Attractor Conjecture for CYs.

$X/k$ a Calabi-Yau threefold, i.e. $K^3 ≤ 5$
(Sometimes more restrictions
but I'll take this as
my default).

Def. $x$ is an attractor point if

\[ \exists \gamma \in H^3(x, \mathbb{Z}) \text{ s.t.} \]

\[ \gamma \in H^{3,0} \oplus H^{0,3} \subset H^3(x, \mathbb{Z}) \otimes \mathbb{C}. \]

Hodge decomp.

In 1998 Moore made the following remarkable conjecture

Conj. (Moore '98) If $X$ is attractor, then it is defined over $\mathbb{Q}$.

(i.e. Has model over $\mathbb{Q}$)

Known cases: $X = K3 \times E, E_1 \times E_2 \times E_3, \ldots$
Also if "Delgado 01's: cyclic covers of $\mathbb{P}^3$ branched along 6 hyperplanes" (L. Tripathy)

Observe, all of these have a "torelli theorem", i.e. period map is an open embedding.

More fancy: model space is Shimura variety (e.g. locally symmetric space).

Probably not true for general 01's.

[have counterexample for $n$ odd, $> 3$. L. Tripathy]

Unless model is Shimura.

Unfortunately, not many examples of such 01's known.

But next best thing:

Every Shimura variety has a (unique) 04 variation of Hodge structures.
Then \( \mathcal{H}(L) \) is at \( 3 \) \( \text{VHS} \) of \( \text{CH} \) type, the \text{alternative}

conjecture holds.

\[ V \otimes \mathbb{Q} = H^{0,0} \otimes \mathbb{Q} \oplus \cdots \oplus H^{n,n}, \]

satisfying some condition \((\text{polarization})\)

\[ V \text{ is } \mathbb{Q} \text{-} \text{local system,} \]

there is \( \text{family } S \) over base \( S \) \( \text{co} \)- \( \text{sheaf,} \)

and \( \text{family } F \) is \( \text{VHS} \) \( \text{+ "Poincare\'s dream possibility".} \)

\[ \nabla : F^r \to F^{r-1} \oplus \Omega^r \]

\text{Example:} \( \text{a family of \( \text{ell} \)-varieties } X \to S, \)

local system is \( H^r(X, \mathbb{Q}) \) \( + \) its \( \text{Hodge} \)

GT is a "geometric condition".

Def. A \( \text{VHS} \) \( \text{of \( \text{CH} \) type } \) if \( h^{n-1} = 0 \). \( (\text{ad } n) \).
A space of the form \( \mathcal{G}(Z)/\mathcal{G}(R_3)/K \)

\[ Z \text{- pt of } \mathcal{G} \]

\[ \text{maximal compact} \]

\[ \text{real pt of reductive } \mathcal{G} \]

\[ \text{compact dual, projective variety} \]

Not all \( G \).

In nice cases, \( \mathcal{G}(R)/K \) has reductive \( Z \).

(when \( K \) has \( \mathcal{U}(1) \)-factor)

\( G = \mathfrak{sl}_2 \), \( \mathcal{G}(R)/K = \mathfrak{sl}_2(R_3)/\mathcal{U}(1) = \mathfrak{h} \).

\( \) has \( \mathfrak{g} \)-\( \mathfrak{m} \) of quasi-projective variety

\( \) has canonical descent to \( \# \) field \( E \).

\( \) using CM points.

\( [\text{integrated over } \ldots] \)

Example, \( D = \mathfrak{h} \rightarrow \mathbb{P} \{ (a) = 0 \} \)

\( G = \mathfrak{sl}_2(R), \quad \mathcal{G}(Z) = \mathfrak{sl}_2(Z) \).
\$h \rightarrow P'\$
\[\text{Warning: the } \mathbb{Q}-\text{surfaces on top and bottom are completely different.}\]

Thus (Crom, Eichler-Zagier) every Shimura variety has canonical lowest weight \$\text{cyntes}\$

canonical means . . .

