Wall finiteness obstruction for DG categories and for algebras over colored DG operads

Weekly seminar of Laboratory of algebraic geometry
Department of Mathematics, HSE

Alexander Efimov

Steklov Mathematical Institute of RAS and HSE

25th September, 2020

Plan of the talk

I will start by recalling the following results of R. Thomason and C.T.C. Wall:

Plan of the talk

I will start by recalling the following results of R. Thomason and C.T.C. Wall:

1) Thomason's classification of dense subcategories of a small triangulated category $\mathcal{T}$ in terms of abelian subgroups of $K_0(\mathcal{T})$.

Plan of the talk

I will start by recalling the following results of R. Thomason and C.T.C. Wall:

1) Thomason's classification of dense subcategories of a small triangulated category $\mathcal{T}$ in terms of abelian subgroups of $K_0(\mathcal{T})$. In particular, it gives a criterion when a perfect module over a small DG category $\mathcal{A}$ is quasi-isomorphic to a semi-free finitely generated $\mathcal{A}$-module.

Plan of the talk

I will start by recalling the following results of R. Thomason and C.T.C. Wall:
1) Thomason's classification of dense subcategories of a small triangulated category $\mathcal{T}$ in terms of abelian subgroups of $K_0(\mathcal{T})$. In particular, it gives a criterion when a perfect module over a small DG category $\mathcal{A}$ is quasi-isomorphic to a semi-free finitely generated $\mathcal{A}$-module.
2) Wall's finiteness obstruction theorem, giving a criterion when a finitely dominated topological space is weakly homotopy equivalent to a finite CW complex.

Plan of the talk

I will start by recalling the following results of R. Thomason and C.T.C. Wall:

1) Thomason's classification of dense subcategories of a small triangulated category $\mathcal{T}$ in terms of abelian subgroups of $K_0(\mathcal{T})$. In particular, it gives a criterion when a perfect module over a small DG category $\mathcal{A}$ is quasi-isomorphic to a semi-free finitely generated $\mathcal{A}$-module.

2) Wall's finiteness obstruction theorem, giving a criterion when a finitely dominated topological space is weakly homotopy equivalent to a finite CW complex.

Then I will formulate the analogue of Wall's finiteness obstruction theorem for (the Morita homotopy category of) small DG categories, formulate some applications (in particular, disprove two conjectures of Orlov) and sketch the proof.

Plan of the talk

I will start by recalling the following results of R. Thomason and C.T.C. Wall:

1) Thomason's classification of dense subcategories of a small triangulated category $\mathcal{T}$ in terms of abelian subgroups of $K_0(\mathcal{T})$. In particular, it gives a criterion when a perfect module over a small DG category $\mathcal{A}$ is quasi-isomorphic to a semi-free finitely generated $\mathcal{A}$-module.

2) Wall's finiteness obstruction theorem, giving a criterion when a finitely dominated topological space is weakly homotopy equivalent to a finite CW complex.

Then I will formulate the analogue of Wall's finiteness obstruction theorem for (the Morita homotopy category of) small DG categories, formulate some applications (in particular, disprove two conjectures of Orlov) and sketch the proof. This result can also be considered as a secondary version of Thomason's theorem.

Plan of the talk

I will start by recalling the following results of R. Thomason and C.T.C. Wall:

1) Thomason's classification of dense subcategories of a small triangulated category $\mathcal{T}$ in terms of abelian subgroups of $K_0(\mathcal{T})$. In particular, it gives a criterion when a perfect module over a small DG category $\mathcal{A}$ is quasi-isomorphic to a semi-free finitely generated $\mathcal{A}$-module.

2) Wall's finiteness obstruction theorem, giving a criterion when a finitely dominated topological space is weakly homotopy equivalent to a finite CW complex.

Then I will formulate the analogue of Wall's finiteness obstruction theorem for (the Morita homotopy category of) small DG categories, formulate some applications (in particular, disprove two conjectures of Orlov) and sketch the proof. This result can also be considered as a secondary version of Thomason's theorem.

Finally, I will (very briefly) sketch a generalization to algebras over colored DG operads.

Thomason's classification of dense subcategories

Let $\mathcal{T}$ be a small triangulated category.

Thomason's classification of dense subcategories

Let $\mathcal{T}$ be a small triangulated category. Its Grothendieck group $K_0(\mathcal{T})$ is generated by the isomorphism classes of objects $X \in \mathcal{T}$, subject to relation $[Y] = [X] + [Z]$ for any exact triangle

$$X \to Y \to Z \to X[1].$$

Thomason's classification of dense subcategories

Let $\mathcal{T}$ be a small triangulated category. Its Grothendieck group $K_0(\mathcal{T})$ is generated by the isomorphism classes of objects $X \in \mathcal{T}$, subject to relation $[Y] = [X] + [Z]$ for any exact triangle

$$X \to Y \to Z \to X[1].$$

Definition

A dense subcategory of $\mathcal{T}$ *is a strictly full triangulated subcategory* $\mathcal{S} \subset \mathcal{T}$ *such that for any object* $X \in \mathcal{T}$ *there exists an object* $X' \in \mathcal{T}$ *such that* $X \oplus X' \in \mathcal{S}$.

Thomason's classification of dense subcategories

Thomason proved a remarkable classification theorem for dense subcategories.

Thomason's classification of dense subcategories

Thomason proved a remarkable classification theorem for dense subcategories.

Theorem (Thomason, 1997)

For a small triangulated category $\mathcal{T}$, its dense subcategories are in bijection with abelian subgroups of $K_0(\mathcal{T})$. Here a subgroup $A \subseteq K_0(\mathcal{T})$ corresponds to the full subcategory

$$\mathcal{S}_A = \{X \in \mathcal{T} \mid [X] \in A\} \subseteq \mathcal{T}.$$

Thomason's classification of dense subcategories

Probably the simplest way of proving Thomason's theorem is via the following

Proposition (Heller's criterion)

Given objects X, Y of a small triangulated category $\mathcal{T}$, the following are equivalent:
(i) : $[X] = [Y]$ *in* $K_0(\mathcal{T})$;
(ii) : *there exist objects* $Z, U, W \in \mathcal{T}$ *and exact triangles of the form*

$$Z \to X \oplus U \to W \to Z[1],$$

$$Z \to Y \oplus U \to W \to Z[1].$$

DG reformulation of Thomason's theorem

Let now $\mathcal{A}$ be a small DG category (over $\mathbb{Z}$). We denote by $D(\mathcal{A})$ the derived category of right $\mathcal{A}$-modules.

DG reformulation of Thomason's theorem

Let now $\mathcal{A}$ be a small DG category (over $\mathbb{Z}$). We denote by $D(\mathcal{A})$ the derived category of right $\mathcal{A}$-modules. For an object $X \in \mathcal{A}$, we denote by h_X the $\mathcal{A}$-module representable by X.

DG reformulation of Thomason's theorem

Let now $\mathcal{A}$ be a small DG category (over $\mathbb{Z}$). We denote by $D(\mathcal{A})$ the derived category of right $\mathcal{A}$-modules. For an object $X \in \mathcal{A}$, we denote by h_X the $\mathcal{A}$-module representable by X. By definition, the full subcategory of perfect $\mathcal{A}$-modules $D_{perf}(\mathcal{A}) \subset D(\mathcal{A})$ is the full triangulated Karoubi complete subcategory, generated by the $\mathcal{A}$-modules h_X, $X \in \mathcal{A}$.

DG reformulation of Thomason's theorem

Let now $\mathcal{A}$ be a small DG category (over $\mathbb{Z}$). We denote by $D(\mathcal{A})$ the derived category of right $\mathcal{A}$-modules. For an object $X \in \mathcal{A}$, we denote by h_X the $\mathcal{A}$-module representable by X. By definition, the full subcategory of perfect $\mathcal{A}$-modules $D_{perf}(\mathcal{A}) \subset D(\mathcal{A})$ is the full triangulated Karoubi complete subcategory, generated by the $\mathcal{A}$-modules h_X, $X \in \mathcal{A}$. Equivalently, $D_{perf}(\mathcal{A})$ is the full subcategory of compact objects in $D(\mathcal{A})$.

DG reformulation of Thomason's theorem

Let now $\mathcal{A}$ be a small DG category (over $\mathbb{Z}$). We denote by $D(\mathcal{A})$ the derived category of right $\mathcal{A}$-modules. For an object $X \in \mathcal{A}$, we denote by h_X the $\mathcal{A}$-module representable by X. By definition, the full subcategory of perfect $\mathcal{A}$-modules $D_{perf}(\mathcal{A}) \subset D(\mathcal{A})$ is the full triangulated Karoubi complete subcategory, generated by the $\mathcal{A}$-modules h_X, $X \in \mathcal{A}$. Equivalently, $D_{perf}(\mathcal{A})$ is the full subcategory of compact objects in $D(\mathcal{A})$.

An $\mathcal{A}$-module M is called semi-free finitely generated if it has a finite filtration

$$0 = F_0 M \subset F_1 M \subset \cdots \subset F_n M = M,$$

such that each subquotient $F_i M / F_{i-1} M$ is isomorphic to an $\mathcal{A}$-module of the form $h_X[n]$, $X \in \mathcal{A}$, $n \in \mathbb{Z}$.

DG reformulation of Thomason's theorem

Let now $\mathcal{A}$ be a small DG category (over $\mathbb{Z}$). We denote by $D(\mathcal{A})$ the derived category of right $\mathcal{A}$-modules. For an object $X \in \mathcal{A}$, we denote by h_X the $\mathcal{A}$-module representable by X. By definition, the full subcategory of perfect $\mathcal{A}$-modules $D_{perf}(\mathcal{A}) \subset D(\mathcal{A})$ is the full triangulated Karoubi complete subcategory, generated by the $\mathcal{A}$-modules h_X, $X \in \mathcal{A}$. Equivalently, $D_{perf}(\mathcal{A})$ is the full subcategory of compact objects in $D(\mathcal{A})$.

An $\mathcal{A}$-module M is called semi-free finitely generated if it has a finite filtration

$$0 = F_0M \subset F_1M \subset \cdots \subset F_nM = M,$$

such that each subquotient $F_iM/F_{i-1}M$ is isomorphic to an $\mathcal{A}$-module of the form $h_X[n]$, $X \in \mathcal{A}$, $n \in \mathbb{Z}$. We denote by $\mathcal{SF}_{f.g.}(\mathcal{A})$ the DG category of semi-free finitely generated $\mathcal{A}$-modules.

DG reformulation of Thomason's theorem

One can formulate the following DG version of Thomason's classification, which is just a special case.

Theorem (Thomason)

For a small DG category $\mathcal{A}$ and a perfect $\mathcal{A}$-module M the following are equivalent
(i) : *M is quasi-isomorphic to a semi-free finitely generated $\mathcal{A}$-module;*
(ii) : *the class $[M] \in K_0(D_{perf}(\mathcal{A}))$ is contained in the abelian subgroup generated by the classes $[h_X]$, $X \in \mathcal{A}$.*

Classical Wall obstruction

Recall that a CW complex X is called *finitely dominated* if there exists a finite CW complex Y and maps $X \xrightarrow{f} Y \xrightarrow{g} X$, such that $gf \sim \mathrm{id}_X$. Equivalently, the identity map id_X is homotopic to some map $r : X \to X$ such that the image $r(X)$ has a compact closure.

Classical Wall obstruction

Recall that a CW complex X is called *finitely dominated* if there exists a finite CW complex Y and maps $X \xrightarrow{f} Y \xrightarrow{g} X$, such that $gf \sim \mathrm{id}_X$. Equivalently, the identity map id_X is homotopic to some map $r : X \to X$ such that the image $r(X)$ has a compact closure.

In 1959 Milnor asked if every finitely dominated CW complex is homotopy equivalent to a finite CW complex. This was already known in the case when each connected component of X is simply connected, but it was considered to be a difficult problem in general.

Classical Wall obstruction

Recall that a CW complex X is called *finitely dominated* if there exists a finite CW complex Y and maps $X \xrightarrow{f} Y \xrightarrow{g} X$, such that $gf \sim \mathrm{id}_X$.
Equivalently, the identity map id_X is homotopic to some map $r : X \to X$ such that the image $r(X)$ has a compact closure.

In 1959 Milnor asked if every finitely dominated CW complex is homotopy equivalent to a finite CW complex. This was already known in the case when each connected component of X is simply connected, but it was considered to be a difficult problem in general.

For simplicity, let us assume that X is connected.

Classical Wall obstruction

In 1965, Wall defined an invariant $w(X) \in \widetilde{K_0}(\mathbb{Z}[\pi_1(X)])$ (see below) for any finitely dominated space X, and proved the following result.

Classical Wall obstruction

In 1965, Wall defined an invariant $w(X) \in \widetilde{K_0}(\mathbb{Z}[\pi_1(X)])$ (see below) for any finitely dominated space X, and proved the following result.

Theorem (C.T.C. Wall, 1965)

A connected finitely dominated space X has a homotopy type of a finite CW complex if and only if $w(X) = 0$.

Classical Wall obstruction

In 1965, Wall defined an invariant $w(X) \in \widetilde{K_0}(\mathbb{Z}[\pi_1(X)])$ (see below) for any finitely dominated space X, and proved the following result.

Theorem (C.T.C. Wall, 1965)

A connected finitely dominated space X has a homotopy type of a finite CW complex if and only if $w(X) = 0$.

Probably the simplest (slightly refined and possibly new) description of the class $w(X)$ is the following. Choose a base point $x_0 \in X$.

Classical Wall obstruction

In 1965, Wall defined an invariant $w(X) \in \widetilde{K_0}(\mathbb{Z}[\pi_1(X)])$ (see below) for any finitely dominated space X, and proved the following result.

Theorem (C.T.C. Wall, 1965)

A connected finitely dominated space X has a homotopy type of a finite CW complex if and only if $w(X) = 0$.

Probably the simplest (slightly refined and possibly new) description of the class $w(X)$ is the following. Choose a base point $x_0 \in X$. Consider the DG algebra $C_\bullet(\Omega_{x_0} X)$ of singular chains on the based loop space.

Classical Wall obstruction

In 1965, Wall defined an invariant $w(X) \in \widetilde{K_0}(\mathbb{Z}[\pi_1(X)])$ (see below) for any finitely dominated space X, and proved the following result.

Theorem (C.T.C. Wall, 1965)

A connected finitely dominated space X has a homotopy type of a finite CW complex if and only if $w(X) = 0$.

Probably the simplest (slightly refined and possibly new) description of the class $w(X)$ is the following. Choose a base point $x_0 \in X$. Consider the DG algebra $C_\bullet(\Omega_{x_0} X)$ of singular chains on the based loop space. By the work of Brav-Dyckerhoff that this DG algebra is homologically smooth (the diagonal bimodule is perfect).

Classical Wall obstruction

In 1965, Wall defined an invariant $w(X) \in \widetilde{K_0}(\mathbb{Z}[\pi_1(X)])$ (see below) for any finitely dominated space X, and proved the following result.

Theorem (C.T.C. Wall, 1965)

A connected finitely dominated space X has a homotopy type of a finite CW complex if and only if $w(X) = 0$.

Probably the simplest (slightly refined and possibly new) description of the class $w(X)$ is the following. Choose a base point $x_0 \in X$. Consider the DG algebra $C_\bullet(\Omega_{x_0} X)$ of singular chains on the based loop space. By the work of Brav-Dyckerhoff that this DG algebra is homologically smooth (the diagonal bimodule is perfect). It follows that the trivial module of rank 1 is perfect: $\mathbb{Z} \in \mathrm{Perf}(C_\bullet(\Omega_{x_0} X))$.

