Lecture 8

First we start with parallel transport in terms of Christoffel symbols.

We will talk about geodesics today. In the course of curves and surfaces we have seen that geodesics are curves in the surface giving locally the shortest path between any two points, and they were defined by the condition that their acceleration field was normal to the surface. Again, we do not have an ambient space to define the condition to be a geodesic so we define things intrinsically. But they still will be the shortest ones!

0.27. **Parallel transport revisited.** Let us recall that a section s is parallel along a path $\gamma : [0, 1] \to M$ if $\nabla_{\gamma'(t)}(s) = 0$ throughout $[0, 1]$. And it gives the isomorphism of $E_{\gamma(0)}$ and $E_{\gamma(1)}$ is called *parallel transport along* γ.

Now we can state

Theorem 0.15. *Let ∇ be a metric connection on a manifold M. Let $\gamma : I \to M$ be a smooth curve, $X_0 \in T_{\gamma(t_0)}M$ where $t_0 \in I$. Then there is a unique parallel vector field $X(t) \in T_{\gamma(t)}$ along γ, with the property $X(t_0) = X_0$. The linear map*

$$T_{\gamma(t_0)}M \to T_{\gamma(t)}M, \quad X_0 \mapsto X(t)$$

is called parallel transport along γ, with respect to the connection ∇.

We have already seen that the proof is due to solving of the first order linear ODEs. Now we can just do it locally using Christoffel symbols as well (for $X = \sum_i b_i \partial_i, \gamma' = \sum x_i' \partial_i$:

$$\frac{db_i}{dt} + \sum_{jk} \Gamma^i_{jk} x_j' b_k = 0$$

Moreover, an affine connection ∇ on a pseudo-Riemannian manifold (M, g) is a metric connection if and only if parallel transport along curves preserves inner products.

0.28. **Second fundamental form.** Suppose $N \subset M$ is a smoothly embedded submanifold. We write ∇^T and $\nabla^\perp$ for the components of ∇ in TN and μN, the normal bundle of N in M (so that $TM|_N = TN \oplus \mu N$).

Definition 0.30. ***(Second fundamental form)*** *For vectors $X, Y \in T_pN$ define the second fundamental form $\boldsymbol{II}(X, Y) \in \mu_p N$ by locally extending X and Y to vector fields on N, and using the formula*

$$\boldsymbol{II}(X, Y) := \nabla^\perp_X Y.$$

Lemma 0.3. *The second fundamental form is well-defined, tensorial, and symmetric in its two terms, what means it is a section $\boldsymbol{II} \in \Gamma(S^2(T^*N) \otimes \mu N)$.*

Proof. Evidently $\mathbf{II}(X, Y)$ is tensorial in X, so it suffices to show that it is symmetric. By computation we have

$$\nabla^\perp_X Y = \nabla_X Y - \nabla^T_X Y = \nabla_Y X - \nabla^T_Y X = \nabla^\perp_Y X$$

where we use the fact that both ∇, ∇^T are torsion-free.

□

Remark. If it is the second fundamental form, then what is the first one? The "first fundamental form" on N is simply the restriction of the inner product on M to TN. Therefore, it is synonymous with the induced Riemannian metric on N.

0.29. **Geodesics.**

Definition 0.31. *Let ∇ be an affine connection on a manifold M. A smooth curve $\gamma : I \to M$ is called a geodesic for the connection γ, if and only if the velocity vector field gamma$'$ is parallel along γ.*

Remark: If $\varphi : \tilde{I} \to I$ is diffeomorphism then $\tilde{\gamma} = \gamma(\varphi(\tilde{t}))$ is geodesic if and only if φ is linear on $\tilde{t}$.

Check it!

Why one come up with this particular definition? Let us compute $\frac{d}{dt}g(\gamma', \gamma')$. It is equal to $\gamma'(g(\gamma', \gamma')) = 2g(\nabla_{\gamma'}\gamma', \gamma')$, that is zero if we on the geodesics. Hence, $|\gamma'| = \sqrt{g(\gamma', \gamma')}$ is constant along the curve. We call γ *normalized* if $|\gamma'| = 1$.

Theorem 0.16. *In local coordinates, geodesics are the solutions of the second order ordinary differential equation,*

$$\frac{d^2 x_i}{dt^2} + \sum_{jk} \Gamma^i_{jk} \dot{x}_j \dot{x}_k = 0 \qquad \textit{(geodesic equation)}$$

Proof. We can just use $b_i = \dot{x}_i$ and plug into the equation of parallel transport. However, let us do the calculation explicitly:

In local chart (U, φ) we have $\varphi \cdot \gamma = (x_1, ..., x_n)$, then $\gamma' = \sum_i x_i'(\varphi_*)^{-1}(\partial_i) = \sum_i x_i' X_i$, where X_i are coordinate vector fields.

