Ricci-flat.*

(iii) *When M is compactr (M, I) is either a torus or a primary Kodaira surface.*

(iv) *M admits a para-hyperkähler structure with metric g and complex structure I.*

In the non-parallel case we have:

Lemma 2 *(Dunajski-West, D.Calderbank) If X is null Killing vector field and $\mathcal{D}$ is a 2-dimensional null-distribution containing X, then $\mathcal{D}$ is integrable, i.e.* $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$

A slight generalization of this is:

Lemma 3 . *Let K and L be two orthogonal null vector fields that are linearly independent at every point.*

(i) If K is conformal Killing, then $\nabla_A B \in span\{K, L\}$ for $A, B \in span\{K, L\}$.

(ii) If K and L are commuting Killing fields, the distribution $span\{K, L\}$ is parallel.

and from here we get:

Theorem 4 *Let (M, g) be a 4-manifold with two orthogonal and linearly independent null conformal Killing vector fields. Then M admits a para-hypercomplex structure (I, S, T), such that (g, I, S, T) is para-hyperhermitian.*

When the fields K, L are parallel they commute. If we only ask $[K, L] = 0$ we get:

Theorem 5 *Let (M, g) be a neutral signature (pseudo) Riemannian 4-manifold with two Killing vector fields K, IK where I is the compatible complex structure from Theorem 4. If K and IK commute, then they are also holomorphic vector fields defining parallel distribution.*

Note In Theorem 5 the fields do not have to be parallel, only the distribution is.

Observation With respect to I (integrable or not) the fundamental form ω_1 is of type $(1,1)$, and because of the commuting relations, $\omega_2 + i\omega_3$ is of type $(2,0)$.

In dimension 4 we have

Theorem 6 *(S.Salamon, Davidov- G -Mushakrov-Yotov)*

i) An oriented 4-dimensional smooth manifold M admits a para-hypercomplex structure if and only if M admits two complex structures with the same orientation I_1 and I_2, such that $I_1I_2 + I_2I_1 = 2pId$ for a constant p with $|p| > 1$.

ii) M admits a para-hyperhermitian structure iff M admits three 2-forms $\omega_1, \omega_2, \omega_3$ such that $-\omega_1^2 = \omega_2^2 = \omega_3^2 = vol, \quad \omega_i \wedge \omega_j = 0$ for $i \neq j$ and a 1-form θ such that $d\omega_i = \theta \wedge \omega_i$.

iii) The structure is parahyperkähler when $\theta = 0$.

If we fix the complex structure, we can reformulate it as:

Lemma 7 *If M, I is an almost complex 4-manifold, then I is integrable and part of a par-hyperhermitian family if and only if there is a nowhere vanishing (2,0)-form Ω a $(1,1)$-form ω and a 1-form θ satisfying $\Omega \wedge \overline{\Omega} = -2\omega^2$, $d\Omega = \theta \wedge \Omega, d\omega = \theta \wedge \omega$*

The Lemma in particular shows that a holomorphic-symplectic 4-manifold with pseudo-Kähler metric is parahyperkähler precisely when $\Omega \wedge \overline{\Omega} = -2\omega^2$

Examples of parahyperkähler compact manifolds and parallel null vector fields

Observation: The last Lemma leads to the fact that the anti-canonical bundle of parahyperk"ahler surface is (holomorphically) trivial.

Proposition (Kamada). *Every compact para-hyperkähler surface is a complex torus or a primary Kodaira surface*

Note that K3 surfaces are excluded. This is due to the fact that they do not admit a symplectic form of opposite orientation - a fact proven by Seiberg-Witten theory.

Moreover, Kamada obtained a description of all para-hyperkähler structures on both types of surfaces.

Complex tori

Proposition (Kamada). *For any para-hyperkähler structure on a complex torus $M = \mathbb{C}^2/\Gamma$ there exist (global) coordinates (z_1, z_2) on $\mathbb{C}^2$ such that the structure is defined by means of the following symplectic forms:*

$$\omega_1 = Im(dz_1 \wedge d\bar{z}_2) + (i/2)\partial\bar{\partial}\varphi,$$

$$\omega_2 = Re(dz_1 \wedge dz_2), \quad \omega_3 = Im(dz_1 \wedge dz_2),$$

where ϕ is a smooth function on M such that

$$4iIm(dz_1 \wedge d\bar{z}_2) \wedge \partial\bar{\partial}\varphi = \partial\bar{\partial}\varphi \wedge \partial\bar{\partial}\varphi.$$

Conversely, any three forms $\Omega_1, \Omega_2, \Omega_3$ of the form given above determine a para-hyperkähler structure on the torus. Moreover, its metric is flat if and only if φ is constant.