Classical Wall obstruction

In 1965, Wall defined an invariant $w(X) \in \widetilde{K_0}(\mathbb{Z}[\pi_1(X)])$ (see below) for any finitely dominated space X, and proved the following result.

Theorem (C.T.C. Wall, 1965)

A connected finitely dominated space X has a homotopy type of a finite CW complex if and only if $w(X) = 0$.

Probably the simplest (slightly refined and possibly new) description of the class $w(X)$ is the following. Choose a base point $x_0 \in X$. Consider the DG algebra $C_\bullet(\Omega_{x_0} X)$ of singular chains on the based loop space. By the work of Brav-Dyckerhoff that this DG algebra is homologically smooth (the diagonal bimodule is perfect). It follows that the trivial module of rank 1 is perfect: $\mathbb{Z} \in \mathrm{Perf}(C_\bullet(\Omega_{x_0} X))$. Thus, we have a well-defined class

$$\widetilde{w}(X) := [\mathbb{Z}] \in K_0(C_\bullet(\Omega_{x_0} X)) \cong K_0(\mathbb{Z}[\pi_1(X, x_0)]).$$

Classical Wall obstruction

In 1965, Wall defined an invariant $w(X) \in \widetilde{K_0}(\mathbb{Z}[\pi_1(X)])$ (see below) for any finitely dominated space X, and proved the following result.

Theorem (C.T.C. Wall, 1965)

A connected finitely dominated space X has a homotopy type of a finite CW complex if and only if $w(X) = 0$.

Probably the simplest (slightly refined and possibly new) description of the class $w(X)$ is the following. Choose a base point $x_0 \in X$. Consider the DG algebra $C_\bullet(\Omega_{x_0} X)$ of singular chains on the based loop space. By the work of Brav-Dyckerhoff that this DG algebra is homologically smooth (the diagonal bimodule is perfect). It follows that the trivial module of rank 1 is perfect: $\mathbb{Z} \in \mathrm{Perf}(C_\bullet(\Omega_{x_0} X))$. Thus, we have a well-defined class

$$\tilde{w}(X) := [\mathbb{Z}] \in K_0(C_\bullet(\Omega_{x_0} X)) \cong K_0(\mathbb{Z}[\pi_1(X, x_0)]).$$

The class $w(X) \in \widetilde{K_0}(\mathbb{Z}[\pi_1(X, x_0)])$ is simply the projection of $\tilde{w}(X)$.

Classical Wall obstruction

Equivalent formulation of Wall's theorem is thus the following: a finitely dominated connected space X has a homotopy type of a finite CW complex if and only if the class $[\mathbb{Z}] \in K_0(C_\bullet(\Omega_{x_0} X))$ is a multiple of the class $[C_\bullet(\Omega_{x_0} X)]$.

Classical Wall obstruction

Equivalent formulation of Wall's theorem is thus the following: a finitely dominated connected space X has a homotopy type of a finite CW complex if and only if the class $[\mathbb{Z}] \in K_0(C_\bullet(\Omega_{x_0}X))$ is a multiple of the class $[C_\bullet(\Omega_{x_0}X)]$.

The "only if" part is easy to see: if X is a finite CW complex, then $C_\bullet(\Omega_{x_0}X)$ is quasi-isomorphic to a semi-free finitely generated DG algebra, hence the trivial module $\mathbb{Z}$ is quasi-isomorphic to a semi-free finitely generated module over $C_\bullet(\Omega_{x_0}X)$.

Classical Wall obstruction

Equivalent formulation of Wall's theorem is thus the following: a finitely dominated connected space X has a homotopy type of a finite CW complex if and only if the class $[\mathbb{Z}] \in K_0(C_\bullet(\Omega_{x_0}X))$ is a multiple of the class $[C_\bullet(\Omega_{x_0}X)]$.

The "only if" part is easy to see: if X is a finite CW complex, then $C_\bullet(\Omega_{x_0}X)$ is quasi-isomorphic to a semi-free finitely generated DG algebra, hence the trivial module $\mathbb{Z}$ is quasi-isomorphic to a semi-free finitely generated module over $C_\bullet(\Omega_{x_0}X)$.

The "if" part is non-trivial, see the original paper of Wall.

Classical Wall obstruction

Equivalent formulation of Wall's theorem is thus the following: a finitely dominated connected space X has a homotopy type of a finite CW complex if and only if the class $[\mathbb{Z}] \in K_0(C_\bullet(\Omega_{x_0}X))$ is a multiple of the class $[C_\bullet(\Omega_{x_0}X)]$.

The "only if" part is easy to see: if X is a finite CW complex, then $C_\bullet(\Omega_{x_0}X)$ is quasi-isomorphic to a semi-free finitely generated DG algebra, hence the trivial module $\mathbb{Z}$ is quasi-isomorphic to a semi-free finitely generated module over $C_\bullet(\Omega_{x_0}X)$.

The "if" part is non-trivial, see the original paper of Wall. It is also worth mentioning that for any finitely presented group G and for any class $\alpha \in \widetilde{K_0}(\mathbb{Z}[G])$, Wall constructs a connected finitely dominated space X such that $\pi_1(X) \cong G$, $\alpha = w(X)$.

Classical Wall obstruction

Equivalent formulation of Wall's theorem is thus the following: a finitely dominated connected space X has a homotopy type of a finite CW complex if and only if the class $[\mathbb{Z}] \in K_0(C_\bullet(\Omega_{x_0}X))$ is a multiple of the class $[C_\bullet(\Omega_{x_0}X)]$.

The "only if" part is easy to see: if X is a finite CW complex, then $C_\bullet(\Omega_{x_0}X)$ is quasi-isomorphic to a semi-free finitely generated DG algebra, hence the trivial module $\mathbb{Z}$ is quasi-isomorphic to a semi-free finitely generated module over $C_\bullet(\Omega_{x_0}X)$.

The "if" part is non-trivial, see the original paper of Wall. It is also worth mentioning that for any finitely presented group G and for any class $\alpha \in \widetilde{K_0}(\mathbb{Z}[G])$, Wall constructs a connected finitely dominated space X such that $\pi_1(X) \cong G$, $\alpha = w(X)$. This in particular gives a negative answer to the question of Milnor.

Preliminaries on DG categories

Fix some base field k. All the DG categories will be k-linear.

Preliminaries on DG categories

Fix some base field k. All the DG categories will be k-linear. For a small DG category $\mathcal{A}$ we denote by $I_{\mathcal{A}}$ the diagonal bimodule:

$$I_{\mathcal{A}}(X, Y) = \mathcal{A}(X, Y), \quad X, Y \in \mathcal{A}.$$

Preliminaries on DG categories

Fix some base field k. All the DG categories will be k-linear. For a small DG category $\mathcal{A}$ we denote by $I_{\mathcal{A}}$ the diagonal bimodule:

$$I_{\mathcal{A}}(X, Y) = \mathcal{A}(X, Y), \quad X, Y \in \mathcal{A}.$$

We will mostly consider small DG categories up to Morita equivalence, and we denote by $\mathrm{Ho}_M(\mathrm{dgcat}_{\mathrm{k}})$ the Morita homotopy category of small DG categories (formally invert Morita equivalences).

Preliminaries on DG categories

Fix some base field k. All the DG categories will be k-linear. For a small DG category $\mathcal{A}$ we denote by $I_{\mathcal{A}}$ the diagonal bimodule:

$$I_{\mathcal{A}}(X, Y) = \mathcal{A}(X, Y), \quad X, Y \in \mathcal{A}.$$

We will mostly consider small DG categories up to Morita equivalence, and we denote by $\mathrm{Ho}_M(\mathrm{dgcat}_{\mathrm{k}})$ the Morita homotopy category of small DG categories (formally invert Morita equivalences).

We recall some definitions:

Definition

Let $\mathcal{A}$ be a small DG category.
1) $\mathcal{A}$ is (homologically) smooth if the diagonal bimodule $I_{\mathcal{A}} \in D(\mathcal{A} \otimes \mathcal{A}^{op})$ is perfect.
2) $\mathcal{A}$ is proper if the complexes of morphisms $\mathcal{A}(X, Y)$ are perfect over k (have finite-dimensional total cohomology), and the triangulated category $D_{perf}(\mathcal{A})$ has a single generator.

Preliminaries on DG categories

The notions of smoothness and properness are invariant under Morita equivalence. Moreover, they are compatible with the corresponding notions for (commutative) schemes:

Preliminaries on DG categories

The notions of smoothness and properness are invariant under Morita equivalence. Moreover, they are compatible with the corresponding notions for (commutative) schemes:

Proposition (Lunts, Orlov)

Let X be a separated scheme of finite type over a field k. Then the DG category $\mathrm{Perf}(X)$ is smooth (resp. proper) if and only if X is smooth (resp. proper).

Finite cell DG categories

The following is a DG categorical analogue of a finite CW complex, and of a semi-free finitely generated DG module.

Finite cell DG categories

The following is a DG categorical analogue of a finite CW complex, and of a semi-free finitely generated DG module.

Definition

A k*-linear DG category* $\mathcal{C}$ *is called "finite cell DG category" if*
(i) *the set of objects* $Ob(\mathcal{C})$ *is finite;*
(ii) *as a graded* k*-linear category,* $\mathcal{C}$ *is freely generated by a finite collection of (homogeneous) morphisms* $f_1, \dots, f_n$*;*
(iii) *for each* $i \in \{1, \dots, n\}$*, the differential* $d(f_i)$ *is contained in the subcategory (with the same objects), generated by* $f_1, \dots, f_{i-1}$.

Homotopically finitely presented DG categories

The following is a DG categorical analogue of a finitely dominated topological space, and of a perfect DG module.

Homotopically finitely presented DG categories

The following is a DG categorical analogue of a finitely dominated topological space, and of a perfect DG module.

Definition

A small DG category $\mathcal{A}$ *is homotopically finitely presented (hfp) if it is a retract in* $\mathrm{Ho}_M(\mathrm{dgcat}_{\mathrm{k}})$ *of a finite cell DG category* $\mathcal{C}$.

Homotopically finitely presented DG categories

The following is a DG categorical analogue of a finitely dominated topological space, and of a perfect DG module.

Definition

A small DG category $\mathcal{A}$ is homotopically finitely presented (hfp) if it is a retract in $\mathrm{Ho}_M(\mathrm{dgcat}_{\mathrm{k}})$ *of a finite cell DG category* $\mathcal{C}$.

We recall some well-known implications:

Proposition (Toën-Vaquié)

Let $\mathcal{A}$ be a small DG category over k.
1) If $\mathcal{A}$ is hfp, then $\mathcal{A}$ is smooth.
2) If $\mathcal{A}$ is smooth and proper, then $\mathcal{A}$ is hfp.

Homotopically finitely presented DG categories

Both implications are non-reversible. For example, the algebra of rational functions $\mathrm{k}(x)$ is smooth but not hfp. We mention the following results.

Homotopically finitely presented DG categories

Both implications are non-reversible. For example, the algebra of rational functions $\mathrm{k}(x)$ is smooth but not hfp. We mention the following results.

Theorem (Lunts)

Let X be a separated scheme of finite type over a perfect field k. Then the DG category $D^b_{coh}(X)$ is smooth. The same holds when k is an arbitrary field and X_{red} has a smooth stratification.

Homotopically finitely presented DG categories

Both implications are non-reversible. For example, the algebra of rational functions $\mathrm{k}(x)$ is smooth but not hfp. We mention the following results.

Theorem (Lunts)

Let X be a separated scheme of finite type over a perfect field k. Then the DG category $D^b_{coh}(X)$ is smooth. The same holds when k is an arbitrary field and X_{red} has a smooth stratification.

Theorem (E)

Let X be a separated scheme of finite type over a field k of characteristic zero. Then $D^b_{coh}(X)$ is hfp.

Homotopically finitely presented DG categories

Both implications are non-reversible. For example, the algebra of rational functions $\mathrm{k}(x)$ is smooth but not hfp. We mention the following results.

Theorem (Lunts)

Let X be a separated scheme of finite type over a perfect field k. Then the DG category $D^b_{coh}(X)$ is smooth. The same holds when k is an arbitrary field and X_{red} has a smooth stratification.

Theorem (E)

Let X be a separated scheme of finite type over a field k of characteristic zero. Then $D^b_{coh}(X)$ is hfp.

The later theorem uses the construction of a categorical resolution of singularities due to Kuznetsov and Lunts.

Summarizing table

Topological spaces	DG modules over a small DG category $\mathcal{A}$	Small DG categories over k

Summarizing table

Topological spaces	DG modules over a small DG category $\mathcal{A}$	Small DG categories over k
weak homotopy equivalences	quasi-isomorphisms	Morita equivalences

Summarizing table

Topological spaces	DG modules over a small DG category $\mathcal{A}$	Small DG categories over k
weak homotopy equivalences	quasi-isomorphisms	Morita equivalences
finite CW complexes	semi-free finitely generated $\mathcal{A}$-modules	finite cell DG categories

Summarizing table

Topological spaces	DG modules over a small DG category $\mathcal{A}$	Small DG categories over k
weak homotopy equivalences	quasi-isomorphisms	Morita equivalences
finite CW complexes	semi-free finitely generated $\mathcal{A}$-modules	finite cell DG categories
finitely dominated spaces	perfect $\mathcal{A}$-modules	homotopically finitely presented DG categories

Summarizing table

Topological spaces	DG modules over a small DG category $\mathcal{A}$	Small DG categories over k
weak homotopy equivalences	quasi-isomorphisms	Morita equivalences
finite CW complexes	semi-free finitely generated $\mathcal{A}$-modules	finite cell DG categories
finitely dominated spaces	perfect $\mathcal{A}$-modules	homotopically finitely presented DG categories
Wall's finiteness obstruction theorem	Thomason's classification of dense subcategories	???

Generating cofibrations

In all the three contexts, the finite cell objects are obtained from the initial object by a finite number of pushouts of the so-called generating cofibrations.

Generating cofibrations

In all the three contexts, the finite cell objects are obtained from the initial object by a finite number of pushouts of the so-called generating cofibrations.

1) In topological spaces, these are $S^{n-1} \hookrightarrow D^n$, $n \geqslant 0$.

Generating cofibrations

In all the three contexts, the finite cell objects are obtained from the initial object by a finite number of pushouts of the so-called generating cofibrations.

1) In topological spaces, these are $S^{n-1} \hookrightarrow D^n$, $n \geqslant 0$.

2) In Mod-$\mathcal{A}$, these are $h_X[n] \to \mathrm{Cone}(h_X \xrightarrow{\mathrm{id}} h_X)[n]$, where $X \in \mathcal{A}$, $n \in \mathbb{Z}$.

Generating cofibrations

In all the three contexts, the finite cell objects are obtained from the initial object by a finite number of pushouts of the so-called generating cofibrations.

1) In topological spaces, these are $S^{n-1} \hookrightarrow D^n$, $n \geqslant 0$.

2) In Mod-$\mathcal{A}$, these are $h_X[n] \to \mathrm{Cone}(h_X \xrightarrow{\mathrm{id}} h_X)[n]$, where $X \in \mathcal{A}$, $n \in \mathbb{Z}$.

