Lecture 7

We will prove the existence of Levi-Civita connection and discuss parallel transport once again.

0.25. **Levi-Civita connection: proof.** I will start with a remark about metric connections.

Remark: One may ask how we can write ∇g if g is (2,0)-tensor and not a vector field. Recall that we can extend connection to all tensor bundles. In particular, connection ∇ induced on T^*M is defined as follows (which I will also denote ∇ given by

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(T^*M), (X, \alpha) \mapsto \nabla_X(\alpha)$$

where $(\nabla_X\alpha)(Y) = X(\alpha(Y)) - \alpha(\nabla_X Y)$. Hence, $\nabla_X(\alpha(Y)) = (\nabla_X\alpha)(Y) + \alpha(\nabla_X Y)$. This generalizes the same way to any tensor bundle. Now, g is a (2,0)-tensor. So if Y, Z are vector fields, $g(Y, Z)$ is a smooth function and hence $\nabla_X g(Y, Z) = X(g(Y, Z))$

On the other hand, by from the extension of connection to the tensor we have $\nabla_X(g(Y, Z)) = (\nabla_X g)(Y, Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.[25]

Using these two equations, we see that $(\nabla g)(X, Y, Z) = \nabla_X g)(Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$. So we see that ∇ is compatible with the metric g if and only if $((\nabla g)(X, Y, Z) = 0$ for all vector fields X, Y, Z.

Now we are ready to prove the existence and uniqueness of metric torsion-free connection.

Proof. We will have some formula which defines ∇ and it will show uniqueness, and then we will show that the object given by this formula satisfy all properties of connection.

1. By the metric property, we have

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

and similarly for the other two cyclic permutations of X, Y, Z.

Adding the first two permutations and subtracting the third, together with the torsion-free property to eliminate expressions of the form $\nabla_X Y - \nabla_Y X$ gives the identity

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X)$$

This identity known as *Koszul identity* shows uniqueness. Any other map ∇' satisfying (i)-(v) is also defined by the same formula so must equal ∇, if it is exist.

Conversely, defining ∇ by this formula one can check that it satisfies the properties of a connection. □

Examples:

1. On $\mathbb{R}^n$, $[\partial_i, \partial_j] = 0$ and $g_o(\partial_i, \partial_j) = \delta_{ij}$ are constant functions, so by the metric properties of Levi-Civita connection $g_0(\nabla\partial_i\partial_j, \partial_k) = 0$ and hence $\nabla_{\partial_i}\partial_j = 0$.

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[25] To be precise we shall write $\nabla_X(g(Y, Z)) = (\nabla_X^{(TM\otimes TM)^*} g)(Y, Z) + g(\nabla_X^{TM} Y, Z) + g(Y, \nabla_X^{TM} Z)$

2. On Euclidean E^n with global coordinates x_i the vector fields ∂_i are parallel in the Levi-Civita connection. If Y is a vector field, X_p is a vector at p and $\gamma : [0,1] \to E^n$ is a smooth path with $\gamma(0) = p$ and $\gamma'(0) = X_p$ then $(\partial_X Y)(p) = \frac{d}{dt}|_{t=0} Y(\gamma(t))$.
3. On $T^n \subset \mathbb{R}^{2n}$ we have that $X_i = -\sin\theta_i \partial_{2i-1} + \cos\theta_i \partial_{2i}$ satisfy $g(X_i, X_j) = \delta_{ij}$ constant and $[X_i, X_j] = 0$, so $\nabla_{X_i} X_j = 0$.
4. Let us define for a smooth submanifold S the "tangential part" of the ambient connection on $\mathbb{E}^n$. There is a natural orthogonal projection map $\pi : T_p\mathbb{E}^n \to T_pS$. We can then define a connection $\nabla^T := \pi \cdot \nabla$ on TS. For $X, Y, Z \in \mathcal{X}(S)$ we have
$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = g(\nabla^T_X Y, Z) + g(Y, \nabla^T_X Z)$$
.

   Moreover, ∇^T is a metric connection. Similarly, since $[X,Y]$ is in $\mathcal{X}(S)$ whenever X and Y are, the perpendicular components of $\nabla_X Y$ and of $\nabla_Y X$ are equal, so ∇^T is torsion-free. Therefore, ∇^T is the Levi-Civita connection on S.

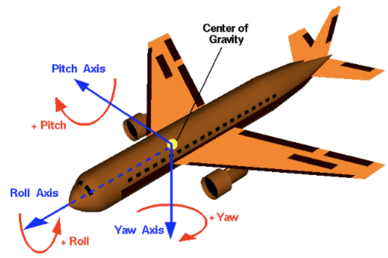
   By the Nash theorem Levi-Civita connection on any manifold can be thought as above.
5. On S^2 we let $f(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ and let $X_1 = f_*\partial_\theta$ and $X_2 = f_*\partial_\varphi$ be the coordinate vector fields on S^2. Then $[X_1, X_2] = 0$. We also have $g(X_1,X_1) = 1, g(X_1,X_2) = 0$ and $g(X_2,X_2) = \sin^2\theta$.