Hence,

$$\nabla_{\gamma'}\gamma' = \sum_i \nabla_{\gamma'}(x_i' X_i) = \sum_i \gamma'(x_i')X_i + x_i' \nabla_{\gamma'} X_i =$$

$$= \sum_i x_i'' X_i + x_i' \sum_j x_j' \nabla_{X_j} X_k = \sum_k \left(\frac{d^2 x_k}{dt^2} + \sum_{ij} \Gamma^k_{ij} \dot{x}_i \dot{x}_j \right) X_k$$

$\square$

Remark: Only $\Gamma^i_{jk} + \Gamma^i_{kj}$ contributes to the geodesic equation. Therefore, if one is interested in the geodesic flow of a metric connection ∇, one might as well assume that ∇ is the Levi-Civita connection.

Examples:

1. $\mathbb{R}^n$. Geodesic equation gives $x_k'' = 0$. The solution is straight lines $x_k(t) = a_k t + b_k$. And normalized means $\sum_i a_i^2 = 1$.
2. On the standard n-torus $T^n \subset R^{2n}$ we saw that $\Gamma^k_{ij} = 0$ and geodesic equations are $\theta_i'' = 0$. We deduce that $\theta_i = a_i t + b_i$, so the geodesics are $\gamma(t) = (\cos(a_1 t + b_1), ..., \sin(a_n t + b_n))$, i.e. the images of the straight lines in T^n.
3. Another example just to confirm something that we have already been aware of for the long time is the 2-dimensional sphere S^2. Let us consider a normalized geodesic in spherical variables $\gamma(t) = (\sin\theta(t)\cos\varphi(t), \sin\theta(t)\sin\varphi(t), \cos\theta(t))$. Recall that $\Gamma^1_{11} = \Gamma^1_{12} = 0, \Gamma^1_{22} = -\sin\theta\cos\theta, \Gamma^2_{11} = \Gamma^2_{22} = 0, \Gamma^2_{12} = \cot\theta$.

   Therefore, for the geodesic equations we have $\theta'' - \sin\theta\cos\theta(\varphi')^2 = 0, \varphi'' + 2\cot\theta\theta'\varphi' = 0$.

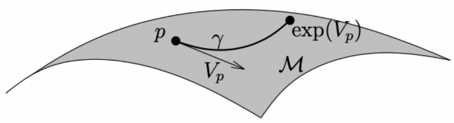
We can see that $\varphi' = 0$ and $\theta'' = 0$ gives a solution if $\theta' = 1$, which is $\gamma(t) = (\sin(t+\theta_0)\cos\varphi_0, \sin(t+\theta_0)\sin\varphi_0, \cos(t+\theta_0))$. *This is a great circle as expected.*

0.30. **Exponential map.** Consider local coordinates $x_i, \frac{dx_i}{dt} =: \xi_i$ on TM (recall it is subspace of $M \times \mathbb{R}^n$). Hence we can define the following vector field using the geodesic equation:

$$\mathcal{S} = \sum_i \xi_i \frac{\partial}{\partial x_i} - \sum_{ijk} \Gamma^i_{jk} \xi_j \xi_k \frac{\partial}{\partial \xi_i}$$

This vector field $\mathcal{S}$ is called the *geodesic spray* of γ, and the corresponding flow is called the *geodesic flow.* If $\mathcal{S}$ is complete then M is called *geodesically complete.*

Definition 0.32. ***(Exponential map)***. *The exponential map exp is a map from a certain open domain U in TM (containing the zero section) to M , defined by $exp(v) = \gamma_v(1)$, where $v \in T_pM$, and $\gamma : [0,1] \to M$ is the unique smooth geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$ (if it exists). The restriction of exp to its domain of definition in T_pM is denoted exp_p.*



This definition is based on the following

Theorem 0.17. *For any $p \in M, v \in T_pM$ there exists a unique maximal geodesic $\gamma_v : I \to M$, where $\gamma_v(0) = p, \gamma_v'(0) = v$.*

Idea of proof is based on the fact that geodesics on M are the projections of the geodesic spray $\mathcal{S}$ by the base point projection $\pi : TM \to M$, which is equivalent to say $\gamma_v(t) = \pi(\varphi_t(v))$ (φ_t is geodesic flow).

What is exponential map geometrically? *It moves points along geodesics starting from the point p.*

First consider our basic example with $(\mathbb{R}^n, g_0)$ we have $\gamma_{p,X}(t) = p + tX$, so $\exp_p(X) = p + X$. It is a translation.

Another example is the sphere S^2. Recently we found that geodesics are given by $\gamma(t) = (\sin(ct+\theta_0)\cos\varphi_0, \sin(ct+\theta_0)\sin\varphi_0, \cos(ct+\theta_0))$. They all start from p but $\gamma'(0) = cX$, hence $\exp_p(2\pi X_1) = \exp_p(X_1)$. It means that the exponential map is not injective in that case.

The origin of the name "exponential" comes from matrix groups (in particular $SO(n)$). In this case exponential map $\exp_I : T_I SO(n) \to SO(n)$ is $\exp_I(A) = exp(A)$. Since A is skew-symmetric, we have $\exp(A) \in SO(n)$. Curves γ_v in that case play the role of 1-parametric subgroups.

Remark: In the above example we have seen that the exponential map is not necessarily diffeomorphism. Even if it is defined on all of T_pM, is typically not a global diffeomorphism, or even a covering map. But locally it is!