3) In $\mathrm{dgcat}_{\mathrm{k}}$, these are $\varnothing \to \mathrm{k}$ and

$$(\bullet \xrightarrow{\mathrm{k}[n]} \bullet) \to (\bullet \xrightarrow{\mathrm{Cone}(\mathrm{k} \xrightarrow{\mathrm{id}} \mathrm{k})[n]} \bullet), \quad n \in \mathbb{Z}.$$

Main theorem

For a small DG category $\mathcal{A}$, we put $K_0(\mathcal{A}) := K_0(D_{perf}(\mathcal{A}))$.

Main theorem

For a small DG category $\mathcal{A}$, we put $K_0(\mathcal{A}) := K_0(D_{perf}(\mathcal{A}))$.

Theorem

For a small DG category $\mathcal{A}$, the following are equivalent:
(i) $\mathcal{A}$ is Morita equivalent to a finite cell DG category;

Main theorem

For a small DG category $\mathcal{A}$, we put $K_0(\mathcal{A}) := K_0(D_{perf}(\mathcal{A}))$.

Theorem

For a small DG category $\mathcal{A}$, the following are equivalent:
(i) $\mathcal{A}$ is Morita equivalent to a finite cell DG category;
(ii) $\mathcal{A}$ is hfp, and moreover $[I_{\mathcal{A}}] \in \mathrm{Im}(K_0(\mathcal{A}) \otimes K_0(\mathcal{A}^{op}) \to K_0(\mathcal{A} \otimes \mathcal{A}^{op}))$.

Main theorem

For a small DG category $\mathcal{A}$, we put $K_0(\mathcal{A}) := K_0(D_{perf}(\mathcal{A}))$.

Theorem

For a small DG category $\mathcal{A}$, the following are equivalent:
(i) $\mathcal{A}$ is Morita equivalent to a finite cell DG category;
(ii) $\mathcal{A}$ is hfp, and moreover $[I_{\mathcal{A}}] \in \mathrm{Im}(K_0(\mathcal{A}) \otimes K_0(\mathcal{A}^{op}) \to K_0(\mathcal{A} \otimes \mathcal{A}^{op}))$.
(iii) $\mathcal{A}$ is Morita equivalent to a DG quotient $\mathcal{E}/\mathcal{S}$, where $\mathcal{E}$ is a pre-triangulated proper DG category with a full exceptional collection, and $\mathcal{S}$ is a subcategory generated by a single object.

Speial case: smooth and proper DG categories

Note that any (quasi-)functor between smooth and proper pre-triangulated Karoubi complete DG categories has both left and right adjoint functors.

Speial case: smooth and proper DG categories

Note that any (quasi-)functor between smooth and proper pre-triangulated Karoubi complete DG categories has both left and right adjoint functors. Also, a (left or right) adjoint to a localization functor is fully faithful. Hence, we have the following corollary.

Speial case: smooth and proper DG categories

Note that any (quasi-)functor between smooth and proper pre-triangulated Karoubi complete DG categories has both left and right adjoint functors. Also, a (left or right) adjoint to a localization functor is fully faithful. Hence, we have the following corollary.

Corollary

Let $\mathcal{A}$ be a smooth and proper DG category. If we have $[I_{\mathcal{A}}] \in \mathrm{Im}(K_0(\mathcal{A}) \otimes K_0(\mathcal{A}^{op}) \to K_0(\mathcal{A} \otimes \mathcal{A}^{op}))$, then there exists a fully faithful quasi-functor $\mathcal{A} \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is a proper DG category with a full exceptional collection.

Speial case: smooth and proper DG categories

Note that any (quasi-)functor between smooth and proper pre-triangulated Karoubi complete DG categories has both left and right adjoint functors. Also, a (left or right) adjoint to a localization functor is fully faithful. Hence, we have the following corollary.

Corollary

Let $\mathcal{A}$ be a smooth and proper DG category. If we have $[I_{\mathcal{A}}] \in \mathrm{Im}(K_0(\mathcal{A}) \otimes K_0(\mathcal{A}^{op}) \to K_0(\mathcal{A} \otimes \mathcal{A}^{op}))$, then there exists a fully faithful quasi-functor $\mathcal{A} \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is a proper DG category with a full exceptional collection.

In other words, if $\mathcal{A}$ satisfies the assumptions of the corollary, then $\mathcal{A}$ is quasi-equivalent to an admissible subcategory of some $\mathcal{E}$ as above.

Application: phantom DG categories

Definition

A smooth and proper DG category $\mathcal{A}$ is called a "phantom category" if we have $[I_{\mathcal{A}}] = 0$ in $K_0(\mathcal{A} \otimes \mathcal{A}^{op})$. This is equivalent to the vanishing of all additive invariants of $\mathcal{A}$.

Application: phantom DG categories

Definition

A smooth and proper DG category $\mathcal{A}$ is called a "phantom category" if we have $[I_{\mathcal{A}}] = 0$ in $K_0(\mathcal{A} \otimes \mathcal{A}^{op})$. This is equivalent to the vanishing of all additive invariants of $\mathcal{A}$.

Corollary

For any phantom category $\mathcal{A}$, there exists a fully faithful quasi-functor $\mathcal{A} \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is a proper DG category with a full exceptional collection.

Application: phantom DG categories

Definition

A smooth and proper DG category $\mathcal{A}$ is called a "phantom category" if we have $[I_{\mathcal{A}}] = 0$ in $K_0(\mathcal{A} \otimes \mathcal{A}^{op})$. This is equivalent to the vanishing of all additive invariants of $\mathcal{A}$.

Corollary

For any phantom category $\mathcal{A}$, there exists a fully faithful quasi-functor $\mathcal{A} \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is a proper DG category with a full exceptional collection.

This disproves a conjecture of Orlov, stating that a proper DG category with a full exceptional collection cannot contain a non-zero phantom category as an admissible subcategory.

Application: Barlow surface

In 1985, R. Barlow constructed (a family of) *simply connected* smooth projective surfaces S of general type having $p_g(S) = 0$. They are homeomorphic but not diffeomorphic to a del Pezzo surface of degree 1.

Application: Barlow surface

In 1985, R. Barlow constructed (a family of) *simply connected* smooth projective surfaces S of general type having $p_g(S) = 0$. They are homeomorphic but not diffeomorphic to a del Pezzo surface of degree 1.

By the work of C. Böhning, H.-C. Graf von Bothmer, L. Katzarkov and P. Sosna, for a generic determinantal Barlow surface S in a small neighborhood (in the moduli space) of the special Barlow surface S_0, there is an exceptional collection of 11 line bundles $\langle L_1, \dots, L_{11}\rangle$ on S such that the right orthogonal category

$$\mathcal{A} = \langle L_1, \dots, L_{11}\rangle^{\perp} \subset D^b_{coh}(S)$$

is a (non-zero) phantom category.

Application: Barlow surface

In 1985, R. Barlow constructed (a family of) *simply connected* smooth projective surfaces S of general type having $p_g(S) = 0$. They are homeomorphic but not diffeomorphic to a del Pezzo surface of degree 1.

By the work of C. Böhning, H.-C. Graf von Bothmer, L. Katzarkov and P. Sosna, for a generic determinantal Barlow surface S in a small neighborhood (in the moduli space) of the special Barlow surface S_0, there is an exceptional collection of 11 line bundles $\langle L_1, \dots, L_{11} \rangle$ on S such that the right orthogonal category

$$\mathcal{A} = \langle L_1, \dots, L_{11} \rangle^{\perp} \subset D^b_{coh}(S)$$

is a (non-zero) phantom category.

In particular, for such S we have $K_0(S \times S) \cong K_0(S) \otimes K_0(S) \cong \mathbb{Z}^{121}$.

Application: Barlow surface

Corollary

If S is a Barlow surface as above, there exists a fully faithful functor $D^b_{coh}(S) \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is a proper DG category with a full exceptional collection.

Application: Barlow surface

Corollary

If S is a Barlow surface as above, there exists a fully faithful functor $D^b_{coh}(S) \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is a proper DG category with a full exceptional collection.

This disproves another conjecture of Orlov which states the following: if X is a smooth projective variety, and we have a fully faithful functor $D^b_{coh}(X) \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is a proper DG category with a full exceptional collection, then X is rational.

Application: Barlow surface

Corollary

If S is a Barlow surface as above, there exists a fully faithful functor $D^b_{coh}(S) \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is a proper DG category with a full exceptional collection.

This disproves another conjecture of Orlov which states the following: if X is a smooth projective variety, and we have a fully faithful functor $D^b_{coh}(X) \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is a proper DG category with a full exceptional collection, then X is rational. The Barlow surface S as above is a counterexample since S is a surface of general type.

Application: varieties with a nice stratification

Corollary

Let X be a smooth proper variety with a stratification such that each stratum is isomorphic to an open subset $U \subset \mathbb{A}^m$ for some m. Then we have a fully faithful functor $D^b_{coh}(X) \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is as above.

Application: varieties with a nice stratification

Corollary

Let X be a smooth proper variety with a stratification such that each stratum is isomorphic to an open subset $U \subset \mathbb{A}^m$ for some m. Then we have a fully faithful functor $D^b_{coh}(X) \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is as above.

Indeed, the assumptions on the stratification imply that for any smooth scheme Y the map $K_0(X) \otimes K_0(Y) \to K_0(X \times Y)$ is surjective. In particular, the condition (ii) of Main Theorem is satisfied for $D^b_{coh}(X)$.

Application: varieties with a nice stratification

Corollary

Let X be a smooth proper variety with a stratification such that each stratum is isomorphic to an open subset $U \subset \mathbb{A}^m$ for some m. Then we have a fully faithful functor $D^b_{coh}(X) \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is as above.

Indeed, the assumptions on the stratification imply that for any smooth scheme Y the map $K_0(X) \otimes K_0(Y) \to K_0(X \times Y)$ is surjective. In particular, the condition (ii) of Main Theorem is satisfied for $D^b_{coh}(X)$.

This corollary confirms a weakened version a conjecture of Lunts which states the following: if a smooth proper variety X has a stratification by affine spaces, then $D^b_{coh}(X)$ has a full exceptional collection.

Application: schemes with a nice stratification

We have a more general corollary in the case when char $\mathrm{k} = 0$.

Corollary

Let X be a separated scheme of finite type over a field k of characteristic zero. Suppose that X_{red} has a stratification as above. Then there is a short exact sequence of pre-triangulated DG categories

$$\mathcal{S} \hookrightarrow \mathcal{E} \to D^b_{coh}(X),$$

where $\mathcal{E}$ is a proper DG category with a full exceptional collection, and $\mathcal{S}$ is generated by a single object.

Application: schemes with a nice stratification

We have a more general corollary in the case when char $\mathrm{k} = 0$.

Corollary

Let X be a separated scheme of finite type over a field k of characteristic zero. Suppose that X_{red} has a stratification as above. Then there is a short exact sequence of pre-triangulated DG categories

$$\mathcal{S} \hookrightarrow \mathcal{E} \to D^b_{coh}(X),$$

where $\mathcal{E}$ is a proper DG category with a full exceptional collection, and $\mathcal{S}$ is generated by a single object.

Indeed, the DG category $D^b_{coh}(X)$ is hfp (by the above theorem), and for any noetherian k-scheme Y we have a surjective map $K_0'(X) \otimes K_0'(Y) \twoheadrightarrow K_0'(X \times Y)$.

Application: schemes with a nice stratification

We have a more general corollary in the case when $\operatorname{char} \mathrm{k} = 0$.

Corollary

Let X be a separated scheme of finite type over a field k of characteristic zero. Suppose that X_{red} has a stratification as above. Then there is a short exact sequence of pre-triangulated DG categories

$$\mathcal{S} \hookrightarrow \mathcal{E} \to D^b_{coh}(X),$$

where $\mathcal{E}$ is a proper DG category with a full exceptional collection, and $\mathcal{S}$ is generated by a single object.

Indeed, the DG category $D^b_{coh}(X)$ is hfp (by the above theorem), and for any noetherian k-scheme Y we have a surjective map $K_0'(X) \otimes K_0'(Y) \twoheadrightarrow K_0'(X \times Y)$. Hence, the condition (ii) of Main theorem is satisfied for $D^b_{coh}(X)$.

Application: proper schemes with a nice stratification

Corollary

Let X be a proper scheme over a field k of characteristic zero. Suppose that X_{red} has a stratification as above. Then there is a fully faithful functor $\mathrm{Perf}(X) \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is a proper DG category with a full exceptional collection.

Application: proper schemes with a nice stratification

Corollary

Let X be a proper scheme over a field k of characteristic zero. Suppose that X_{red} has a stratification as above. Then there is a fully faithful functor $\mathrm{Perf}(X) \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is a proper DG category with a full exceptional collection.

Indeed, recall that if $\mathcal{A} = D^b_{coh}(X)$ for a proper scheme X, then we have an equivalence $\mathrm{Perf}(X) \simeq \mathrm{PsPerf}(\mathcal{A})$, where $\mathrm{PsPerf}(\mathcal{A})$ is the DG category of (say, h-projective or cofibrant) pseudo-perfect $\mathcal{A}$-modules (that is, $\mathcal{A}$-modules with values in $\mathrm{Perf}(\mathrm{k})$).

Application: proper schemes with a nice stratification

Corollary

Let X be a proper scheme over a field k of characteristic zero. Suppose that X_{red} has a stratification as above. Then there is a fully faithful functor $\mathrm{Perf}(X) \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is a proper DG category with a full exceptional collection.

Indeed, recall that if $\mathcal{A} = D^b_{coh}(X)$ for a proper scheme X, then we have an equivalence $\mathrm{Perf}(X) \simeq \mathrm{PsPerf}(\mathcal{A})$, where $\mathrm{PsPerf}(\mathcal{A})$ is the DG category of (say, h-projective or cofibrant) pseudo-perfect $\mathcal{A}$-modules (that is, $\mathcal{A}$-modules with values in $\mathrm{Perf}(\mathrm{k})$).

More precisely, since $\mathcal{A}$ is smooth, we have an inclusion $\mathrm{PsPerf}(\mathcal{A}) \subset \mathrm{Perf}(\mathcal{A}) \simeq D^b_{coh}(X)$, and the essential image is given by $\mathbb{D}_X \otimes_{\mathcal{O}_X} \mathrm{Perf}(X)$, where $\mathbb{D}_X$ is the dualizing complex.

Application: proper schemes with a nice stratification

Thus, if $\mathcal{E} \to \mathcal{A} = D^b_{coh}(X)$ is a quotient functor from the above corollary, we can take the restriction of scalars to get a fully faithful functor

$$\mathrm{Perf}(X) \simeq \mathrm{PsPerf}(\mathcal{A}) \hookrightarrow \mathrm{PsPerf}(\mathcal{E}) = \mathrm{Perf}(\mathcal{E}) \simeq \mathcal{E}.$$

Application: proper schemes with a nice stratification

Thus, if $\mathcal{E} \to \mathcal{A} = D^b_{coh}(X)$ is a quotient functor from the above corollary, we can take the restriction of scalars to get a fully faithful functor

$$\mathrm{Perf}(X) \simeq \mathrm{PsPerf}(\mathcal{A}) \hookrightarrow \mathrm{PsPerf}(\mathcal{E}) = \mathrm{Perf}(\mathcal{E}) \simeq \mathcal{E}.$$

Remark

By Orlov's results on the semi-orthogonal gluings of geometric categories, in all of the above corollaries we may assume that $\mathcal{E} = D^b_{coh}(Y)$, where Y is a sequence of projective bundles over a point.