Primary Kodaira surfaces

Consider the affine transformations $\rho_i(z_1, z_2) = (z_1 + a_i, z_2 + \overline{a}_i z_1 + b_i)$ of $\mathbb{C}^2$, where a_i, b_i, $i = 1, 2, 3, 4$, are complex numbers such that $a_1 = a_2 = 0$, $Im(a_3\overline{a}_4) = b_1$. Then ρ_i generate a group G of affine transformations acting freely and properly discontinuously on $\mathbb{C}^2$. The quotient space $M = \mathbb{C}^2/G$ is called a primary Kodaira surface.

Proposition (Kamada). *For any para-hyperkähler structure on a primary Kodaira surface $M = \mathbb{C}^2/G$ there exist (global) coordinates (z_1, z_2) on $\mathbb{C}^2$ such that the structure is defined by means of the following symplectic forms:*

$$\omega_1 = Im(dz_1 \wedge d\overline{z}_2) + iRe(z_1)dz_1 \wedge d\overline{z}_1 + (i/2)\partial\overline{\partial}\phi,$$

$$\omega_2 = Re(e^{i\theta}dz_1 \wedge dz_2), \quad \omega_3 = Im(e^{i\theta}dz_1 \wedge dz_2),$$

where θ is a real number and ϕ is a smooth function on M such that

$$4i(Im(dz_1 \wedge d\overline{z}_2) + iRe(z_1)(dz_1 \wedge d\overline{z}_1)) \wedge \partial\overline{\partial}\phi = \partial\overline{\partial}\phi \wedge \partial\overline{\partial}\phi.$$

Conversely, any three forms $\Omega_1, \Omega_2, \Omega_3$ of the form given above determine a para-hyperkähler structure on M. Moreover, its metric is flat if and only if φ is constant.

Examples.

• The metric is flat $\Longleftrightarrow \phi = const.$

• Every primary Kodaira surface M is a toric bundle over an elliptic complex curve L and the lift ϕ of any smooth function on L to the surface M satisfies the above equation.

Existence of null parallel vector fields: In both examples, the function ϕ which has vanishing derivatives along two null vector fields exists. In the case of primary Kodaira, the fields are along the fibers of the elliptic fibration. Inn both cases the fields are parallel, so in both cases we have infinite dimensional families of metrics with two fixed null parallel vector fields. In fact the question whether another solutions of the parahyperkähler equations for ϕ exist is open, but the metric with two parallel null vector fields on compact 4-manifold is among the ones just described.

Compact 4-manifolds with two null Killing vector fields

From the existence of nowhere vanishing (2,0)-form follows that $c_1(I) = 0$ for any almost parahyperhermitian structure g, I, S, T. When I is integrable s M, I is a compact complex surface one can follow the Kodaira's classification and determine all possible such surfaces. Note that from the adjunction formula follows that the surface is minimal. Also for Kodaira dimension $k = -\infty$ $c_1^2(I) = -b_2(M)$, so the more "exotic" surfaces of type VII_0 are excluded. This leads to:

Theorem 8 *Suppose that M, g is an oriented compact 4-manifold with neutral metric g and two orthogonal null Killing vector fields, independent at each point. Then up to a finite cover M is diffeomorphic to one the following:*

i) 4-torus

ii) $S^1 \times H_3$, where H_3 is a compact quotient of the 3-dimensional Heisenberg group under an action of a lattice

iii) $S^1 \times S^3$

iv) $S^1 \times M^3$, where $M^3 = \widetilde{SL(2,R)}/\Gamma$ with Γ - a co-compact discrete subgroup of $\widetilde{SL(2,R)}$. (Such M^3 is a Seifert bundle)

v) A compact quotient of a 4-dimensional solvable Lie group of type S_0, S^+ or S^-.

If in addition the two vector fields commute, then in v) only S^+ is allowed.

Now we mention what the commuting relartions between X and Y are:

Theorem 9 *Suppose M is a complete 4-dimensional pseudo-Riemannian manifold with two independent Killing vector fields spanning an integrable 2-dimensional distribution $\mathcal{D}$ (so when they are null $\mathcal{D}$ is automatically integrable). Then we have one of the two cases:*

i) There are non-vanishing independent vector fields X, Y generating $\mathcal{D}$ at every point, such that either they commute, or $[X, Y] = Y$.

ii) There are non-vanishing vector fields X, Y_1, Y_2 generating $\mathcal{D}$ at every point, such that $[X, Y_1] = Y_2, [X, Y_2] = -Y_1$.

