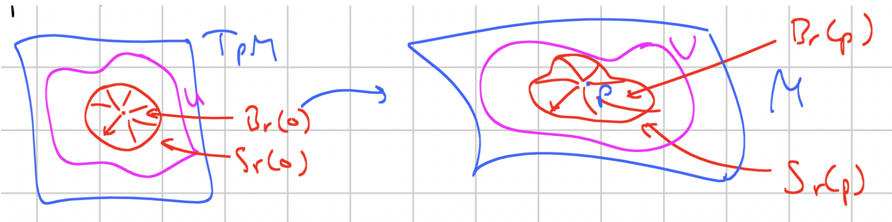
Lecture 9

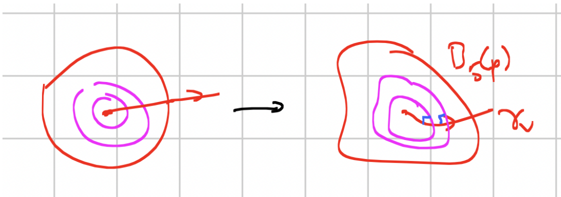
We will prove Gauss' lemma today.

0.33. Gauss' lemma proof.



Let $U \subset T_pM$ and $V \ni p$ be neighborhoods where the exponential map $\exp_p : U \to V$ is diffeomorphism. Consider the ball $B_r(0)$ such that $B_r(p) := \exp_p B_r(0)$ with its closure lies in V as in the picture above. Denote $S_r(0)$ the border of the ball $B_r(0)$ and $S_r(p) := \exp_p(S_r(0))$.

Here how the radial geodesics $\exp_p(tv)$ intersect the spheres $\exp_p(S_r(0))$ (in a picture below).



Proof. 1. Introduce polar coordinates $(\rho, \varphi_1, ..., \varphi_{m-1})$ on T_pM, thus $\rho^2 = \sum_i x_i^2$ and φ_i are local coordinates on the unit sphere $S^{m-1} \subset T_pM$. Using $\exp_p$, we can view these as coordinates on (suitable open subsets of) $\exp_p(B_r(0))$ for $r < i_p(M)$. We have a well-defined vector field $\frac{\partial}{\partial \rho}$ on $\exp_p(B_r(0) \setminus \{0\})$[28].

2. In these coordinates we have $g_{\rho\rho} = 1, g_{\rho\varphi_j} = 0$.

We will skip the proof here, the second equation follows from the fact that the connection is torsion-free.

3. From the above we have that the radial geodesics $\exp_p(tv)$ are orthogonal to the spheres $\exp_p(S_r(0))$, for all $0 < r < i_p(M)$.

Indeed, define $Z := (\exp_p)_* \frac{\partial}{\partial \rho}$. Then $Z_{\gamma_v(t)} = \gamma_v'(t)$ (we use $|Z| = 1$).

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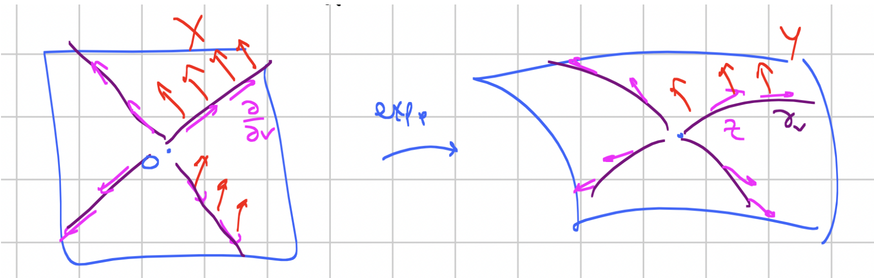
[28]Note that its integral curves are exactly the unit speed radial geodesics.



We want any vector tangent to $S_r(p)$ at $\gamma_v(t)$ be orthogonal to $Z_{\gamma_v(t)}$. Let X be any vector field defined on $S_1(0)$. Extend this field to $B_1(0) \setminus \{0\}$ by $X_{t\omega} = X_\omega$, for $\omega \in S_1(0)$.

It suffices to show that $Y := (\exp_p)_* X$ and γ_v' are orthogonal along γ_v.

Fix v and consider function $f(t) = g(Y_{\gamma_v(t)}, Z_{\gamma_v(t)})$. We are going to prove that $f(t) = 0$.

$$\frac{d}{dt} f(t) = Zg(Y, Z) = g(\nabla_Z Y, Z) + g(Y, \nabla_Z Z),$$

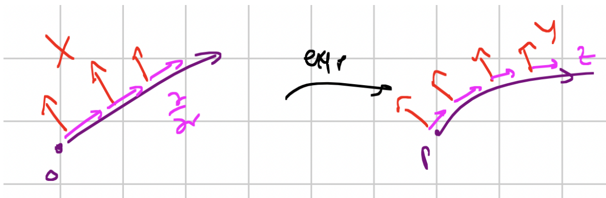
the second term is zero by definition since γ_v is geodesic.

$$g(\nabla_Z Y, Z) = g(\nabla_Y Z, Z) + g([Z, Y], Z)$$

the first term vanishes since $2g(\nabla_Y Z, Z) = Yg(Z, Z) = 0$.

Moreover, $[Z, Y] = [(\exp_p)_* \partial_\rho, (\exp_p)_* X] = (\exp_p)_* [\partial_\rho, X] = 0$.