Application: proper schemes with a nice stratification

Thus, if $\mathcal{E} \to \mathcal{A} = D^b_{coh}(X)$ is a quotient functor from the above corollary, we can take the restriction of scalars to get a fully faithful functor

$$\mathrm{Perf}(X) \simeq \mathrm{PsPerf}(\mathcal{A}) \hookrightarrow \mathrm{PsPerf}(\mathcal{E}) = \mathrm{Perf}(\mathcal{E}) \simeq \mathcal{E}.$$

Remark

By Orlov's results on the semi-orthogonal gluings of geometric categories, in all of the above corollaries we may assume that $\mathcal{E} = D^b_{coh}(Y)$, where Y is a sequence of projective bundles over a point.

We will now sketch the proof of Main Theorem.

Deformed tensor algebra of a bimodule

We recall the notion of a (deformed) tensor algebra of a bimodule over a DG category.

Deformed tensor algebra of a bimodule

We recall the notion of a (deformed) tensor algebra of a bimodule over a DG category. Let $\mathcal{D}$ be a small DG category, and $M \in \mathcal{D}\text{-}\operatorname{Mod}\text{-}\mathcal{D}$.

Deformed tensor algebra of a bimodule

We recall the notion of a (deformed) tensor algebra of a bimodule over a DG category. Let $\mathcal{D}$ be a small DG category, and $M \in \mathcal{D}\text{-}\mathrm{Mod}\text{-}\mathcal{D}$. Then $T_{\mathcal{D}}(M)$ is a DG category with the same objects, and its morphisms are given by

$$T_{\mathcal{D}}(M)(X,Y) = \mathcal{D}(X,Y) \oplus M(X,Y) \oplus \bigoplus_{n \geqslant 2} M(-,Y) \underset{\mathcal{D}}{\otimes} M^{\otimes_{\mathcal{D}}(n-2)} \underset{\mathcal{D}}{\otimes} M(X,-).$$

Deformed tensor algebra of a bimodule

We recall the notion of a (deformed) tensor algebra of a bimodule over a DG category. Let $\mathcal{D}$ be a small DG category, and $M \in \mathcal{D}\text{-}\operatorname{Mod}\text{-}\mathcal{D}$. Then $T_{\mathcal{D}}(M)$ is a DG category with the same objects, and its morphisms are given by

$$T_{\mathcal{D}}(M)(X,Y) = \mathcal{D}(X,Y) \oplus M(X,Y) \oplus \bigoplus_{n\geqslant 2} M(-,Y) \underset{\mathcal{D}}{\otimes} M^{\otimes_{\mathcal{D}}(n-2)} \underset{\mathcal{D}}{\otimes} M(X,-).$$

Now, if $\alpha : M \to \mathcal{I}_{\mathcal{A}}[1]$ is a morphism of $\mathcal{D}$-$\mathcal{D}$-bimodules, then we have a DG category $T_{\mathcal{D},\alpha}(M)$ which is defined as follows.

Deformed tensor algebra of a bimodule

We recall the notion of a (deformed) tensor algebra of a bimodule over a DG category. Let $\mathcal{D}$ be a small DG category, and $M \in \mathcal{D}\text{-}\mathrm{Mod}\text{-}\mathcal{D}$. Then $T_{\mathcal{D}}(M)$ is a DG category with the same objects, and its morphisms are given by

$$T_{\mathcal{D}}(M)(X,Y) = \mathcal{D}(X,Y) \oplus M(X,Y) \oplus \bigoplus_{n\geqslant 2} M(-,Y) \underset{\mathcal{D}}{\otimes} M^{\otimes_{\mathcal{D}}(n-2)} \underset{\mathcal{D}}{\otimes} M(X,-).$$

Now, if $\alpha : M \to \mathcal{I}_{\mathcal{A}}[1]$ is a morphism of $\mathcal{D}$-$\mathcal{D}$-bimodules, then we have a DG category $T_{\mathcal{D},\alpha}(M)$ which is defined as follows. As a graded k-linear category, it equals $T_{\mathcal{D}}(M)$.

Deformed tensor algebra of a bimodule

We recall the notion of a (deformed) tensor algebra of a bimodule over a DG category. Let $\mathcal{D}$ be a small DG category, and $M \in \mathcal{D}\text{-}\operatorname{Mod}\text{-}\mathcal{D}$. Then $T_{\mathcal{D}}(M)$ is a DG category with the same objects, and its morphisms are given by

$$T_{\mathcal{D}}(M)(X,Y) = \mathcal{D}(X,Y) \oplus M(X,Y) \oplus \bigoplus_{n \geqslant 2} M(-,Y) \underset{\mathcal{D}}{\otimes} M^{\otimes_{\mathcal{D}}(n-2)} \underset{\mathcal{D}}{\otimes} M(X,-).$$

Now, if $\alpha : M \to \mathcal{I}_{\mathcal{A}}[1]$ is a morphism of $\mathcal{D}$-$\mathcal{D}$-bimodules, then we have a DG category $T_{\mathcal{D},\alpha}(M)$ which is defined as follows. As a graded k-linear category, it equals $T_{\mathcal{D}}(M)$. The deformed differential d_α is defined as follows:

$$d_{\alpha|\mathcal{D}} = d_{\mathcal{D}}, \quad d_{\alpha|M} = d_M + \alpha$$

(extend to the tensor powers of M by the Leibniz rule).

Deformed tensor algebra of a bimodule

We will in fact apply the construction $T_{\mathcal{D},\alpha}(M)$ only in the case when all the modules $M(-,X)$ are cofibrant.

Deformed tensor algebra of a bimodule

We will in fact apply the construction $T_{\mathcal{D},\alpha}(M)$ only in the case when all the modules $M(-,X)$ are cofibrant. If $M' \xrightarrow{f} M$ is a quasi-isomorphism of such bimodules, and $\alpha : M \to \mathcal{I}_{\mathcal{D}}[1]$ is a morphism, then the induced DG functor $T_{\mathcal{D},\alpha\circ f}(M') \to T_{\mathcal{D},\alpha}(M)$ is a quasi-equivalence.

Deformed tensor algebra of a bimodule

We will in fact apply the construction $T_{\mathcal{D},\alpha}(M)$ only in the case when all the modules $M(-,X)$ are cofibrant. If $M' \xrightarrow{f} M$ is a quasi-isomorphism of such bimodules, and $\alpha : M \to \mathcal{I}_{\mathcal{D}}[1]$ is a morphism, then the induced DG functor $T_{\mathcal{D},\alpha\circ f}(M') \to T_{\mathcal{D},\alpha}(M)$ is a quasi-equivalence.

Moreover, if $F : \mathcal{D} \to \mathcal{D}'$ is a Morita equivalence of small DG categories, M is a cofibrant $\mathcal{D}$-$\mathcal{D}$-bimodule and $\alpha : M \to I_{\mathcal{A}}[1]$ is a morphism, then the induced DG functor

$$T_{\mathcal{D},\alpha}(M) \to T_{\mathcal{D}',\beta}(\mathcal{D}' \underset{\mathcal{D}}{\otimes} M \underset{\mathcal{D}}{\otimes} \mathcal{D}')$$

is a Morita equivalence, where β is the composition

$$\mathcal{D}' \underset{\mathcal{D}}{\otimes} M \underset{\mathcal{D}}{\otimes} \mathcal{D}' \xrightarrow{\mathcal{D}'\otimes_{\mathcal{D}}\alpha\otimes_{\mathcal{D}}\mathcal{D}'} \mathcal{D}' \underset{\mathcal{D}}{\otimes} I_{\mathcal{D}}[1] \underset{\mathcal{D}}{\otimes} \mathcal{D}' \to I_{\mathcal{D}'}[1].$$

Example: Drinfeld DG quotient

Consider the following special case: $\mathcal{D}$ is a small DG category, $X \in \mathcal{D}$ an object, and $M = h_{(X,X^{op})}[1] = h_X \otimes h_X^{\vee}[1]$ – the shift of the representable bimodule.

Example: Drinfeld DG quotient

Consider the following special case: $\mathcal{D}$ is a small DG category, $X \in \mathcal{D}$ an object, and $M = h_{(X,X^{op})}[1] = h_X \otimes h_X^{\vee}[1]$ – the shift of the representable bimodule. Take $\alpha = \mathrm{id}_X : M \to I_{\mathcal{D}}[1]$.

Example: Drinfeld DG quotient

Consider the following special case: $\mathcal{D}$ is a small DG category, $X \in \mathcal{D}$ an object, and $M = h_{(X,X^{op})}[1] = h_X \otimes h_X^{\vee}[1]$ – the shift of the representable bimodule. Take $\alpha = \mathrm{id}_X : M \to I_{\mathcal{D}}[1]$.

Then we have a (chain level) isomorphism of DG categories $T_{\mathcal{D},\mathrm{id}_X}(h_X \otimes h_X^{\vee}[1]) \cong \mathcal{D}/\{X\}$, where the RHS is the Drinfeld DG quotient of $\mathcal{D}$ by the subcategory $\{X\}$ (with a single object).

Example: Drinfeld DG quotient

Consider the following special case: $\mathcal{D}$ is a small DG category, $X \in \mathcal{D}$ an object, and $M = h_{(X,X^{op})}[1] = h_X \otimes h_X^{\vee}[1]$ – the shift of the representable bimodule. Take $\alpha = \mathrm{id}_X : M \to I_{\mathcal{D}}[1]$.

Then we have a (chain level) isomorphism of DG categories $T_{\mathcal{D},\mathrm{id}_X}(h_X \otimes h_X^{\vee}[1]) \cong \mathcal{D}/\{X\}$, where the RHS is the Drinfeld DG quotient of $\mathcal{D}$ by the subcategory $\{X\}$ (with a single object).

The same holds for an arbitrary full DG subcategory $\mathcal{S} \subset \mathcal{D}$, if we take M to be the direct sum of shifts of representable bimodules $h_{(X,X^{op})}[1]$, $X \in \mathcal{S}$.

Example: finite cell DG categories

Another special case of interest for us is the following.

Proposition

Let $\mathcal{C}$ be a finite cell DG category, and $M \in \mathcal{SF}_{f.g.}(\mathcal{C} \otimes \mathcal{C}^{op})$ – a semi-free finitely generated bimodule. Then for any morphism $\alpha : M \to I_{\mathcal{C}}[1]$ the deformed tensor algebra $T_{\mathcal{C},\alpha}(M)$ is also a finite cell DG category.

Example: finite cell DG categories

Another special case of interest for us is the following.

Proposition

Let $\mathcal{C}$ be a finite cell DG category, and $M \in \mathcal{SF}_{f.g.}(\mathcal{C} \otimes \mathcal{C}^{op})$ – a semi-free finitely generated bimodule. Then for any morphism $\alpha : M \to I_{\mathcal{C}}[1]$ the deformed tensor algebra $T_{\mathcal{C},\alpha}(M)$ is also a finite cell DG category.

Indeed, the DG category $T_{\mathcal{C},\alpha}(M)$ is obtained from $\mathcal{C}$ by (consecutively) freely adding new morphisms (that is, without relations).

Example: finite cell DG categories

Another special case of interest for us is the following.

Proposition

Let $\mathcal{C}$ be a finite cell DG category, and $M \in \mathcal{SF}_{f.g.}(\mathcal{C} \otimes \mathcal{C}^{op})$ – a semi-free finitely generated bimodule. Then for any morphism $\alpha : M \to I_{\mathcal{C}}[1]$ the deformed tensor algebra $T_{\mathcal{C},\alpha}(M)$ is also a finite cell DG category.

Indeed, the DG category $T_{\mathcal{C},\alpha}(M)$ is obtained from $\mathcal{C}$ by (consecutively) freely adding new morphisms (that is, without relations). In other words, the functor $\mathcal{C} \to T_{\mathcal{C},\alpha}(M)$ is a finite composition of pushouts of generating cofibrations.

Implication $(iii) \Rightarrow (i)$ in Main Theorem

Suppose that $\mathcal{E}$ is a pre-triangulated proper DG category with a full exceptional collection, and $\mathcal{S} \subset \mathcal{E}$ a subcategory generated by a single object.

Implication $(iii) \Rightarrow (i)$ in Main Theorem

Suppose that $\mathcal{E}$ is a pre-triangulated proper DG category with a full exceptional collection, and $\mathcal{S} \subset \mathcal{E}$ a subcategory generated by a single object. Then $\mathcal{E}$ is Morita equivalent to a directed DG category $\mathcal{E}'$, i.e. $Ob(\mathcal{E}') = \{X_1, \dots, X_n\}$, where $\mathcal{E}'(X_i, X_j) = 0$ for $i > j$, and $\mathcal{E}'(X_i, X_i) = \mathrm{k}$.

Implication $(iii) \Rightarrow (i)$ in Main Theorem

Suppose that $\mathcal{E}$ is a pre-triangulated proper DG category with a full exceptional collection, and $\mathcal{S} \subset \mathcal{E}$ a subcategory generated by a single object. Then $\mathcal{E}$ is Morita equivalent to a directed DG category $\mathcal{E}'$, i.e. $Ob(\mathcal{E}') = \{X_1, \dots, X_n\}$, where $\mathcal{E}'(X_i, X_j) = 0$ for $i > j$, and $\mathcal{E}'(X_i, X_i) = \mathrm{k}$.

Moreover, we may assume that $\mathcal{E}'$ is a finite cell DG category (e.g. by replacing $\mathcal{E}'$ with the bar-cobar construction of the minimal A_∞-model of $\mathcal{E}'$).

Implication $(iii) \Rightarrow (i)$ in Main Theorem

Suppose that $\mathcal{E}$ is a pre-triangulated proper DG category with a full exceptional collection, and $\mathcal{S} \subset \mathcal{E}$ a subcategory generated by a single object. Then $\mathcal{E}$ is Morita equivalent to a directed DG category $\mathcal{E}'$, i.e. $Ob(\mathcal{E}') = \{X_1, \dots, X_n\}$, where $\mathcal{E}'(X_i, X_j) = 0$ for $i > j$, and $\mathcal{E}'(X_i, X_i) = \mathrm{k}$.

Moreover, we may assume that $\mathcal{E}'$ is a finite cell DG category (e.g. by replacing $\mathcal{E}'$ with the bar-cobar construction of the minimal A_∞-model of $\mathcal{E}'$).

Now, take a semi-free finitely generated module M over $\mathcal{E}'$, such that the equivalence $D_{perf}(\mathcal{E}') \xrightarrow{\sim} H^0(\mathcal{E})$ sends M to a generator of $\mathcal{S}$.

Implication $(iii) \Rightarrow (i)$ in Main Theorem

Suppose that $\mathcal{E}$ is a pre-triangulated proper DG category with a full exceptional collection, and $\mathcal{S} \subset \mathcal{E}$ a subcategory generated by a single object. Then $\mathcal{E}$ is Morita equivalent to a directed DG category $\mathcal{E}'$, i.e. $Ob(\mathcal{E}') = \{X_1, \dots, X_n\}$, where $\mathcal{E}'(X_i, X_j) = 0$ for $i > j$, and $\mathcal{E}'(X_i, X_i) = \mathrm{k}$.

Moreover, we may assume that $\mathcal{E}'$ is a finite cell DG category (e.g. by replacing $\mathcal{E}'$ with the bar-cobar construction of the minimal A_∞-model of $\mathcal{E}'$).