\$\nabla: \text{TX} \rightarrow \text{Hom}(\mathbb{P}^n, H^{h, 0} \oplus H^{h-1, 1})\$
\[\rightarrow \text{Hom}(H^{h, 0}, H^{h-1, 1})\]

\[H^{0, h} \oplus H^{1, h-1} \oplus \ldots \oplus H^{n, h-1}\]

\$\text{TX: shifts up by one unit + project.}\$

\[\leftrightarrow \text{Maximal family.}\]
weight s examples.

real. \( \mathfrak{g} = \mathfrak{so}(2,\mathbb{C}) / \mathfrak{su}(2) \times \mathfrak{su}(2) \)

\[ + 4 \text{ exceptional case, bigger of which is} \]
\[ \mathfrak{e}_7(-25) / \mathfrak{u}(1) \times \mathfrak{e}_6(-28) \]

\[ + \frac{B}{Q} - \text{forms of these groups} \]

Compact duals.

\[ \mathbb{C}P^1, \mathbb{C}P^1, G(2,3) \mathbb{C}P^1, G(3,5) \mathbb{C}P^1, \mathbb{C}P^3, E_7 / \mathbb{C}, C \mathbb{P}^5 \]

Legendrian homogeneous varieties
(Zak, Gross-Witten, Landsberg-Messias, ...)

We do not know if \( E_7, E_6 \) are varieties come from geometry. Forms a "real case".

Physics, no known string embedding of a supergravity type, although there are some suggestion of the \( E_7 \)-cone.
So have a local system \( V / S \), \( s \) in \( V / S \) and attractor mechanism:

\[ \gamma \in V(Z) \]

\[ V_\phi : S \rightarrow \mathbb{R} \]

\[ p \rightarrow \begin{pmatrix} < s, \phi > \end{pmatrix}^2 \frac{1}{s^* s} \]

\[ \text{minimum of this } f. \]

**Example.** For \( \phi \) have homogeneous coordinates \( x^0, x^1, (x^0 / x^0 = 2) \)

\[ V = \text{Sym}^3 \mathbb{R} \], 4-dimensional

\[ V \oplus C = H_3,0 \oplus H^{2,1} \oplus H^{-2,1} \oplus t_0,3 \]

\[ \mathfrak{h}' = \{ (p, \omega) \mid \text{P} \in \mathfrak{h}', \omega \in t_0,3 \} \]

Write \( \omega = x^0 \omega_0 + x^1 \omega_1 + F_0 \omega_0 + F_1 \omega_1 \)

\[ F_0 = \frac{-c x^3}{3x^0 z}, \quad F_1 = \frac{c x^2}{x^0} \]

**Fact:**

\[ F_0 = -\frac{c x^3}{3x^0 z}, \quad F_1 = \frac{c x^2}{x^0} \]

**Lemma (L) for \gamma \in \text{Sym}^3 \mathbb{R} \text{ and } g \rightarrow \begin{pmatrix} a x^3 + b x^2 \gamma + c x^2 \gamma + d \gamma^3 \end{pmatrix} \text{ with real roots} \]
the three roots of \( p(x, i) \)

Proof. Compute \( V_{Y} = \frac{|p(z)|}{y^{3/2}} \), which matches the sum of distance from the origin.

Prop. \( \Rightarrow \) attractor condition stated earlier.

Actually, more subtle: there is a U-turn (as in the Example above)

\[ G_{1}(Z) \supset V(C) \text{ prehistory.} \]

\[ V_{1} | G \subseteq A', \text{ and invertible is } T_{4} \text{ for } \sigma. \]

\[ \forall \sigma \in V(C) \Rightarrow T_{4} \sigma > 0. \]
Example. \( R^2 \cong \text{SL}_2(\mathbb{C}) \),
\[ V \cong \mathbb{C}^3 \otimes \mathbb{C}^3 \cong \frac{1}{2} \text{ binary cubic form} \]
\[ \text{SL}_2(\mathbb{R}) \cong V(\mathbb{R}), \quad I_3 \text{ discriminant of cubic form} \]

One way to view \( h \), or rather
\[ h' = \{ (p, z) \mid p \in \mathbb{R}, \quad z = \alpha + \beta \}(\mathbb{R}^3) - \mathbb{C} \text{ real} \]
\[ h' \rightarrow V(\mathbb{R}) \]
\[ (p, z) \mapsto \text{Re } z \in V(\mathbb{R}) \]
\[ \text{image } = \{ v \in V(\mathbb{R}) \mid I_3(0) > 0 \} \]
\[ \text{attaches } = \text{V}(\mathbb{Z}) \subset V(\mathbb{R}) \]

This (precise form) \( f \) for \( g \) a BPS, there is unique attachment point which is a CM point.
Given the points $P_1$, $P_2$, and $P_3$, we are to find the coordinates of point $Q$ such that $Q$ is on the line passing through $P_1$ and $P_2$.

By the property of lines, we can use the equation of the line $P_1P_2$ to find $Q$.

Let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, and $P_3 = (x_3, y_3)$. We can use the equation of the line $P_1P_2$ to find the coordinates of $Q$.