This is a simple direct consequence of the classification of finite dimensional Lie algebras of vector fields on a plane (Komrakov-Churyumov-Doubrov). Note that all three options appear locally for a flat metric. In the compact case option ii) is an open question.

Theorem 10 *The smooth 4-manifolds of the types $i), ii)$, and the compact quotients of solvable Lie groups of type S^+ from iv) in Theorem 8 admit (infinite-dimensional families of) neutral metrics with 2 commuting independent orthogonal null vector fields. The properly elliptic elliptic surfaces from v) in Theorem 8 admit metrics with null Killing vector fields X, Y such that $[X, Y] = Y$.*

The proof is case by case and the examples are next.

Inoue surfaces of type S^+ Since we are looking at examples with 2 null Killing fields, we can select the complex structure appropriately. Consider the 4-dimensional solvable Lie algebra sol_4^1 defined via relations

$$[X_2, X_3] = X_1 \quad [X_2, X_4] = X_2 \quad [X_3, X_4] = -X_3$$

and X_1 commuting with all $X_i, i = 1, 2, 3$. The dual 1-forms on the corresponding simply-connected Lie3 group satisfy

$$d\alpha_1 = \alpha_3 \wedge \alpha_2 \quad d\alpha_2 = \alpha_4 \wedge \alpha_2 \quad d\alpha_3 = \alpha_3 \wedge \alpha_4 \quad d\alpha_4 = 0$$

Consider the complex structure defined via $IX_1 = X_2, IX_3 = X_4$ which has (1,0)-forms $\alpha_1 + i\alpha_2, \alpha_3 + i\alpha_4$. Since

$$d(\alpha_1 + i\alpha_2) = (\alpha_3 + i\alpha) \wedge \alpha_2 \quad d(\alpha_3 + i\alpha_4) = \frac{i}{2}(\alpha_3 + i\alpha_4) \wedge (\alpha_3 - i\alpha_4)$$

the structure I is integrable.

Such structure defines an invariant complex structure on the simply-connected solvable Lie group with Lie algebra sol_1^4. It is know that the group admits a cocompact discrete subgroup Γ and the quotient equipped with the complex structure induced by I is an Inoue surface of type S^+.

Consider $\omega = \alpha_1 \wedge \alpha_3 + \alpha_2 \wedge \alpha_4$. Then it is non-degenerate $(1, 1)$ form defining a neutral metric via $g(X, Y) = \omega(X, IY)$. Let $K = X_1, IK = IX_1 = X_2$. Then $\mathcal{L}_K I = \mathcal{L}_K \omega = 0$, so K is KIlling and real part of a holomorphic vector field. Also one can check that IK is again Killing and holomorphic with $[K, IK] = 0$. We can directly check that replacing ω by $\omega_f = \omega + f\alpha_3 \wedge \alpha_4$, with $X_1(f) = X_2(f) = 0$ provides for small enough f again $(1, 1)$-form defining neutral metric with K, IK still null and Killing.

Note that $\Omega = (\alpha_1 + i\alpha_2) \wedge (\alpha_3 + i\alpha_4)$ is (2,0)-form and $d\Omega = \alpha_4 \wedge \Omega, d\omega_f = \alpha_4 \wedge \omega_f$, defines a parahyperhermitian structure

Properly elliptic surfaces

Consider a basis of $sl(2,R)$ by taking the matrices $X = 1/2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $Y = 1/2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $Z = 1/2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ the commutation relations are

$$[X,Y] = Z, [Y,Z] = -X, [Z,X] = Y$$

and in the basis of the dual 1-forms

$$d\alpha = \beta \wedge \gamma, \quad d\beta = \alpha \wedge \gamma, \quad d\gamma = -\alpha \wedge \beta$$