Now, take a semi-free finitely generated module M over $\mathcal{E}'$, such that the equivalence $D_{perf}(\mathcal{E}') \xrightarrow{\sim} H^0(\mathcal{E})$ sends M to a generator of $\mathcal{S}$. By the above discussion, the DG quotient $\mathcal{E}/\mathcal{S}$ is Morita equivalent to the deformed tensor algebra $T_{\mathcal{E}',\mathrm{id}_M}(M \otimes M^\vee[1])$, which is a finite cell DG category.

Implication $(i) \Rightarrow (iii)$ (sketch)

Consider the following general situation. Let $\mathcal{A}$ be a small DG category, $X, Y \in \mathcal{A}$ – a pair of objets, and $u \in \mathcal{A}(X, Y)^n$ – a closed morphism of degree n.

Implication $(i) \Rightarrow (iii)$ (sketch)

Consider the following general situation. Let $\mathcal{A}$ be a small DG category, $X, Y \in \mathcal{A}$ – a pair of objets, and $u \in \mathcal{A}(X, Y)^n$ – a closed morphism of degree n. Denote by $\mathcal{A}\langle v\rangle$ the DG category with the same objects, which is obtained from $\mathcal{A}$ by freely adding a morphism $v : X \to Y$ of degree $n-1$, such that $dv = u$. Equivalently, $\mathcal{A}\langle v\rangle = T_{\mathcal{A},u}(h_X \otimes h_Y^{\vee}[1-n])$.

Implication $(i) \Rightarrow (iii)$ (sketch)

Consider the following general situation. Let $\mathcal{A}$ be a small DG category, $X, Y \in \mathcal{A}$ – a pair of objets, and $u \in \mathcal{A}(X, Y)^n$ – a closed morphism of degree n. Denote by $\mathcal{A}\langle v \rangle$ the DG category with the same objects, which is obtained from $\mathcal{A}$ by freely adding a morphism $v : X \to Y$ of degree $n-1$, such that $dv = u$. Equivalently, $\mathcal{A}\langle v \rangle = T_{\mathcal{A},u}(h_X \otimes h_Y^{\vee}[1-n])$.

Then the DG category $\mathcal{A}\langle v \rangle$ can be described as follows. Define $\mathcal{B}$ to be the following semi-orthogonal gluing:

$$\mathcal{B} := \begin{pmatrix} \mathcal{A} & 0 \\ \mathrm{Cone}(h_X \otimes h_Y^{\vee}[-n] \xrightarrow{u} I_{\mathcal{A}}) & \mathcal{A} \end{pmatrix}$$

Implication $(i) \Rightarrow (iii)$ (sketch)

Consider the following general situation. Let $\mathcal{A}$ be a small DG category, $X, Y \in \mathcal{A}$ – a pair of objets, and $u \in \mathcal{A}(X, Y)^n$ – a closed morphism of degree n. Denote by $\mathcal{A}\langle v\rangle$ the DG category with the same objects, which is obtained from $\mathcal{A}$ by freely adding a morphism $v : X \to Y$ of degree $n-1$, such that $dv = u$. Equivalently, $\mathcal{A}\langle v\rangle = T_{\mathcal{A},u}(h_X \otimes h_Y^{\vee}[1-n])$.

Then the DG category $\mathcal{A}\langle v\rangle$ can be described as follows. Define $\mathcal{B}$ to be the following semi-orthogonal gluing:

$$\mathcal{B} := \begin{pmatrix} \mathcal{A} & 0 \\ \mathrm{Cone}(h_X \otimes h_Y^{\vee}[-n] \xrightarrow{u} I_{\mathcal{A}}) & \mathcal{A} \end{pmatrix}$$

Denoting by $\iota_1, \iota_2 : \mathcal{A} \to \mathcal{B}$ the natural inclusions, we have natural morphisms $w_Z : \iota_1(Z) \to \iota_2(Z)$, $Z \in \mathcal{A}$, corresponding to $\mathrm{id}_Z \in I_{\mathcal{A}}(Z, Z)$.

Implication $(i) \Rightarrow (iii)$ (sketch)

Now, the DG category $\mathrm{Perf}(\mathcal{A})$ is quasi-equivalent to (the Karoubi completion of) the quotient of $\mathrm{Perf}(\mathcal{B})$ by the cones $\mathrm{Cone}(w_Z)$, where Z runs through any generating set of objects of $\mathcal{A}$.

Implication $(i) \Rightarrow (iii)$ (sketch)

Now, the DG category $\mathrm{Perf}(\mathcal{A})$ is quasi-equivalent to (the Karoubi completion of) the quotient of $\mathrm{Perf}(\mathcal{B})$ by the cones $\mathrm{Cone}(w_Z)$, where Z runs through any generating set of objects of $\mathcal{A}$.

Now let $\mathcal{C}$ be a finite cell DG category, and denote by $f_1, \dots, f_n$ the (ordered) generating set of morphisms.

Implication $(i) \Rightarrow (iii)$ (sketch)

Now, the DG category $\mathrm{Perf}(\mathcal{A})$ is quasi-equivalent to (the Karoubi completion of) the quotient of $\mathrm{Perf}(\mathcal{B})$ by the cones $\mathrm{Cone}(w_Z)$, where Z runs through any generating set of objects of $\mathcal{A}$.

Now let $\mathcal{C}$ be a finite cell DG category, and denote by $f_1, \dots, f_n$ the (ordered) generating set of morphisms. We have a chain of (non-full) DG subcategories $\mathcal{C}_0 \subset \mathcal{C}_1 \dots \subset \mathcal{C}_n = \mathcal{C}$, where $Ob(\mathcal{C}_i) = Ob(\mathcal{C})$, and the morphisms in $\mathcal{C}_i$ are freely generated by $f_1, \dots, f_i$.

Implication $(i) \Rightarrow (iii)$ (sketch)

Now, the DG category $\mathrm{Perf}(\mathcal{A})$ is quasi-equivalent to (the Karoubi completion of) the quotient of $\mathrm{Perf}(\mathcal{B})$ by the cones $\mathrm{Cone}(w_Z)$, where Z runs through any generating set of objects of $\mathcal{A}$.

Now let $\mathcal{C}$ be a finite cell DG category, and denote by $f_1, \dots, f_n$ the (ordered) generating set of morphisms. We have a chain of (non-full) DG subcategories $\mathcal{C}_0 \subset \mathcal{C}_1 \dots \subset \mathcal{C}_n = \mathcal{C}$, where $Ob(\mathcal{C}_i) = Ob(\mathcal{C})$, and the morphisms in $\mathcal{C}_i$ are freely generated by $f_1, \dots, f_i$. Using the above discussion, we can construct inductively the functors $\mathcal{E}_i \to \mathcal{C}_i$, such that

Implication $(i) \Rightarrow (iii)$ (sketch)

Now, the DG category $\mathrm{Perf}(\mathcal{A})$ is quasi-equivalent to (the Karoubi completion of) the quotient of $\mathrm{Perf}(\mathcal{B})$ by the cones $\mathrm{Cone}(w_Z)$, where Z runs through any generating set of objects of $\mathcal{A}$.

Now let $\mathcal{C}$ be a finite cell DG category, and denote by $f_1, \dots, f_n$ the (ordered) generating set of morphisms. We have a chain of (non-full) DG subcategories $\mathcal{C}_0 \subset \mathcal{C}_1 \dots \subset \mathcal{C}_n = \mathcal{C}$, where $Ob(\mathcal{C}_i) = Ob(\mathcal{C})$, and the morphisms in $\mathcal{C}_i$ are freely generated by $f_1, \dots, f_i$. Using the above discussion, we can construct inductively the functors $\mathcal{E}_i \to \mathcal{C}_i$, such that

• the underlying graded category of $\mathcal{E}_i$ is a (k-linear) path category of a finite directed graded quiver;

Implication $(i) \Rightarrow (iii)$ (sketch)

Now, the DG category $\mathrm{Perf}(\mathcal{A})$ is quasi-equivalent to (the Karoubi completion of) the quotient of $\mathrm{Perf}(\mathcal{B})$ by the cones $\mathrm{Cone}(w_Z)$, where Z runs through any generating set of objects of $\mathcal{A}$.

Now let $\mathcal{C}$ be a finite cell DG category, and denote by $f_1, \dots, f_n$ the (ordered) generating set of morphisms. We have a chain of (non-full) DG subcategories $\mathcal{C}_0 \subset \mathcal{C}_1 \dots \subset \mathcal{C}_n = \mathcal{C}$, where $Ob(\mathcal{C}_i) = Ob(\mathcal{C})$, and the morphisms in $\mathcal{C}_i$ are freely generated by $f_1, \dots, f_i$. Using the above discussion, we can construct inductively the functors $\mathcal{E}_i \to \mathcal{C}_i$, such that
• the underlying graded category of $\mathcal{E}_i$ is a (k-linear) path category of a finite directed graded quiver;
• the functor $D_{perf}(\mathcal{E}_i) \to D_{perf}(\mathcal{C}_i)$ is a quotient functor (up to direct summands), and its kernel is generated by a single object;

Implication $(i) \Rightarrow (iii)$ (sketch)

Now, the DG category $\mathrm{Perf}(\mathcal{A})$ is quasi-equivalent to (the Karoubi completion of) the quotient of $\mathrm{Perf}(\mathcal{B})$ by the cones $\mathrm{Cone}(w_Z)$, where Z runs through any generating set of objects of $\mathcal{A}$.

Now let $\mathcal{C}$ be a finite cell DG category, and denote by $f_1, \dots, f_n$ the (ordered) generating set of morphisms. We have a chain of (non-full) DG subcategories $\mathcal{C}_0 \subset \mathcal{C}_1 \dots \subset \mathcal{C}_n = \mathcal{C}$, where $Ob(\mathcal{C}_i) = Ob(\mathcal{C})$, and the morphisms in $\mathcal{C}_i$ are freely generated by $f_1, \dots, f_i$. Using the above discussion, we can construct inductively the functors $\mathcal{E}_i \to \mathcal{C}_i$, such that

• the underlying graded category of $\mathcal{E}_i$ is a (k-linear) path category of a finite directed graded quiver;

• the functor $D_{perf}(\mathcal{E}_i) \to D_{perf}(\mathcal{C}_i)$ is a quotient functor (up to direct summands), and its kernel is generated by a single object;

• the map $Ob(\mathcal{E}_i) \to Ob(\mathcal{C}_i)$ is surjective.

Implication $(i) \Rightarrow (iii)$ (sketch)

Now, the DG category $\mathrm{Perf}(\mathcal{A})$ is quasi-equivalent to (the Karoubi completion of) the quotient of $\mathrm{Perf}(\mathcal{B})$ by the cones $\mathrm{Cone}(w_Z)$, where Z runs through any generating set of objects of $\mathcal{A}$.

Now let $\mathcal{C}$ be a finite cell DG category, and denote by $f_1, \dots, f_n$ the (ordered) generating set of morphisms. We have a chain of (non-full) DG subcategories $\mathcal{C}_0 \subset \mathcal{C}_1 \dots \subset \mathcal{C}_n = \mathcal{C}$, where $Ob(\mathcal{C}_i) = Ob(\mathcal{C})$, and the morphisms in $\mathcal{C}_i$ are freely generated by $f_1, \dots, f_i$. Using the above discussion, we can construct inductively the functors $\mathcal{E}_i \to \mathcal{C}_i$, such that

• the underlying graded category of $\mathcal{E}_i$ is a (k-linear) path category of a finite directed graded quiver;

• the functor $D_{perf}(\mathcal{E}_i) \to D_{perf}(\mathcal{C}_i)$ is a quotient functor (up to direct summands), and its kernel is generated by a single object;

• the map $Ob(\mathcal{E}_i) \to Ob(\mathcal{C}_i)$ is surjective.

In the most straightforward construction, we have $|Ob(\mathcal{E}_i)| = 2^i |Ob(\mathcal{C})|$.

Example: free algebra

For each $n \geqslant 0$, let us consider the free algebra $\mathrm{k}\langle x_1, \dots, x_n\rangle$, where $\deg(x_i) = 0$, and the generalized Kronecker quiver Q_{n+1} with arrows $u_0, \dots, u_n$.

Example: free algebra

For each $n \geqslant 0$, let us consider the free algebra $\mathrm{k}\langle x_1, \dots, x_n \rangle$, where $\deg(x_i) = 0$, and the generalized Kronecker quiver Q_{n+1} with arrows $u_0, \dots, u_n$. Then we have

$$\mathrm{Perf}(\mathrm{k}\langle x_1, \dots, x_n \rangle) \simeq \mathrm{Perf}(\mathrm{k}Q_{n+1})/\langle \mathrm{Cone}(u_0) \rangle.$$

Informally, we have $x_i = u_i u_0^{-1}$.

Implication $(iii) \Rightarrow (ii)$

If we have $\mathcal{A} = \mathcal{E}/\mathcal{S}$, where $\mathcal{E}$ is a proper DG category with a full exceptional collection, and $\mathcal{S}$ is generated by a single object, then we already know that $\mathcal{A}$ is hfp.

Implication $(iii) \Rightarrow (ii)$

If we have $\mathcal{A} = \mathcal{E}/\mathcal{S}$, where $\mathcal{E}$ is a proper DG category with a full exceptional collection, and $\mathcal{S}$ is generated by a single object, then we already know that $\mathcal{A}$ is hfp. Further, $K_0(\mathcal{E} \otimes \mathcal{E}^{op}) = K_0(\mathcal{E}) \otimes K_0(\mathcal{E}^{op})$.

Implication $(iii) \Rightarrow (ii)$

If we have $\mathcal{A} = \mathcal{E}/\mathcal{S}$, where $\mathcal{E}$ is a proper DG category with a full exceptional collection, and $\mathcal{S}$ is generated by a single object, then we already know that $\mathcal{A}$ is hfp. Further, $K_0(\mathcal{E} \otimes \mathcal{E}^{op}) = K_0(\mathcal{E}) \otimes K_0(\mathcal{E}^{op})$. Note that the natural map

$$\mathcal{A} \overset{\mathbf{L}}{\underset{\mathcal{E}}{\otimes}} I_{\mathcal{E}} \overset{\mathbf{L}}{\underset{\mathcal{E}}{\otimes}} \mathcal{A} \to I_{\mathcal{A}}$$

in $D_{perf}(\mathcal{A} \otimes \mathcal{A}^{op})$ is an isomorphism.

Implication $(iii) \Rightarrow (ii)$

If we have $\mathcal{A} = \mathcal{E}/\mathcal{S}$, where $\mathcal{E}$ is a proper DG category with a full exceptional collection, and $\mathcal{S}$ is generated by a single object, then we already know that $\mathcal{A}$ is hfp. Further, $K_0(\mathcal{E} \otimes \mathcal{E}^{op}) = K_0(\mathcal{E}) \otimes K_0(\mathcal{E}^{op})$. Note that the natural map

$$\mathcal{A} \overset{\mathbf{L}}{\underset{\mathcal{E}}{\otimes}} I_{\mathcal{E}} \overset{\mathbf{L}}{\underset{\mathcal{E}}{\otimes}} \mathcal{A} \to I_{\mathcal{A}}$$

in $D_{perf}(\mathcal{A} \otimes \mathcal{A}^{op})$ is an isomorphism. Therefore, the class $[I_{\mathcal{A}}]$ is in the image of the composition

$$K_0(\mathcal{E}) \otimes K_0(\mathcal{E}^{op}) \to K_0(\mathcal{A}) \otimes K_0(\mathcal{A}^{op}) \to K_0(\mathcal{A} \otimes \mathcal{A}^{op}).$$

Implication $(ii) \Rightarrow (i)$: plan

The remaining (and the most) implication $(ii) \Rightarrow (i)$ is proved in two steps.