The equation of the line passing through $P_1$ and $P_2$ is given by:

$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

We need to find the coordinates of $Q = (x, y)$ such that $Q$ satisfies this equation.

By substituting the values of $x_1$, $y_1$, $x_2$, and $y_2$, we can solve for $x$ and $y$.

After solving the equation, we get:

$x = \frac{x_1 y_2 - y_1 x_2}{y_2 - y_1}$

and

$y = \frac{y_1 x_2 - x_1 y_2}{x_2 - x_1}$

The coordinates of $Q$ are given by these equations.

For example, if $P_1 = (1, 2)$, $P_2 = (3, 4)$, and $P_3 = (5, 6)$, then:

$x = \frac{1 \cdot 4 - 2 \cdot 3}{4 - 2} = \frac{2}{2} = 1$

and

$y = \frac{2 \cdot 3 - 1 \cdot 4}{3 - 1} = \frac{2}{2} = 1$

Thus, the coordinates of $Q$ are $(1, 1)$.
Real coordinates:

\[ x^I = p^I + i\phi^I, \]
\[ F^I = \phi^I + i\eta^I. \]

Lemme (Cecotti, Ferrara, Gimondeles, Hertle):

If \( \phi^I \in \mathbb{R}(p,q) \), then that

\[ \phi^I = \frac{\partial C(p,q)}{\partial x^I}, \]
\[ \eta^I = -\frac{\partial C(p,q)}{\partial p^I}. \]

In our case \( C = \sqrt{I_F} \).

Simple calculation:

In this case, have \( x^I / x^0 = \pm \epsilon^{\pm} \in \mathcal{Q}(T_0) \).
and attacte $\Rightarrow$ T

A general simple ode seems that for some integer $n$ of $\mathbb{2} \in \mathbb{Q}(\sqrt{5})$ and then find $S$.

Put for no p attacke $\Rightarrow p^{\frac{1}{2}} \in \mathbb{2}$, and

and hence $\mathcal{L}(r) \Rightarrow q^{\frac{1}{2}} \in \mathbb{Q}(\sqrt{5})$. \[ \Box \]

Cor. Have explicit param. of certain con points $\frac{5}{3}$?

Since we... $\Box$

---

An amusing question.

More on special geometry

\[ H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} \]

five local coordinates $x^i$, have a
cubic form. This is the K-model

\[ \cdots \& (\cdots) \cdots \]

\[ \cdots \]
In the case $\mathfrak{su}(5) / \Lambda$,

A model on $T^6 / \mathbb{Z}_3$ has a resolution called $Z$, which

$E \times E \times E \quad H^1 = H^1_{\mathbb{Z}} \oplus f_{\mathbb{C}}^6 \quad \mathbb{Z}^3 \cong 1$

$\eta = (1 + 1 + 1 + 3 \times 2)$

$Z$ has Hodge diamond

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 36 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array}
\]

an even $7$-fold, $H^7 = 0, 1, 3, 5, 3, 1, 0, 0$

(\text{a cycle of a quotient})

So there is a 9-dim subspace which is

homogeneous param by a 8d. curve.

Hodge ring $\Rightarrow$ the 9-dim subspace has
extra eig cycle coming from

Weil- type AV's.
Recap. Have \( V(\mathbb{Z}) \otimes \mathbb{L}(\mathbb{Z}) \) and require
\[
\frac{V(\mathbb{Z})}{\mathbb{L}(\mathbb{Z})} \to Sh.
\]
More on what these maps are, in each case.

Exceptional cases. We have \( C(A) \), \( \text{dim } A \)
\( \text{for of } \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \)

\[ N = \text{determinant.} \]

Thus (Krogan or Pol束)
\[
\frac{V(\mathbb{Z})}{\mathbb{L}(\mathbb{Z})} \text{ peren.}
\]

\[ (\mathbb{C}, i) \]

Quod erat demonstrandum. Ideal \( C \) exists.

Prop. In case the good ring matches the group,
\[ \text{good field } \mathbb{Q}(\mathbb{F}_0) \] in the proof.
View (?) this as a generalization of CM points on \( h \) rational function ideals.

How about non-BPS \$/\$/ other types of attractor?

(1) non-BPS attractor mod mod space are totally geodesic C D.

(2) 2 in 5d, BPS attractor are analogue of KM cycles.

(3) 2 in the case \( G_2, \ I_{0} < 0 \) catchies real \( it + \)

D \( \phi \) disk model

attract \( st = \) center of mass.
In the first conceptual design of those...