Suppose that V is the invariant vector field tangent to the S^1-factor and define an invariant complex structure I by $IV = Z, IX = Y$. If θ is dual to V with $d\theta = 0$, then the basis of $(1,0)$-forms for an invariant complex structure is given by $\theta + i\alpha, \beta + i\gamma$. Notice also that

$$d(\beta + i\gamma) = -i\alpha \wedge (\beta + i\gamma), d(\theta + i\alpha) = \beta \wedge \gamma$$

so I is integrable. . Now consider the form

$$\Omega = (\theta + i\alpha) \wedge (\beta + i\gamma)$$

This is a $(2,0)$-form and from above follows $d\Omega = -\theta \wedge \Omega$. Note that $\omega = \theta \wedge \alpha - \beta \wedge \gamma$ also satisfies $d\omega = -\theta \wedge \omega$, so it defines a para-hyperhermitian structure, which is ;eft invariant (and locally conformally para-hyperkähler flat). In fact the metric defined by ω and I is the bi-invariant metric $g = \theta^2 + \alpha^2 - \beta^2 - \gamma^2$. One can see directly that

$$\mathcal{L}_V g = \mathcal{L}_X g = \mathcal{L}_Y g = \mathcal{L}_Z g = 0$$

which also follows from the fact that V, X, Y, Z are (left-)invariant. Then we can choose

$$K = V + Y, IK = X + Z$$

Pseudo-Hermitian surfaces with nonvanishing null Killing vector field

We can modify the observation from before about the relation between Killing vector fields and geometric structures on 4-manifolds in the following way:

Lemma 11 *If* $V, <,>$ *is a vector space with* $(2,2)$ *scalar product* $<,>$ *and* I *is a compatible complex structure and* X *is a null vector in* V*, Then there is a unique endomorphism* S*, such that* $S^2 = 1, IS = -SI, < SX, SY > = - < X, Y >$*, and* $SX = X$*.*

The proof is obvious - take S to be identity on the null plane containing X and different from X, IX, and -identity on the same plane containing IX. Using this construction we have:

Theorem 12 *If* (M, g, I) *is a pseudo-Hermitian surface with nowhere vanishing null vector field* X*. If* M *is compact, then the complex surface* (M, I) *is one of the following:*

i) a tori

ii) a primary Kodaira surface

iii) A Hopf surface

iv) an Inoue surface

v) a properly elliptic surface

For any M*, if* X *is Killing and holomorphic, then* $\mathcal{L}_X S = 0$ *for the structure* S *defined above.*

Example on $\mathbf{S}^1 \times \mathbf{S}^3$: Consider $S^3 = SU(2)$ as a Lie group and X_1, X_2, X_3, X_4 the left-invariant vector fields defining its Lie algebra, where

$$[X_2, X_3] = X_4, \quad [X_3, X_4] = X_2, \quad [X_4, X_2] = X_3$$

and X_1 commutes with all others. Again the dual basis of invariant 1-forms α_i defines a complex structure with $\alpha_1 + i\alpha_2, \alpha_3 + i\alpha_4$ being (1,0)-forms Then the form $\omega = Re((\alpha_1 + i\alpha_2) \wedge (\alpha_3 - i\alpha_4))$ defines a left-invariant pseudo-Hermitian metric. One can check that $\mathcal{L}_X \alpha_i = 0$ because of the commuting relations, so X is null Killing and real holomorphic. Here IX is not Killing.

Curvature of 4-dimensional (pseudo-)Riemannian manifolds

Let M be an oriented 4-manifold with a metric g of signature (2,2) or (4,0). Then g induces an inner product on the bundle Λ^2 and the Hodge star operator $* : \Lambda^2 \to \Lambda^2$ is an involution, so $\Lambda^2 = \Lambda_+ + \Lambda_-$.

Let $\mathcal{R} : \Lambda^2 \longrightarrow \Lambda^2$ be the curvature operator of (M, g), related to the curvature tensor R by

$$g(\mathcal{R}(X \wedge Y), Z \wedge T) = g(R(X, Y)Z, T); \quad X, Y, Z, T \in TM.$$

Then $\mathcal{R}$ admits an $SO(4)$ or $SO(2,2)$-irreducible decomposition

$$\mathcal{R} = \frac{\tau}{6} I + \mathcal{B} + \mathcal{W}_+ + \mathcal{W}_-$$

. Here τ is the scalar curvature, $\mathcal{B}$ represents the traceless Ricci tensor, $\mathcal{W} = \mathcal{W}_+ + \mathcal{W}_-$ corresponds to the Weyl conformal tensor, and $\mathcal{W}_\pm = \mathcal{W}|\Lambda_\pm = \frac{1}{2}(\mathcal{W} \pm *\mathcal{W}) \in End(\Lambda_\pm)$. In matrix form:

$$\mathcal{R} = \begin{bmatrix} W_+ + \frac{\tau}{6} & B \\ {}^tB & W_- + \frac{\tau}{6} \end{bmatrix}$$

The metric g is Einstein exactly when $\mathcal{B} = 0$ and is conformally flat when $\mathcal{W} = 0$. It is said to be *self-dual*, resp. *anti-self-dual*, if $\mathcal{W}_- = 0$, resp. $\mathcal{W}_+ = 0$.