Implication $(ii) \Rightarrow (i)$: plan

The remaining (and the most) implication $(ii) \Rightarrow (i)$ is proved in two steps.

Step 1. For any hfp DG category $\mathcal{A}$, the tensor product $\mathcal{A} \otimes \mathrm{k}[x^{\pm 1}]$ is Morita equivalent to a finite cell DG category $\mathcal{C}$.

Implication $(ii) \Rightarrow (i)$: plan

The remaining (and the most) implication $(ii) \Rightarrow (i)$ is proved in two steps.

Step 1. For any hfp DG category $\mathcal{A}$, the tensor product $\mathcal{A} \otimes \mathrm{k}[x^{\pm 1}]$ is Morita equivalent to a finite cell DG category $\mathcal{C}$. Moreover, for any finite subset $\{\gamma_1, \dots, \gamma_n\} \subset K_0(\mathcal{A})$, the DG category $\mathcal{C}$ can be chosen in such a way that the image of $Ob(\mathcal{C}) \to K_0(\mathcal{C}) \cong K_0(\mathcal{A} \otimes \mathrm{k}[x^{\pm 1}])$ contains the classes $\gamma_i \otimes \mathrm{k}[x^{\pm 1}]$, for $i = 1, \dots, n$.

Implication $(ii) \Rightarrow (i)$: plan

The remaining (and the most) implication $(ii) \Rightarrow (i)$ is proved in two steps.

Step 1. For any hfp DG category $\mathcal{A}$, the tensor product $\mathcal{A} \otimes \mathrm{k}[x^{\pm 1}]$ is Morita equivalent to a finite cell DG category $\mathcal{C}$. Moreover, for any finite subset $\{\gamma_1, \dots, \gamma_n\} \subset K_0(\mathcal{A})$, the DG category $\mathcal{C}$ can be chosen in such a way that the image of $Ob(\mathcal{C}) \to K_0(\mathcal{C}) \cong K_0(\mathcal{A} \otimes \mathrm{k}[x^{\pm 1}])$ contains the classes $\gamma_i \otimes \mathrm{k}[x^{\pm 1}]$, for $i = 1, \dots, n$.

Step 2. If $\mathcal{A}$ satisfies the condition (ii), then, roughly speaking, $\mathcal{A}$ is obtained from $\mathcal{A} \otimes \mathrm{k}[x^{\pm 1}]$ by "attaching a finite number of cells", up to a Morita equivalence.

A result of Mather

Before proving the statement of Step 1, we recall the following result from topology, which was known before Wall finiteness obstruction theorem.

A result of Mather

Before proving the statement of Step 1, we recall the following result from topology, which was known before Wall finiteness obstruction theorem.

Proposition (M. Mather, 1965)

A topological space X is finitely dominated if and only if $X \times S^1$ is weakly homotopy equivalent to a finite CW complex.

A result of Mather

Before proving the statement of Step 1, we recall the following result from topology, which was known before Wall finiteness obstruction theorem.

Proposition (M. Mather, 1965)

A topological space X is finitely dominated if and only if $X \times S^1$ is weakly homotopy equivalent to a finite CW complex.

The "if" part is obvious.

A result of Mather

Before proving the statement of Step 1, we recall the following result from topology, which was known before Wall finiteness obstruction theorem.

Proposition (M. Mather, 1965)

A topological space X is finitely dominated if and only if $X \times S^1$ is weakly homotopy equivalent to a finite CW complex.

The "if" part is obvious.
For the "only if" part, recall that a mapping torus of a self-map $\varphi : Z \to Z$ is defined by the formula

$$T(\varphi) = Z \times [0,1]/\{(x,0) \sim (\varphi(x),1)\} \cong Z \times \mathbb{R}_{\geqslant 0}/\{(x,t) \sim (\varphi(x),t+1)\}.$$

A result of Mather

Before proving the statement of Step 1, we recall the following result from topology, which was known before Wall finiteness obstruction theorem.

Proposition (M. Mather, 1965)

A topological space X is finitely dominated if and only if $X \times S^1$ is weakly homotopy equivalent to a finite CW complex.

The "if" part is obvious.
For the "only if" part, recall that a mapping torus of a self-map $\varphi : Z \to Z$ is defined by the formula

$$T(\varphi) = Z \times [0,1]/\{(x,0) \sim (\varphi(x),1)\} \cong Z \times \mathbb{R}_{\geqslant 0}/\{(x,t) \sim (\varphi(x), t+1)\}.$$

Note that the map $\tilde{\varphi} : T(\varphi) \to T(\varphi)$, $\tilde{\varphi}(x,t) = (\varphi(x), t)$, is homotopic to the identity map $\mathrm{id}_{T(\varphi)}$.

A result of Mather

It follows that for any maps $f : X \to Y$, $g : Y \to X$, the maps $\widetilde{f} : T(gf) \to T(fg)$ and $\widetilde{g} : T(fg) \to T(gf)$ are homotopy inverse to each other.

A result of Mather

It follows that for any maps $f : X \to Y$, $g : Y \to X$, the maps $\widetilde{f} : T(gf) \to T(fg)$ and $\widetilde{g} : T(fg) \to T(gf)$ are homotopy inverse to each other. In particular, if gf is homotopic to id_X, we have the homotopy equivalences

$$X \times S^1 = T(\mathrm{id}_X) \simeq T(gf) \simeq T(fg).$$

A result of Mather

It follows that for any maps $f: X \to Y$, $g: Y \to X$, the maps $\widetilde{f}: T(gf) \to T(fg)$ and $\widetilde{g}: T(fg) \to T(gf)$ are homotopy inverse to each other. In particular, if gf is homotopic to id_X, we have the homotopy equivalences

$$X \times S^1 = T(\mathrm{id}_X) \simeq T(gf) \simeq T(fg).$$

If moreover Y is a finite CW complex, then $T(fg)$ is also a finite CW complex. This proves the "only if" part.

A generalization of Mather's argument

Note that the mapping torus $T(\varphi)$ of a self-map φ can be described (in the homotopy category) as a homotopy coequalizer:
$T(\varphi) \simeq \mathrm{coeq}^h(\mathrm{id}_Z, \varphi)$.

A generalization of Mather's argument

Note that the mapping torus $T(\varphi)$ of a self-map φ can be described (in the homotopy category) as a homotopy coequalizer:
$T(\varphi) \simeq \mathrm{coeq}^h(\mathrm{id}_Z, \varphi)$. Moreover, for any maps $f : X \to Y$, $g : Y \to X$, the following natural maps in the homotopy category are isomorphisms:

$$\mathrm{coeq}^h(\mathrm{id}_X, gf) \xrightarrow{\sim} \mathrm{hocolim}(X \underset{g}{\overset{f}{\leftrightarrows}} Y) \xleftarrow{\sim} \mathrm{coeq}^h(\mathrm{id}_Y, fg).$$

A generalization of Mather's argument

Note that the mapping torus $T(\varphi)$ of a self-map φ can be described (in the homotopy category) as a homotopy coequalizer:
$T(\varphi) \simeq \mathrm{coeq}^h(\mathrm{id}_Z, \varphi)$. Moreover, for any maps $f : X \to Y$, $g : Y \to X$, the following natural maps in the homotopy category are isomorphisms:

$$\mathrm{coeq}^h(\mathrm{id}_X, gf) \xrightarrow{\sim} \mathrm{hocolim}(X \underset{g}{\overset{f}{\leftrightarrows}} Y) \xleftarrow{\sim} \mathrm{coeq}^h(\mathrm{id}_Y, fg).$$

In fact, the same isomorphisms hold in the homotopy category of any model category (and in any ∞-category, if at least one of the colimits exists).

A generalization of Mather's argument

Note that the mapping torus $T(\varphi)$ of a self-map φ can be described (in the homotopy category) as a homotopy coequalizer:
$T(\varphi) \simeq \mathrm{coeq}^h(\mathrm{id}_Z, \varphi)$. Moreover, for any maps $f : X \to Y$, $g : Y \to X$, the following natural maps in the homotopy category are isomorphisms:

$$\mathrm{coeq}^h(\mathrm{id}_X, gf) \xrightarrow{\sim} \mathrm{hocolim}(X \underset{g}{\overset{f}{\leftrightarrows}} Y) \xleftarrow{\sim} \mathrm{coeq}^h(\mathrm{id}_Y, fg).$$

In fact, the same isomorphisms hold in the homotopy category of any model category (and in any ∞-category, if at least one of the colimits exists). Equivalently, the corresponding functors between the index categories are homotopy cofinal, which is easy to check.

Special case: stable ∞-category

The above isomorphisms between the homotopy colimits are especially easy to see in the case of a stable ∞-category.

Special case: stable ∞-category

The above isomorphisms between the homotopy colimits are especially easy to see in the case of a stable ∞-category. Indeed, if $\mathcal{T}$ is its (triangulated) homotopy category, and $f : X \to Y$, $g : Y \to X$ – a pair of morphisms, then we have the isomorphisms

$$\mathrm{Cone}(\mathrm{id}_X - gf) \simeq \mathrm{Cone}\left(\begin{pmatrix} \mathrm{id}_X & g \\ f & \mathrm{id}_Y \end{pmatrix}\right) \simeq \mathrm{Cone}(\mathrm{id}_Y - fg).$$

Special case: stable ∞-category

The above isomorphisms between the homotopy colimits are especially easy to see in the case of a stable ∞-category. Indeed, if $\mathcal{T}$ is its (triangulated) homotopy category, and $f : X \to Y$, $g : Y \to X$ – a pair of morphisms, then we have the isomorphisms

$$\mathrm{Cone}(\mathrm{id}_X - gf) \simeq \mathrm{Cone}\left(\begin{pmatrix}\mathrm{id}_X & g \\ f & \mathrm{id}_Y\end{pmatrix}\right) \simeq \mathrm{Cone}(\mathrm{id}_Y - fg).$$

This is standard (in particular, if $(\mathrm{id}_X - gf)$ is invertible, then so is $(\mathrm{id}_Y - fg)$).

Special case: DG categories

Let $\mathcal{D}$ be a small DG category, and $\Phi : \mathcal{D} \to \mathcal{D}$ a DG functor. We would like to describe the homotopy coequalizer $\mathrm{coeq}^h(\mathrm{id}_{\mathcal{D}}, \Phi)$.

Special case: DG categories

Let $\mathcal{D}$ be a small DG category, and $\Phi : \mathcal{D} \to \mathcal{D}$ a DG functor. We would like to describe the homotopy coequalizer $\mathrm{coeq}^h(\mathrm{id}_{\mathcal{D}}, \Phi)$. For any functor $F : \mathcal{D} \to \mathcal{D}'$, let us denote by $N_F \in \mathcal{D}\text{-}\mathrm{Mod}\text{-}\mathcal{D}'$ the associated bimodule:

$$N_F(X, Y) = \mathcal{D}'(X, F(Y)), \quad X \in \mathcal{D}',\ Y \in \mathcal{D}.$$

Special case: DG categories

Let $\mathcal{D}$ be a small DG category, and $\Phi : \mathcal{D} \to \mathcal{D}$ a DG functor. We would like to describe the homotopy coequalizer $\mathrm{coeq}^h(\mathrm{id}_{\mathcal{D}}, \Phi)$. For any functor $F : \mathcal{D} \to \mathcal{D}'$, let us denote by $N_F \in \mathcal{D}\text{-}\mathrm{Mod}\text{-}\mathcal{D}'$ the associated bimodule:

$$N_F(X, Y) = \mathcal{D}'(X, F(Y)), \quad X \in \mathcal{D}', \ Y \in \mathcal{D}.$$

Consider the tensor algebra $T_{\mathcal{D}}(N_\Phi)$.

Special case: DG categories

Let $\mathcal{D}$ be a small DG category, and $\Phi : \mathcal{D} \to \mathcal{D}$ a DG functor. We would like to describe the homotopy coequalizer $\mathrm{coeq}^h(\mathrm{id}_{\mathcal{D}}, \Phi)$. For any functor $F : \mathcal{D} \to \mathcal{D}'$, let us denote by $N_F \in \mathcal{D}\text{-}\mathrm{Mod}\text{-}\mathcal{D}'$ the associated bimodule:

$$N_F(X, Y) = \mathcal{D}'(X, F(Y)), \quad X \in \mathcal{D}',\ Y \in \mathcal{D}.$$

Consider the tensor algebra $T_{\mathcal{D}}(N_\Phi)$. The restriction of scalars functor $\mathrm{Res} : \mathrm{Mod}\text{-}\mathcal{D} \to \mathrm{Mod}\text{-}T_{\mathcal{D}}(N_\Phi)$ preserves perfect modules.

Special case: DG categories

Let $\mathcal{D}$ be a small DG category, and $\Phi : \mathcal{D} \to \mathcal{D}$ a DG functor. We would like to describe the homotopy coequalizer $\mathrm{coeq}^h(\mathrm{id}_{\mathcal{D}}, \Phi)$. For any functor $F : \mathcal{D} \to \mathcal{D}'$, let us denote by $N_F \in \mathcal{D}\text{-}\mathrm{Mod}\text{-}\mathcal{D}'$ the associated bimodule:

$$N_F(X, Y) = \mathcal{D}'(X, F(Y)), \quad X \in \mathcal{D}',\ Y \in \mathcal{D}.$$

Consider the tensor algebra $T_{\mathcal{D}}(N_\Phi)$. The restriction of scalars functor $\mathrm{Res} : \mathrm{Mod}\text{-}\mathcal{D} \to \mathrm{Mod}\text{-}T_{\mathcal{D}}(N_\Phi)$ preserves perfect modules. For example, if $\mathcal{D}$ is a DG algebra, then we have a short exact sequence of $T_{\mathcal{D}}(N_\Phi)$-modules:

$$0 \to N_\Phi \underset{\mathcal{D}}{\otimes} T_{\mathcal{D}}(N_\Phi) \to T_{\mathcal{D}}(N_\Phi) \to \mathrm{Res}(\mathcal{D}) \to 0.$$

Special case: DG categories

It is easy to check that

$$\mathrm{coeq}^h(\mathrm{id}_{\mathcal{D}}, N_\Phi) \simeq \mathrm{Perf}(T_{\mathcal{D}}(N_\Phi))/\mathrm{Res}(\mathcal{D}).$$

Special case: DG categories

It is easy to check that

$$\mathrm{coeq}^h(\mathrm{id}_{\mathcal{D}}, N_\Phi) \simeq \mathrm{Perf}(T_{\mathcal{D}}(N_\Phi))/\mathrm{Res}(\mathcal{D}).$$

In particular, we have $\mathrm{coeq}^h(\mathrm{id}_{\mathcal{D}}, \mathrm{id}_{\mathcal{D}}) \simeq \mathcal{D} \otimes \mathrm{k}[x^{\pm 1}]$.

Special case: DG categories

It is easy to check that

$$\mathrm{coeq}^h(\mathrm{id}_{\mathcal{D}}, N_\Phi) \simeq \mathrm{Perf}(T_{\mathcal{D}}(N_\Phi))/\mathrm{Res}(\mathcal{D}).$$

In particular, we have $\mathrm{coeq}^h(\mathrm{id}_{\mathcal{D}}, \mathrm{id}_{\mathcal{D}}) \simeq \mathcal{D} \otimes \mathrm{k}[x^{\pm 1}]$.