Thereofore, $f(t) = c$ for some constant c.



Now we extend X and ∂_ρ along the arc $\{tv\}$ continuously to 0, hence we can extend Y, Z along γ_v to p. As we have seen in the proof of proposition 0.4 the map $(\exp_p)_* : T_0(T_pM) = T_pM \to T_pM$ is the identity, so $Y_p = X_0$ and $Z_p = (\partial_\rho)_0$. Therefore, $c = \lim_{t\to 0} g(Y, Z)_{\gamma_v(t)} = g(Y, Z)_p = g(X, \partial_\rho)_0 = 0$.

4. Consider any curve $\gamma(t)$ be any curve with $\gamma(0) = p, \gamma(1) = q$. Suppose that $\gamma(t) \in \exp_p(B_r(0)\ \{0\})$. The derivative in geodesic polar coordinates is

$$\dot\gamma = \dot\rho \frac{\partial}{\partial \rho} + \sum_j \dot\varphi_j \frac{\partial}{\partial \varphi_j}$$

5. We have $g(\dot\gamma, \dot\gamma) \geq |\dot\rho|^2$ and equality holds iff $\varphi_j = const$.

6. Then we have

$$L(\gamma) = \in_0^1 g(\dot\gamma, \dot\gamma)^{1/2} dt \geq \int_0^1 |\dot\rho| dt \geq \int_0^1 \dot\rho dt = \rho(1) = r$$

. $\square$

Corollary 0.2. *Let $p, q \in M$. Suppose there exists a piecewise smooth curve $\gamma : [0,1] \to M$ of length $d(p,q)$ from p to q. Then γ is a reparametrization of a smooth geodesic of length $d(p,q)$.*

Proof. Compactness of $\gamma([0,1]) \subset M$ implies that the infimum of the set of all injectivity radii $i_{\gamma(t)}(M)$ is strictly positive. Choose number $\epsilon > 0$ to be smaller than this infimum.

Then for any two points on the curve, of distance less than ϵ, the unique shortest curve connecting these points is the geodesic given by the exponential map (Gauss' lemma). In particular, γ must coincide with that geodesic up to reparametrization. $\square$

Example: Let us describe geodesics in $\mathbb{C}P^n$. Consider vector field $E = \sum_j^{n+1} x_{2j-1}\partial_{2j} - x_{2j}\partial_{2j-1}$ on S^{2n+1}.Consider H_z – the set of fields in $T_z S^{2n+1}$ which are orthogonal to $E(z)$. The map $\Phi_z = d\pi_z : H_z \to T_{\pi(z)}\mathbb{C}P^n$[29] is invertible so we an define the Riemannian metric on $\mathbb{C}P^n$:

$$h(X,Y) = g_z(\Phi^{-1}(X), \Phi^{-1}(Y))$$

This metric is known Fubini-Study metric.

If γ is geodesic in this metric ($\gamma(0) = \pi(z), \gamma'(0) = X$) then there is geodesic $\tilde{\gamma} \subset S^{2n+1}$ with $\tilde{\gamma} = Z = \Phi^{-1}(X)$. It has a form $\tilde{\gamma}(t) = z\cos t + Z\sin t$. Moreover, $\tilde{\gamma}'(t) \in H_{\tilde{\gamma}(t)}$. Then $\gamma = \pi \cdot \tilde{\gamma}$. Therefore, all geodesics in $(\mathbb{C}P^n, h)$ are the projections of geodesics in (S^{2n+1}, g) so that $\tilde{\gamma}'(t) \in H_{\tilde{\gamma}(t)}$ for all t.

0.34. **Hopf-Rinow.** Since geodesics are so important, and their short time existence and uniquess are so useful, it is important to know when they can be extended for all time.

Definition 0.33. *(M,g) is (geodesically) complete if $exp_p(X)$ is defined for all $X \in T_pM$ and for all $p \in M$.*

Equivalently, normalised geodesics $\gamma_{(p,X)}(t)$ =$\exp_p(tX)$ are defined for all $X \in T_pM$ with $|X| = 1$, for all $t \in \mathbb{R}$ and for all $p \in M$.

Examples:

1. We see that $(\mathbb{R}^2, g_0)$ is complete because straight lines $\gamma(t) = (x_1 + ty_1, x_2 + ty_2)$ are defined for all $t \in \mathbb{R}$ and any $y_1, y_2 \in \mathbb{R}$. The same argument true for $\mathbb{R}^n$. And for torus $T^n \subset \mathbb{R}^n$.
2. On S^n with the standard induced Riemannian metric g normalised geodesics are great circles, they are parameterized by t and so are certainly defined for all points and tangent vectors, hence (S^n, g) is complete.
3. If we remove a point from the sphere then the geodesics that passed through that point are now no longer defined for all $t \in \mathbb{R}$. In fact, we see that if we take any Riemannian manifold and remove a point then it cannot be complete with the induced Riemannian metric.

Proposition 0.18. *If $p, q \in (M,g)$, define $d(p,q) \equiv dist(p,q) = inf\{L(\gamma) : \gamma$ is a piecewise smooth curve from p to $q\}$. Then (M,d) is a metric space.*

For the proof we need to check that the metric ball is usual ball and back.

[29] π is the Hopf fibration.