In the presence of Hermitian structure - an integrable compatible complex structure I with fundamental form $\omega \in \Lambda_+$, there is a further decomposition of the curvature operator under the group $U(2)$ or $U(1,1)$. In particular $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$ in J-invariant and anti-invariant parts. Also $\mathcal{W}_+ = \mathcal{W}_+^1 + \mathcal{W}_+^2 + \mathcal{W}_+^3$, where $\mathcal{W}_+^1 = \frac{3}{8}(g(\mathcal{W}_+(\omega), \omega))(\omega \otimes \omega - \frac{3}{2}Id)$, $\mathcal{W}_+^2 = \frac{1}{2}(\mathcal{W}_+ + J \circ \mathcal{W}_+ \circ J)$, $\mathcal{W}_+^3 = \mathcal{W}_+ - \mathcal{W}_+^1 - \mathcal{W}_+^2$. If J is integrable $\mathcal{W}_+^3 = 0$. Moreover $Spec(\mathcal{W}_+)$ is degenerate iff $\mathcal{W}_+^2 = 0$.

If I is integrable, then $-I$ is again integrable and positively oriented. We say that complex structures are independent if they are not equal up to sign. In this terms one has the following:

Theorem 13 *(S.Salamon for positive g) Assume that an oriented Riemannian 4-manifold admits 3 independent compatible positively-oriented complex structures. Then g is anti-selfdual*

The proof is unchanged for the neutral signature metrics. So one has:

Corollary 14 *If (M, g, I) is a 4-dimensional manifold with complex structure I compaatible with the Hermitian metric g of signature (2,2) which admits a Killing vector field X, then either X is real-holomorphic, or g is anti-selfdual.*

Existence of special positive Hermitian metrics, known results

If g is positive and M is compact, then:

• $\mathcal{W}_+^2 = 0$ iff (g, J) is locally conformally Kähler ($d\theta = 0$).

• If (M, g, J) is Hermitian Einstein and non-Kähler, then $c_1(J) > 0$ and (g, J) is conformally Kähler(C. LeBrun). Existence on $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$ is known for $k = 1$ (Page-Pope), $k = 2$ (Chen-LeBrun-Weber). For $k > 2$ Hermitian Einstein metric is Kähler (LeBrun).

• (M, g, J) is Hermitian self-dual ($\mathcal{W}_- = 0$) iff it is conformally equivalent to one of the following (Apostolov-Davidov-Mushkarov): $(\mathbb{C}P^2, g_{FS})$, compact quotient of the unit ball with the Bergman metric, complex torus or hyperelliptic surface with flat metric, minimal ruled surface with conformaly flat metric, Hopf surface with conformally flat metric.

• If (M, g, J) is anti-selfdual and b_1 is even, then there is a Kähler scalar flat metric in $[g]$. If b_1 is odd, (M, J, g) is locally conformally Kähler and is of class VII. On Hopf surface the metric is necessary conformally flat (Boyer). There is a construction on some parabolic Inoue surfaces (LeBrun).

• If Λ_+ is trivial, there is hyperhermitian structure $(M, g, I, J, , K)$ and (M, I) is either conformally hyperkähler or quaternionic Hopf surface (Boyer).

• If $Spec(\mathcal{W}_+) = 3$, then there are at most 2 integrable J compatible with g and the orientation - bihermitian structure (twisted generalized Kähler).

Known results in the indefinite case:

If g is neutral, most of the topological restrictions above are no longer valid, since they are based on vanishing of the L_2-norms or positivity. In particular most of the known results are local. Existence of neutral metric g leads to topological restrictions itself. When g is neutral or pseudo - Kähler, some restrictions arise from the Seiberg-Witten theory. Here are the main known global results:

• (Petean) Let (M, g, J) be a compact neutral Kähler surface and $k(M, J)$ its Kodaira dimension.

(i) If $k(M, J) = -\infty$, then (M, J) is either a ruled surface, or a surface of class VII_0 with no global spherical shell and with positive even second Betti number.