Moreover, the above equivalences between homotopy colimits can be checked explicitly: for any DG functors $F : \mathcal{D} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{D}$, we have

$$\mathrm{Perf}(T_{\mathcal{D}}(N_{GF}))/\mathrm{Res}(\mathcal{D}) \simeq \mathrm{Perf}(T_{\mathcal{C}\sqcup\mathcal{D}}(N_F \oplus N_G))/\mathrm{Res}(\mathcal{C} \sqcup \mathcal{D})$$
$$\simeq \mathrm{Perf}(T_{\mathcal{C}}(N_{FG}))/\mathrm{Res}(\mathcal{C}).$$

Finishing Step 1

In the special case when GF is homotopic to $\mathrm{id}_{\mathcal{D}}$ (i.e. the bimodule N_{GF} is quasi-isomorphic to $I_{\mathcal{C}}$), we have a chain of Morita equivalences

$$\mathcal{D} \otimes \mathrm{k}[x^{\pm 1}] \simeq \mathrm{Perf}(T_{\mathcal{D}}(N_{GF}))/\mathrm{Res}(\mathcal{D}) \simeq \mathrm{Perf}(T_{\mathcal{C}}(N_{FG}))/\mathrm{Res}(\mathcal{C}). \quad (0.1)$$

Finishing Step 1

In the special case when GF is homotopic to $\mathrm{id}_{\mathcal{D}}$ (i.e. the bimodule N_{GF} is quasi-isomorphic to $I_{\mathcal{C}}$), we have a chain of Morita equivalences

$$\mathcal{D} \otimes \mathrm{k}[x^{\pm 1}] \simeq \mathrm{Perf}(T_{\mathcal{D}}(N_{GF}))/\mathrm{Res}(\mathcal{D}) \simeq \mathrm{Perf}(T_{\mathcal{C}}(N_{FG}))/\mathrm{Res}(\mathcal{C}). \quad (0.1)$$

Now, if $\mathcal{C}$ is a finite cell DG category, then $I_{\mathcal{C}}$ is quasi-isomorphic to a semi-free finitely generated $\mathcal{C}$-$\mathcal{C}$-bimodule, hence so is N_{FG} (by the standard argument).

Finishing Step 1

In the special case when GF is homotopic to $\mathrm{id}_{\mathcal{D}}$ (i.e. the bimodule N_{GF} is quasi-isomorphic to $I_{\mathcal{C}}$), we have a chain of Morita equivalences

$$\mathcal{D}\otimes \mathrm{k}[x^{\pm 1}] \simeq \mathrm{Perf}(T_{\mathcal{D}}(N_{GF}))/\mathrm{Res}(\mathcal{D}) \simeq \mathrm{Perf}(T_{\mathcal{C}}(N_{FG}))/\mathrm{Res}(\mathcal{C}). \quad (0.1)$$

Now, if $\mathcal{C}$ is a finite cell DG category, then $I_{\mathcal{C}}$ is quasi-isomorphic to a semi-free finitely generated $\mathcal{C}$-$\mathcal{C}$-bimodule, hence so is N_{FG} (by the standard argument). Choosing a quasi-isomorphism $M \xrightarrow{\sim} N_{FG}$ with $M \in \mathcal{SF}_{f.g.}(\mathcal{C}\otimes\mathcal{C}^{op})$, we get a quasi-equivalence $T_{\mathcal{C}}(M) \to T_{\mathcal{C}}(N_{FG})$.

Finishing Step 1

In the special case when GF is homotopic to $\mathrm{id}_{\mathcal{D}}$ (i.e. the bimodule N_{GF} is quasi-isomorphic to $I_{\mathcal{C}}$), we have a chain of Morita equivalences

$$\mathcal{D} \otimes \mathrm{k}[x^{\pm 1}] \simeq \mathrm{Perf}(T_{\mathcal{D}}(N_{GF}))/\mathrm{Res}(\mathcal{D}) \simeq \mathrm{Perf}(T_{\mathcal{C}}(N_{FG}))/\mathrm{Res}(\mathcal{C}). \quad (0.1)$$

Now, if $\mathcal{C}$ is a finite cell DG category, then $I_{\mathcal{C}}$ is quasi-isomorphic to a semi-free finitely generated $\mathcal{C}$-$\mathcal{C}$-bimodule, hence so is N_{FG} (by the standard argument). Choosing a quasi-isomorphism $M \xrightarrow{\sim} N_{FG}$ with $M \in \mathcal{SF}_{f.g.}(\mathcal{C} \otimes \mathcal{C}^{op})$, we get a quasi-equivalence $T_{\mathcal{C}}(M) \to T_{\mathcal{C}}(N_{FG})$. Finally, choosing $P \in \mathcal{SF}_{f.g.}(T_{\mathcal{C}}(M))$ to be the module corresponding to the generator of triangulated subcategory $\langle \mathrm{Res}(\mathcal{D}) \rangle$, we see that the finite cell DG category $T_{T_{\mathcal{C}}(M),\mathrm{id}_P}(P \otimes P^{\vee}[1])$ is Morita equivalent to the RHS of (0.1), hence also to $\mathcal{D} \otimes \mathrm{k}[x^{\pm 1}]$.

Finishing Step 1

In the special case when GF is homotopic to $\mathrm{id}_{\mathcal{D}}$ (i.e. the bimodule N_{GF} is quasi-isomorphic to $I_{\mathcal{C}}$), we have a chain of Morita equivalences

$$\mathcal{D} \otimes \mathrm{k}[x^{\pm 1}] \simeq \mathrm{Perf}(T_{\mathcal{D}}(N_{GF}))/\mathrm{Res}(\mathcal{D}) \simeq \mathrm{Perf}(T_{\mathcal{C}}(N_{FG}))/\mathrm{Res}(\mathcal{C}). \quad (0.1)$$

Now, if $\mathcal{C}$ is a finite cell DG category, then $I_{\mathcal{C}}$ is quasi-isomorphic to a semi-free finitely generated $\mathcal{C}$-$\mathcal{C}$-bimodule, hence so is N_{FG} (by the standard argument). Choosing a quasi-isomorphism $M \xrightarrow{\sim} N_{FG}$ with $M \in \mathcal{SF}_{f.g.}(\mathcal{C} \otimes \mathcal{C}^{op})$, we get a quasi-equivalence $T_{\mathcal{C}}(M) \to T_{\mathcal{C}}(N_{FG})$. Finally, choosing $P \in \mathcal{SF}_{f.g.}(T_{\mathcal{C}}(M))$ to be the module corresponding to the generator of triangulated subcategory $\langle \mathrm{Res}(\mathcal{D}) \rangle$, we see that the finite cell DG category $T_{T_{\mathcal{C}}(M),\mathrm{id}_P}(P \otimes P^{\vee}[1])$ is Morita equivalent to the RHS of (0.1), hence also to $\mathcal{D} \otimes \mathrm{k}[x^{\pm 1}]$.
This essentially finishes Step 1 (the assertion about a finite subset of classes in K_0 is straightforward).

Step 2

It is easy to obtain k from $\mathrm{k}[x^{\pm 1}]$ by "attaching a cell": we have a quasi-isomorphism

$$T_{\mathrm{k}[x^{\pm 1}],x-1}(\mathrm{k}[x^{\pm 1}] \otimes \mathrm{k}[x^{\pm 1}][1]) \xrightarrow{\sim} \mathrm{k}.$$

Step 2

It is easy to obtain k from $\mathrm{k}[x^{\pm 1}]$ by "attaching a cell": we have a quasi-isomorphism

$$T_{\mathrm{k}[x^{\pm 1}],x-1}(\mathrm{k}[x^{\pm 1}]\otimes \mathrm{k}[x^{\pm 1}][1]) \xrightarrow{\sim} \mathrm{k}.$$

Multiplying it by an arbitrary small DG category $\mathcal{A}$, we get a quasi-equivalence

$$T_{\mathcal{A}\otimes \mathrm{k}[x^{\pm 1}],I_{\mathcal{A}}\otimes (x-1)}(I_{\mathcal{A}}\otimes \mathrm{k}[x^{\pm 1}]\otimes \mathrm{k}[x^{\pm 1}][1]) \xrightarrow{\sim} \mathcal{A}.$$

Step 2

It is easy to obtain k from $\mathrm{k}[x^{\pm 1}]$ by "attaching a cell": we have a quasi-isomorphism

$$T_{\mathrm{k}[x^{\pm 1}],x-1}(\mathrm{k}[x^{\pm 1}] \otimes \mathrm{k}[x^{\pm 1}][1]) \xrightarrow{\sim} \mathrm{k}.$$

Multiplying it by an arbitrary small DG category $\mathcal{A}$, we get a quasi-equivalence

$$T_{\mathcal{A}\otimes \mathrm{k}[x^{\pm 1}],I_{\mathcal{A}}\otimes(x-1)}(I_{\mathcal{A}} \otimes \mathrm{k}[x^{\pm 1}] \otimes \mathrm{k}[x^{\pm 1}][1]) \xrightarrow{\sim} \mathcal{A}.$$

Now suppose that $\mathcal{A}$ satisfies the condition (ii) of Main Theorem.

Step 2

It is easy to obtain k from $\mathrm{k}[x^{\pm 1}]$ by "attaching a cell": we have a quasi-isomorphism

$$T_{\mathrm{k}[x^{\pm 1}],x-1}(\mathrm{k}[x^{\pm 1}]\otimes \mathrm{k}[x^{\pm 1}][1]) \xrightarrow{\sim} \mathrm{k}.$$

Multiplying it by an arbitrary small DG category $\mathcal{A}$, we get a quasi-equivalence

$$T_{\mathcal{A}\otimes\mathrm{k}[x^{\pm 1}],I_{\mathcal{A}}\otimes(x-1)}(I_{\mathcal{A}}\otimes \mathrm{k}[x^{\pm 1}]\otimes \mathrm{k}[x^{\pm 1}][1]) \xrightarrow{\sim} \mathcal{A}.$$

Now suppose that $\mathcal{A}$ satisfies the condition (ii) of Main Theorem. By Step 1 and by Thomason theorem, we can find a finite cell DG category $\mathcal{C}$, which is Morita equivalent to $\mathcal{A}\otimes \mathrm{k}[x^{\pm 1}]$, such that the image of $I_{\mathcal{A}}\otimes \mathrm{k}[x^{\pm 1}]\otimes \mathrm{k}[x^{\pm 1}][1]$ under the equivalence

$$D_{perf}((\mathcal{A}\otimes \mathrm{k}[x^{\pm 1}])\otimes(\mathcal{A}\otimes \mathrm{k}[x^{\pm 1}])^{op}) \simeq D_{perf}(\mathcal{C}\otimes\mathcal{C}^{op})$$

is quasi-isomorphic to a semi-free finitely generated DG module $M \in \mathcal{SF}_{f.g.}(\mathcal{C}\otimes\mathcal{C}^{op})$.

Step 2

If now $\alpha : M \to I_{\mathcal{C}}[1]$ is the morphism corresponding to $I_{\mathcal{A}} \otimes (x - 1)$, we get a Morita equivalence between the finite cell DG category $T_{\mathcal{C},\alpha}(M)$ and the DG category $\mathcal{A}$, which finishes Step 2 and proves Main Theorem.

Related results

Using similar arguments, one can prove some new results about smooth and proper DG categories, and about compactifications. We will formulate them without proofs.

Related results

Using similar arguments, one can prove some new results about smooth and proper DG categories, and about compactifications. We will formulate them without proofs.

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be pre-triangulated Karoubi complete smooth and proper DG categories, and $\mathcal{B} \neq 0$. TFAE:

Related results

Using similar arguments, one can prove some new results about smooth and proper DG categories, and about compactifications. We will formulate them without proofs.

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be pre-triangulated Karoubi complete smooth and proper DG categories, and $\mathcal{B} \neq 0$. TFAE:
1) The class $[I_{\mathcal{A}}] \in K_0(\mathcal{A} \otimes \mathcal{A}^{op})$ is generated by the classes $[N_{GF}]$, where $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$ are quasi-functors.

Related results

Using similar arguments, one can prove some new results about smooth and proper DG categories, and about compactifications. We will formulate them without proofs.

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be pre-triangulated Karoubi complete smooth and proper DG categories, and $\mathcal{B} \neq 0$. TFAE:
1) The class $[I_{\mathcal{A}}] \in K_0(\mathcal{A} \otimes \mathcal{A}^{op})$ is generated by the classes $[N_{GF}]$, where $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$ are quasi-functors.
2) $\mathcal{A}$ is a homotopy retract of some (proper) semi-orthogonal gluing of a finite number of copies of $\mathcal{B}$.

Related results

Using similar arguments, one can prove some new results about smooth and proper DG categories, and about compactifications. We will formulate them without proofs.

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be pre-triangulated Karoubi complete smooth and proper DG categories, and $\mathcal{B} \neq 0$. TFAE:
1) The class $[I_{\mathcal{A}}] \in K_0(\mathcal{A} \otimes \mathcal{A}^{op})$ is generated by the classes $[N_{GF}]$, where $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$ are quasi-functors.
2) $\mathcal{A}$ is a homotopy retract of some (proper) semi-orthogonal gluing of a finite number of copies of $\mathcal{B}$.
3) $\mathcal{A}$ is quasi-equivalent to an admissible subcategory of a (proper) semi-orthogonal gluing of a finite number of copies of $\mathcal{B}$.

Related results

Moreover, we have the following more precise statement about retracts

Related results

Moreover, we have the following more precise statement about retracts

Proposition

If $\mathcal{A}$ and $\mathcal{B}$ are as above, and $\mathcal{A}$ is a homotopy retract of $\mathcal{B}$, then we have a fully faithful quasi-functor

$$\mathcal{A} \hookrightarrow \mathcal{C} = \langle \mathcal{B}, \mathcal{B}, \mathcal{B}, \mathcal{B} \rangle,$$

where $\mathcal{C}$ is a (proper) semi-orthogonal gluing of 4 copies of $\mathcal{B}$.

Related results

Moreover, we have the following more precise statement about retracts

Proposition

If $\mathcal{A}$ and $\mathcal{B}$ are as above, and $\mathcal{A}$ is a homotopy retract of $\mathcal{B}$, then we have a fully faithful quasi-functor

$$\mathcal{A} \hookrightarrow \mathcal{C} = \langle \mathcal{B}, \mathcal{B}, \mathcal{B}, \mathcal{B} \rangle,$$

where $\mathcal{C}$ is a (proper) semi-orthogonal gluing of 4 copies of $\mathcal{B}$.

Maybe 4 is not minimal.

Related results

Recall that a smooth categorical compactification of a pre-triangulated DG category $\mathcal{A}$ is a quotient quasi-functor $F : \mathcal{C} \to \mathcal{A}$ (up to direct summands), where $\mathcal{C}$ is a pre-triangulated smooth and proper DG category, and the kernel of F is generated by a single object.

Related results

Recall that a smooth categorical compactification of a pre-triangulated DG category $\mathcal{A}$ is a quotient quasi-functor $F : \mathcal{C} \to \mathcal{A}$ (up to direct summands), where $\mathcal{C}$ is a pre-triangulated smooth and proper DG category, and the kernel of F is generated by a single object. Such a compactification implies that $\mathcal{A}$ is hfp.