   By Koszul identity we have $g(\nabla_{X_1}X_1, X_1) = \frac{1}{2}X_1(g(X_1,X_1)) = 0$. And $g((\nabla_{X_1}X_1, X_2) = \frac{1}{2}\left(2X_1 g(X_1,X_2) - X_2 g(X_1,X_1)\right) = 0$. Therefore, $\nabla_{X_1}X_1 = 0$.

   Moreover,
$$g(\nabla_{X_2}X_2, X_1) = \frac{1}{2}\left(2X_2(g(X_1,X_1)) - X_1 g(X_2,X_2)\right) = -\frac{1}{2}\frac{\partial}{\partial\theta}\sin^2\theta = -\sin\theta\cos\theta$$
   and
$$g(\nabla_{X_2}X_2, X_2) = \frac{1}{2}(X_2 g(X_2,X_2)) = \frac{1}{2}\frac{\partial}{\partial\varphi}\sin^2\theta = 0$$
   The last is computation of $\nabla_{X_1}X_2$ and $\nabla_{X_2}X_1$:
$$g(\nabla_{X_1}X_2, X_1) = 0, \quad g(\nabla_{X_1}X_2, X_2) = \frac{1}{2}\left(X_1 g(X_2,X_2) + X_2 g(X_2,X_1) - X_2 g(X_1,X_2)\right) = \sin\theta\cos\theta$$
   Therefore, $\nabla_{X_1}X_2 = \nabla_{X_2}X_1 = \cot\theta X_2$

Remark: Let us talk about Levi-Civita connection in terms of dynamics (example 4 above). We may assume the Riemannian manifold is an embedded submanifold of Euclidean space: its metric at any point is just the restriction of the Euclidean inner product to the tangent plane. Imagine we live on this submanifold (just like we live on a sphere called Earth) and we want to calculate things, such as our acceleration as we run around our planet.

Remember, the metric gives us a means of measuring distances and angles, but no direct way of computing rates-of-change of vector fields. A connection is what determines the rates-of-change of vector fields (such as acceleration, which is the rate-of-change of velocity vectors). And connections are just "infinitesimal limits" of parallel transport. The case of a smooth submanifold is instructive. We can imagine defining parallel transport along a path γ in S by "rolling" the tangent plane to S along γ, infinitesimally projecting it to TS as we go. Since the projection is orthogonal, the plane does not "twist" in the direction of TS as it is rolled; this is the geometric meaning of the fact that this connection is torsion-free. In the language of flight dynamics, there is pitch where the submanifold S is not flat and roll where the curve γ is not "straight", but no yaw.



0.26. **Christoffel symbols.** How to compute connections in local coordinates? The answer is given by Christoffel symbols.

Definition 0.29. *With respect to local coordinates x_i, the Christoffel symbols of a connection ∇ on TM are the functions Γ^k_{ij} defined by the formula*

$$\nabla_i \partial_j = \sum_k \Gamma^k_{ij} \partial_k$$

Remark: The Christoffel symbols depend on the choice of coordinates![26]

1. On $\mathbb{R}^n$ we have $\nabla_{\partial_i} \partial_j = 0$ so $\Gamma^k_{ij} = 0$. Similarly on T^n.
2. For S^2 we see that $\nabla_{X_1} X_1 = 0$ so

$$\Gamma^1_{11} = \Gamma^2_{11} = 0.$$

Also we have $\nabla_{X_2} X_2 = -\sin\theta \cos\theta X_1$, so

$$\Gamma^1_{22} = -\sin\theta \cos\theta, \Gamma^2_{22} = 0.$$

Also, $\nabla_{X_1} X_2 = \nabla_{X_2} X_1 = \cot\theta X_2$ so

$$\Gamma^1_{12} = \Gamma^1_{21} = 0, \quad \Gamma^2_{12} = \Gamma^2_{21} = \cot\theta.$$

Now we are ready to compute the Levi-Civita connection ∇ locally when using the coordinate vector fields.

Let us take a look on Koszul formula from the the proof of theorem 0.13.

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X)$$

Taking $X = \partial_x, Y = \partial_y, Z = \partial_z$ to be coordinate vector fields in the formula above, we obtain a formula for the Christoffel symbols Γ^i_{jk} of the Levi-Civita connection:

$$2\sum_i \Gamma^i_{jk} g_{il} = \frac{\partial g_{kl}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_k} - \frac{\partial g_{jk}}{\partial x_l}$$

[26]This is why a lot of people try not to use them. Historically, they have been used much since it is possible to compute a lot of things locally.

Theorem 0.14. *In local coordinates, the Christoffel symbols for the Levi-Civita connection are given by*

$$\Gamma^i_{jk} = \frac{1}{2}\sum_i (g^{-1})_{il}\left(\frac{\partial g_{kl}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_k} - \frac{\partial g_{jk}}{\partial x_l}\right),$$

where $(g^{-1})_{il}$ *denote the inverse matrix to* g_{ij}.

Remark: Of course it is possible to start with this formula to define LC-connection, and then check that the local definitions patch together.

Remark: Since the Christoffel symbols are symmetric in j, k, it is immediate from this formula that ∇ is torsion-free. Indeed, $\nabla_{X_j} X_k - \nabla_{X_k} X_j = [X_j, X_k]$ which is equivalent to $\sum_i (\Gamma^i_{jk} - \Gamma^i_{kj}) X_i = 0$.