(ii) If $k(M, J) = 0$, then (M, J) is either a hyperelliptic surface, a primary Kodaira surface or a complex torus.

(iii) If $k(M, J) = 1$, then (M, J) is a minimal properly elliptic surface with zero signature.

(iv) If $k(M, J) = 2$, then (M,J) is a minimal surface of general type with nonnegative even signature.

• (Petean) If in addition M is Kähler Einstein then M is one of the following:

i) a Complex Torus;

ii) a Hyperelliptic surface;

iii) a Primary Kodaira surface;

iv) a minimal ruled surface over a curve of genus g 2; or

v) a minimal surface of class V II0 with no global spherical shell, and with second Betti number even and positive.

• A Zollfrei manifold is a manifold in which all maximally extended null geodesics are closed. Using a global twistor approach LeBrun and Mason showed that a self-dual Zollfrei 4-manifold is diffeomorphic to either $S^2 \times S^2$ or $S^2 \times S^2/Z_2$.

Parahyperhermitian reduction and instanton moduli spaces

The reduction of para-hypercomplex structures is similar to the hypercomplex reduction as developed by D. Joyce and is based on the reduction of hypersymplectic structures considered by N.Hitchin. Let G be a compact group of hypercomplex automorphisms of (M, I, S, T) with Lie algebra $\mathfrak{g}$ and denote the algebra of the induced hyper-holomorphic vector fields with the same $\mathfrak{g}$. Suppose that $\nu = (\nu_1, \nu_2, \nu_3) : M \to \mathbf{R^3} \otimes \mathfrak{g}^*$ is a G-equivariant map satisfying the following:

i) The Cauchy-Riemann condition $Id\nu_1 = -Sd\nu_2 = -Td\nu_3$, and

ii) The transversality condition $Id\nu_1(X) \neq 0$ for all $X \in \mathfrak{g}$.

Any map satisfying these conditions is called a G-moment map. Given a point $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ in $\mathbf{R^3} \otimes \mathfrak{g}$, denote the level set $\nu^{-1}(\zeta)$ by P. If ζ_i are in the center of $\mathfrak{g}$ then the level set P is invariant. Then we have:

Theorem 15 *Let ν be a G-moment map for a group G which acts properly and freely on $P = \nu^{-1}(0)$. Suppose that on $\nu^{-1}(0)$ there is no non-zero solution to the equation $IX + SY + TZ = 0$ for $X, Y, Z \in \mathfrak{g}$. Then the quotient manifold $N = P/G$ is smooth and inherits a neutral hypercomplex structure.*

We first notice that in the moment map definition one can use any 3 complex structures I_1, I_2, I_3 of the family $K_{(a,b,c)}$ instead of I, S, T. Then the Cauchy-Riemann condition is $I_1 d\nu_1 = I_2 d\nu_2 = I_3 d\nu_3$. Here the anti-commutators of I_1, I_2, I_3 satisfy Theorem 1 i). Then the reduction theorem is still valid.

Now consider a compact complex surface with neutral hypercomplex structure and a neutral hyperhermitian metric. Then we fix complex structures I_1, I_2, I_3 and their Kähler forms $\omega_1, \omega_2, \omega_3$, which define a basis for the self-dual forms Λ^+ at each point. Now a 2-form F is ASD if and only if $F \wedge \omega_i = 0$ for $i = 1, 2, 3$. In particular a connection A on a SU(k)-bundle is an instanton if its curvature F_A satisfies this condition. Then we have:

Corollary 16 *The "smooth part" of the moduli space of SU(k)-instantons on a compact para-hypercomplex four manifold admits a para- hypercomplex structure.*

Here the group G is the gauge group of the bundle. The moment maps are given by $\nu_i(A) = F_A \wedge \omega_i$.

If $a \in \Omega^1(M, su(k))$ is any tangent vector at A generated by $Lie(G)$, then $d(\nu_i)_A(a) = d_A a \wedge \omega_i$. In this case the main identity is $\omega_i \wedge d^c_A a = d_A * a - d^c\omega \wedge a$ for any complex structure where $d^c_A = I^{-1} d_A I$. The Cauchy-Riemann condition follows from the identity $d^1\omega_1 = d^2\omega_2 = d^3\omega^3 = *\theta$ satisfied for any neutral hyperhermitian structure. Then the subset in the smooth part of the moduli space where this structure is degenerate is given by $[A]$ such that $d^1_A a + d^2_A b + d^3_A c = 0$ has a nonzero solution (a, b, c) for some $A \in [A]$.