Related results

Recall that a smooth categorical compactification of a pre-triangulated DG category $\mathcal{A}$ is a quotient quasi-functor $F : \mathcal{C} \to \mathcal{A}$ (up to direct summands), where $\mathcal{C}$ is a pre-triangulated smooth and proper DG category, and the kernel of F is generated by a single object. Such a compactification implies that $\mathcal{A}$ is hfp.

Recently, I've constructed an example of a hfp DG category which does not admit a smooth categorical compactification (a counterexample to one of the Kontsevich's conjectures).

Related results

Recall that a smooth categorical compactification of a pre-triangulated DG category $\mathcal{A}$ is a quotient quasi-functor $F : \mathcal{C} \to \mathcal{A}$ (up to direct summands), where $\mathcal{C}$ is a pre-triangulated smooth and proper DG category, and the kernel of F is generated by a single object. Such a compactification implies that $\mathcal{A}$ is hfp.

Recently, I've constructed an example of a hfp DG category which does not admit a smooth categorical compactification (a counterexample to one of the Kontsevich's conjectures). Using the above methods, we obtain the following criterion for existence of a smooth compactification.

Related results

Recall that a smooth categorical compactification of a pre-triangulated DG category $\mathcal{A}$ is a quotient quasi-functor $F : \mathcal{C} \to \mathcal{A}$ (up to direct summands), where $\mathcal{C}$ is a pre-triangulated smooth and proper DG category, and the kernel of F is generated by a single object. Such a compactification implies that $\mathcal{A}$ is hfp.

Recently, I've constructed an example of a hfp DG category which does not admit a smooth categorical compactification (a counterexample to one of the Kontsevich's conjectures). Using the above methods, we obtain the following criterion for existence of a smooth compactification.

Theorem

Let $\mathcal{A}$ be a hfp pre-triangulated DG category. TFAE:
1) $\mathcal{A}$ admits a smooth categorical compactification.
2) There exists a DG functor $\mathcal{C} \to \mathcal{A}$, where $\mathcal{C}$ is smooth and proper, such that $[I_{\mathcal{A}}] \in \mathrm{Im}(K_0(\mathcal{C} \otimes \mathcal{C}^{op}) \to K_0(\mathcal{A} \otimes \mathcal{A}^{op}))$.

Generalization for algebras over DG operads

The above methods imply the following result for DG algebras:

Corollary

Let A be a hfp DG algebra. TFAE:
1) A is quasi-isomorphic to a finite cell DG algebra.

Generalization for algebras over DG operads

The above methods imply the following result for DG algebras:

Corollary

Let A be a hfp DG algebra. TFAE:
1) A is quasi-isomorphic to a finite cell DG algebra.
2) $[A] \in \mathbb{Z} \cdot [A \otimes A]$ in $K_0(A \otimes A^{op})$.

Generalization for algebras over DG operads

The above methods imply the following result for DG algebras:

Corollary

Let A be a hfp DG algebra. TFAE:
1) A is quasi-isomorphic to a finite cell DG algebra.
2) $[A] \in \mathbb{Z} \cdot [A \otimes A]$ in $K_0(A \otimes A^{op})$.

Note that the condition 2) is equivalent to the following:
$[\Omega_A] \in \mathbb{Z} \cdot [A \otimes A]$ in $K_0(A \otimes A^{op})$, where $\Omega_A = \ker(A \otimes A \xrightarrow{m_A} A)$ is the bimodule of differentials.

Generalization for algebras over DG operads

The above methods imply the following result for DG algebras:

Corollary

Let A be a hfp DG algebra. TFAE:
1) A is quasi-isomorphic to a finite cell DG algebra.
2) $[A] \in \mathbb{Z} \cdot [A \otimes A]$ *in* $K_0(A \otimes A^{op})$.

Note that the condition 2) is equivalent to the following:
$[\Omega_A] \in \mathbb{Z} \cdot [A \otimes A]$ in $K_0(A \otimes A^{op})$, where $\Omega_A = \ker(A \otimes A \xrightarrow{m_A} A)$ is the bimodule of differentials.
Since Ω_A is the cotangent complex of A (in the operadic sense, for the associative operad), it is natural to ask whether a similar result holds for a general (colored) DG operad.

Generalization for algebras over DG operads

The above methods imply the following result for DG algebras:

Corollary

Let A be a hfp DG algebra. TFAE:
1) A is quasi-isomorphic to a finite cell DG algebra.
2) $[A] \in \mathbb{Z} \cdot [A \otimes A]$ in $K_0(A \otimes A^{op})$.

Note that the condition 2) is equivalent to the following:
$[\Omega_A] \in \mathbb{Z} \cdot [A \otimes A]$ in $K_0(A \otimes A^{op})$, where $\Omega_A = \ker(A \otimes A \xrightarrow{m_A} A)$ is the bimodule of differentials.
Since Ω_A is the cotangent complex of A (in the operadic sense, for the associative operad), it is natural to ask whether a similar result holds for a general (colored) DG operad. In fact, it does, and we briefly illustrate it for the commutative operad.

Wall obstruction for commutative DG algebras

Suppose that the base field k has characteristic zero.

Wall obstruction for commutative DG algebras

Suppose that the base field k has characteristic zero. Commutative finite cell DG algebras are defined as free (super-)commutative DG algebras $\mathrm{k}[x_1, \ldots, x_n]$, where again dx_i is generated by $x_1, \ldots, x_{i-1}$.

Wall obstruction for commutative DG algebras

Suppose that the base field k has characteristic zero. Commutative finite cell DG algebras are defined as free (super-)commutative DG algebras $\mathrm{k}[x_1, \ldots, x_n]$, where again dx_i is generated by $x_1, \ldots, x_{i-1}$. Similarly, hfp commutative DG algebras are their homotopy retracts.

Wall obstruction for commutative DG algebras

Suppose that the base field k has characteristic zero. Commutative finite cell DG algebras are defined as free (super-)commutative DG algebras $\mathrm{k}[x_1, \dots, x_n]$, where again dx_i is generated by $x_1, \dots, x_{i-1}$. Similarly, hfp commutative DG algebras are their homotopy retracts.

We denote by $L_{A/\mathrm{k}}$ the cotangent complex of a cdga A. It can be computed for example as $\Omega_{B/\mathrm{k}} \otimes_B A$, where $B \xrightarrow{\sim} A$ is any cofibrant (say, semi-free) replacement.

Wall obstruction for commutative DG algebras

Suppose that the base field k has characteristic zero. Commutative finite cell DG algebras are defined as free (super-)commutative DG algebras $\mathrm{k}[x_1, \dots, x_n]$, where again dx_i is generated by $x_1, \dots, x_{i-1}$. Similarly, hfp commutative DG algebras are their homotopy retracts.
We denote by $L_{A/\mathrm{k}}$ the cotangent complex of a cdga A. It can be computed for example as $\Omega_{B/\mathrm{k}} \otimes_B A$, where $B \xrightarrow{\sim} A$ is any cofibrant (say, semi-free) replacement.

Theorem

Let A be a homotopically finitely presented cdga. TFAE:
1) A is quasi-isomorphic to a finite cell DG algebra.

Wall obstruction for commutative DG algebras

Suppose that the base field k has characteristic zero. Commutative finite cell DG algebras are defined as free (super-)commutative DG algebras $\mathrm{k}[x_1, \dots, x_n]$, where again dx_i is generated by $x_1, \dots, x_{i-1}$. Similarly, hfp commutative DG algebras are their homotopy retracts.
We denote by $L_{A/\mathrm{k}}$ the cotangent complex of a cdga A. It can be computed for example as $\Omega_{B/\mathrm{k}} \otimes_B A$, where $B \xrightarrow{\sim} A$ is any cofibrant (say, semi-free) replacement.

Theorem

Let A be a homotopically finitely presented cdga. TFAE:
1) A is quasi-isomorphic to a finite cell DG algebra.
2) We have $[L_{A/\mathrm{k}}] \in \mathbb{Z} \cdot [A]$ in $K_0(A)$.

Wall obstruction for commutative DG algebras

Here the implication 1)⇒2) is obvious.

Wall obstruction for commutative DG algebras

Here the implication 1)$\Rightarrow$2) is obvious.
For an arbitrary cofibrant hfp cdga A, we see (applying the argument involving coeqalizers) that $\mathrm{Sym}_A(\Omega_{A/\mathrm{k}}[1]) \simeq \mathrm{coeq}^h(\mathrm{id}_A, \mathrm{id}_A)$ is quasi-isomorphic to a finite cell cdga B.

Wall obstruction for commutative DG algebras

Here the implication 1)⇒2) is obvious.
For an arbitrary cofibrant hfp cdga A, we see (applying the argument involving coeqalizers) that $\mathrm{Sym}_A(\Omega_{A/\mathrm{k}}[1]) \simeq \mathrm{coeq}^h(\mathrm{id}_A, \mathrm{id}_A)$ is quasi-isomorphic to a finite cell cdga B. If moreover A satisfies 2), we can find (by Thomason) a semi-free DG B-module M, such that M corresponds to $\Omega_{A/\mathrm{k}} \otimes_A Sym_A(\Omega_{A/\mathrm{k}}[1])$ under the equivalence $D_{perf}(B) \simeq D_{perf}(Sym_A(\Omega_{A/\mathrm{k}}[1]))$.

Wall obstruction for commutative DG algebras

Here the implication 1)⇒2) is obvious.

For an arbitrary cofibrant hfp cdga A, we see (applying the argument involving coeqalizers) that $\mathrm{Sym}_A(\Omega_{A/\mathrm{k}}[1]) \simeq \mathrm{coeq}^h(\mathrm{id}_A, \mathrm{id}_A)$ is quasi-isomorphic to a finite cell cdga B. If moreover A satisfies 2), we can find (by Thomason) a semi-free DG B-module M, such that M corresponds to $\Omega_{A/\mathrm{k}} \otimes_A Sym_A(\Omega_{A/\mathrm{k}}[1])$ under the equivalence $D_{perf}(B) \simeq D_{perf}(Sym_A(\Omega_{A/\mathrm{k}}[1]))$.

Then the corresponding deformed symmetric algebra $\mathrm{Sym}_{B,\alpha}(M[2])$ is quasi-isomorphic to A.

Wall obstruction for commutative DG algebras

Here the implication 1)⇒2) is obvious.

For an arbitrary cofibrant hfp cdga A, we see (applying the argument involving coeqalizers) that $\mathrm{Sym}_A(\Omega_{A/\mathrm{k}}[1]) \simeq \mathrm{coeq}^h(\mathrm{id}_A, \mathrm{id}_A)$ is quasi-isomorphic to a finite cell cdga B. If moreover A satisfies 2), we can find (by Thomason) a semi-free DG B-module M, such that M corresponds to $\Omega_{A/\mathrm{k}} \otimes_A Sym_A(\Omega_{A/\mathrm{k}}[1])$ under the equivalence $D_{perf}(B) \simeq D_{perf}(Sym_A(\Omega_{A/\mathrm{k}}[1]))$.

Then the corresponding deformed symmetric algebra $\mathrm{Sym}_{B,\alpha}(M[2])$ is quasi-isomorphic to A. But again the cdga $\mathrm{Sym}_{B,\alpha}(M[2])$ is finite cell. This proves the implication 2)⇒1).

Arbitrary colored DG operads

Similar arguments apply for an arbitrary colored DG operad (a DG multicategory) $\mathcal{C}$, (if char$(\mathrm{k}) > 0$, one needs to make certain standard assumptions).

Arbitrary colored DG operads

Similar arguments apply for an arbitrary colored DG operad (a DG multicategory) $\mathcal{C}$, (if $\mathrm{char}(\mathrm{k}) > 0$, one needs to make certain standard assumptions). We choose the generating cofibrations to be of the form

$$\mathrm{Free}(P) \to \mathrm{Free}(\mathrm{Cone}(P \xrightarrow{\mathrm{id}} P)),$$

where P runs through some collection of cofibrant perfect left $\mathcal{C}(1)$-modules, closed under shifts, which includes representable modules.

Arbitrary colored DG operads

Similar arguments apply for an arbitrary colored DG operad (a DG multicategory) $\mathcal{C}$, (if $\mathrm{char}(\mathrm{k}) > 0$, one needs to make certain standard assumptions). We choose the generating cofibrations to be of the form

$$\mathrm{Free}(P) \to \mathrm{Free}(\mathrm{Cone}(P \xrightarrow{\mathrm{id}} P)),$$

where P runs through some collection of cofibrant perfect left $\mathcal{C}(1)$-modules, closed under shifts, which includes representable modules.

The condition 2) will be of the form $[L_A] \in S \subset K_0(U(A))$, where $U(A)$ is the enveloping DG category of A,

Arbitrary colored DG operads

Similar arguments apply for an arbitrary colored DG operad (a DG multicategory) $\mathcal{C}$, (if $\mathrm{char}(\mathrm{k}) > 0$, one needs to make certain standard assumptions). We choose the generating cofibrations to be of the form

$$\mathrm{Free}(P) \to \mathrm{Free}(\mathrm{Cone}(P \xrightarrow{\mathrm{id}} P)),$$

where P runs through some collection of cofibrant perfect left $\mathcal{C}(1)$-modules, closed under shifts, which includes representable modules.

The condition 2) will be of the form $[L_A] \in S \subset K_0(U(A))$, where $U(A)$ is the enveloping DG category of A, and L_A is the cotangent complex (a left $U(A)$-module),

Arbitrary colored DG operads

Similar arguments apply for an arbitrary colored DG operad (a DG multicategory) $\mathcal{C}$, (if $\mathrm{char}(\mathrm{k}) > 0$, one needs to make certain standard assumptions). We choose the generating cofibrations to be of the form

$$\mathrm{Free}(P) \to \mathrm{Free}(\mathrm{Cone}(P \xrightarrow{\mathrm{id}} P)),$$

where P runs through some collection of cofibrant perfect left $\mathcal{C}(1)$-modules, closed under shifts, which includes representable modules.

The condition 2) will be of the form $[L_A] \in S \subset K_0(U(A))$, where $U(A)$ is the enveloping DG category of A, and L_A is the cotangent complex (a left $U(A)$-module), and S is an abelian subgroup of $K_0(U(A))$, generated by the classes of $U(A) \otimes_{\mathcal{C}(1)} P$, where P is as above (participates in the collection of generating cofibrations).

Arbitrary colored DG operads

Similar arguments apply for an arbitrary colored DG operad (a DG multicategory) $\mathcal{C}$, (if $\mathrm{char}(\mathrm{k}) > 0$, one needs to make certain standard assumptions). We choose the generating cofibrations to be of the form

$$\mathrm{Free}(P) \to \mathrm{Free}(\mathrm{Cone}(P \xrightarrow{\mathrm{id}} P)),$$

where P runs through some collection of cofibrant perfect left $\mathcal{C}(1)$-modules, closed under shifts, which includes representable modules.

The condition 2) will be of the form $[L_A] \in S \subset K_0(U(A))$, where $U(A)$ is the enveloping DG category of A, and L_A is the cotangent complex (a left $U(A)$-module), and S is an abelian subgroup of $K_0(U(A))$, generated by the classes of $U(A) \otimes_{\mathcal{C}(1)} P$, where P is as above (participates in the collection of generating cofibrations).

This general formulation formally implies all the (new) results mentioned in the talk.