Lemma 0.4. *For all p there is an open subset $U \subset T_pM$ containing the origin so that the restriction $exp_p : U \to M$ is a diffeomorphism onto its image.*

Proof. Since $\exp_p$ is smooth. Indeed, exponential map is (as we know) restriction of the map $\pi \cdot \varphi$ to the submanifold $\{1\} \times T_pM$ and hence smooth.

So it suffices to show that $d\exp_p : T_0T_pM \to T_pM$ is nonsingular. But if we identify T_0T_pM with T_pM using its linear structure, the definition of exp implies that $d \exp_p : T_pM \to T_pM$ is the identity map: $T_0\exp_p(v) = \frac{d}{dt}|_{t=0}\exp_p(tv) = \frac{d}{dt}|_{t=0}\gamma_v(t) = v$.

Hence, it is nonsingular. We need to use the Inverse function theorem to prove the existence of such neighborhood. □

Remember, one time I mentioned we have normal coordinates (local). Now we have it as just a consequence of the above!

If one chooses a basis in T_pM (ie identifying T_pM and $\mathbb{R}^m$), the exponential map gives a system of local coordinates $x_1, ..., x_m$ on a neighborhood of p. These coordinates are called *normal coordinates* at p:

Theorem 0.18. *In normal coordinates $x_1, ..., x_m$ based at $p \in M$, the geodesics through p are given by straight lines, $x_i(t) = ta_i, a_i \in \mathbb{R}$. Moreover, all Christoffel symbols Γ^i_{jk} vanish at 0.*

0.31. **Energy functional.** We have stated that geodesic are indeed minimizing in a classic sense. First one could ask what are we minimizing here. We will start with some "physics".

Define the *energy function* $E(v) = \frac{1}{2}g_p(v, v)$, in the other words $E(\gamma) = \frac{1}{2}\int_a^b |\dot\gamma|^2 dt$[27]. Let us work now with LC connection. Since parallel transport for a metric connection preserves inner products, the geodesic flow preserves the energy: That is, $\mathcal{S}(E) = 0$. It follows that $\mathcal{S}$ is tangent to the level surfaces of the energy functional.

If we are looking for curve minimizing length (a posteriori energy) we can write down the *length functional*:

$$L(\gamma) = \int_a^b |\dot\gamma| dt$$

And the minimizing the length functional is the the same as solving Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial x'_k} - \frac{\partial L}{\partial x_k}$$

Question: How the energy and length are connected?

Let us apply **the Cauchy-Schwartz inequality**:

$$L(\gamma)^2 \le \int_a^b 1^2 dt \cdot \int_a^b |\dot\gamma|^2 dt = 2(b-a)E(\gamma),$$

where the equality holds iff $|\dot\gamma|$ is constant. Therefore,

Proposition 0.17. *A curve γ minimize $E(\gamma)$ if and only if it minimize $L(\gamma)$ and $|\dot\gamma|$ is constant.*

Since any curve can be reparametrized to have constant $|\dot\gamma|$, to minimize $L(\gamma)$ is equivalent to minimize $E(\gamma)$, whose integrand is much simpler.

[27] Sometimes this functional is called *action*.

0.32. **Gauss' lemma.** Gauss' Lemma is the key to showing that geodesics are locally unique distance minimizers. That is,

Lemma 0.5. ***(Gauss lemma)*** *Let $p \in M$ and $X \in T_pM$ such that $exp_p(X)$ defined. If $Y \in T_X(T_pM)$ (which is once again identified with another copy of $\mathbb{R}^n$ at p – T_pM) then*

$$g_{exp_p(X)}\left(d(exp_p)_X(X), d(exp_p)_X(Y)\right) = g_p(X, Y).$$

The proof of lemma involves "geodesic polar coordinates". In these words, Gauss' Lemma says that if we exponentiate (orthogonal) polar coordinates r, θ_i on T_pM, then the image of the radial vector field $dexp_p(\partial_r)$ (which is tangent to the geodesics through p) is perpendicular to the level sets $exp(r = \text{constant})$.

Remark: For each $p \in M$, the injectivity radius of (M, g) at p is $inj_p(M, g) = sup\{r : \exp_p$ is a diffeomorphism on $B_r(0) \subset T_pM\}$, and the injectivity radius of (M, g) is $inj(M, g) = inf\{inj_p(M, g) | p \in M\}$.

For example, $inj(S^n, g_{S^n}) = \pi$.

Lemma 0.6. ***(Gauss' lemma reformulated)*** *For all $0 < r < i_p(M)$, the radial geodesics $exp_p(tv)$ intersect the spheres $exp_p(S_r(0))$ orthogonally.*

For any $v \in S_r(0)$, the point $q = exp_p(v)$ has distance $d(p, q) = r$ from p, and the geodesic $exp_p(tv)$ is the unique (up to reparametrization) curve of length $d(p, q)$ connecting p, q.

In particular, $exp_p(S_r(0)) = S_r(p)$ for any $0 < r < i_p(M)$.