Lecture 6

We will start to discuss the main subject of the course – Riemannian manifolds. First the metric comes. Then we will define Levi-Civita connection, but first we will come up with the notion of connection.

0.19. **Linear algebra recap.**

- Let us quickly recall some linear algebra. A bilinear form on a vector space V is a bilinear map $g : V \times V \to \mathbb{R}$. Such a bilinear form is called **symmetric** if $g(v, w) = g(w, v)$ for all v, w, and in this case is completely determined by the associated quadratic form $q(v) = g(v, v)$. Form g is called an **inner product** if it is positive definite, i.e. $g(v, v) > 0$ for all $v \in V$. More generally, a symmetric form g is called **non-degenerate** if $g(v, w) = 0$ for all w implies $v = 0$. Non-degenerate symmetric bilinear forms are also called **indefinite inner products**.
- Let U, V be vector spaces. We can define a vector space $U \otimes V$ as the free vector space generated by the symbols $u \otimes v$ as $u \in U, v \in V$, modulo the subspace generated by $u \otimes (av + b\tilde{v}) - au \otimes v - bu \otimes \tilde{v}$ and $(au + b\tilde{u}) \otimes v - au \otimes v - b\tilde{u} \otimes v$.

0.20. **Riemannian metric.** Since an inner product is a bilinear map $T_pM \times T_pM \to \mathbb{R}$, which is also symmetric and positive definite, then it can be considered an element of $(T_pM \otimes T_pM)^*$, and thus, $T_p^*M \otimes T_p^*M$.

Definition 0.21. *Let M be a smooth manifold. A Riemannian metric is a symmetric positive definite inner product $g_p := \langle \cdot, \cdot \rangle_p$ on T_pM for each $p \in M$ so that for any two smooth vector fields X, Y the function $p \to \langle X, Y \rangle_p$ is smooth. A* ***Riemannian manifold*** *is a smooth manifold with a Riemannian metric. For $v \in T_pM$ the* ***length*** *of v is $\langle v, v \rangle^{1/2}$ and is denoted $|v|$.*

The smoothness on p one can write as that the quadratic form $q : TM \to \mathbb{R}, v \mapsto g_p(v, v)$ for $v \in T_pM$ is a smooth map $q \in C^\infty(TM)$.

Remark. With the definition above g is an element of $\Gamma(T^*M \otimes T^*M)$

In local coordinates x_i[20], a Riemannian metric can be written as a symmetric tensor $g := g_{ij} dx_i dx_j$. The notion of a Riemannian metric is supposed to capture the idea that a Riemannian manifold should look like Euclidean space "to first order".

Example.

1. In $\mathbb{R}^n$ a choice of basis determines a positive definite inner product by declaring that the basis elements are orthonormal. The linear structure on $\mathbb{R}^n$ lets us identify $T_p\mathbb{R}^n$ with $\mathbb{R}^n$, and we can use this to give a Riemannian metric on $\mathbb{R}^n$. Endowed with Riemannian metric $\mathbb{R}^n$ becomes Euclidean space $\mathbb{E}^n$. Moreover, in $\mathbb{R}^n$ it means that any inner product can be written as symmetric matrix[21] as in $\langle x, y \rangle = x^T Ay$.
2. **(Smooth submanifold).** Let S be a smooth submanifold of Euclidean space. For $p \in S$ the inner product on $T_p\mathbb{E}^n$ restricts to an inner product on T_pS. In particular, let us consider $S^2 \setminus \{S, N\}$ with the induced

[20] There is technically a theorem on the existence of such coordinates.

[21] The metric does not depend on the choice of basis, however, the matrix changes.

metric, and vector fields $X_1 = \cos\theta\cos\varphi\partial_1 + \cos\theta\sin\varphi\partial_2 - \sin\theta\partial_3, X_2 = -\sin\theta\sin\varphi\partial_1 + \sin\theta\cos\varphi\partial_2$. Then induced metric on S^2 has matrix $\begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$

In our standard idea of constructing objects first, and then maps between them on the second we now need to define maps.

Definition 0.22. *A smooth map $f : M \to N$ between Riemannian manifolds is an isometric immersion if for all $p \in M$, the map $df : T_pM \to T_{f(p)}N$ preserves inner products.*

Remark. More generally, a pseudo-Riemannian metric of signature (d_+, d_-) is defined by letting the g_p be indefinite inner products of signature (d_+, d_-). For example, the case of signature (3, 1) is relevant to general relativity, with 3 space dimensions and 1 time dimension[22].

Lemma 0.2. *Any pseudo-Riemannian metric defines a symmetric C^∞-bilinear map*

$$g : \mathcal{X} \times \mathcal{X} \to C^\infty(M), g(X, Y)_p = g_p(X_p, Y_p)$$

Conversely, every symmetric C^∞-bilinear map $g : \mathcal{X} \times \mathcal{X} \to C^\infty(M)$ satisfying the property that $g(X, Y)_p = 0$ for all Y implies $X_p = 0$, defines a psedo-Riemannian metric.

Just an idea of how proof can work here: Consider $g(X, Y) = \frac{1}{2}(g(X + Y, X + Y) - g(X, X) - g(Y, Y))$. Conversely, $g(X, Y)_p$ depends only on X_p, Y_p. Hence we can define $g_p(X_p, Y_p) := g(X, Y)_p$. Then check all properties, passing to local coordinates, in particular that g_p depends smoothly on p.

0.21. **Construction of Riemannian metrics.**

0.21.1. *Overkill method.* We can construct a Riemannian metric by applying the following:

Theorem 0.7. ***(Whitney)*** *Let M be second countable. Then there exists an embedding (homeomorphic to its image with subspace topology) $F : M \hookrightarrow \mathbb{R}^{2n+1}$.*

Actually, it turns out that all Riemannian metrics arise in this way:

Theorem 0.8. ***(Nash)*** *Let (M, g) be a Riemannian manifold. Then there exists an embedding $F : M \to \mathbb{R}^{(n+2)(n+3)/2}$ such that $g = F^*(e)$ where e denotes the euclidean metric on $R^{(n+2)(n+3)/2}$.*

Embeddings of the above form are known as *isometric embeddings.*

Although by the above Riemannian geometry is nothing other than the study of submanifolds of $\mathbb{R}^N$ with the induced metric from Euclidean space,

0.21.2. *Via partition of unity.* This is a classic way how to construct a metric (and only metric).

Theorem 0.9. ***(Partitions of unity)***. *Let M be a manifold.*

a) Any open cover$\{U_\alpha\}$ of M has a locally finite ***refinement*** *$\{V_\beta\}$. That is, $\{V_\beta\}$ is an open cover, each V_β is contained in some U_α and the cover is locally*

[22]There is no distinguished splitting into "space"/"time"

finite in the sense that each point in M has an open neighborhood meeting only finitely many V_β's.

b) For any locally finite cover U_α of M, there exists a ***partition of unity****, that is a collection of functions $\chi_\alpha \in C^\infty(M)$ with $supp(\chi_\alpha) \subset U_\alpha$, such that $0 \leq \chi_\alpha \leq 1$ and $\sum_\alpha \chi_\alpha = 1$.*

Remark. (a) says that M is *paracompact.*

Remark. Evaluated at any point p, this sum is to be interpreted as a finite sum, over the (by assumption!) finitely many indices α where $\chi_\alpha(p) \neq 0$.

Idea of a proof of theorem 0.9: (i) One constructs a "shrinking" of the open cover U_α to a new cover V_α, such that $\bar{V_\alpha} \subset U_\alpha$. The new cover is still locally finite.

(ii) One constructs functions $f_\alpha \in C^\infty(M)$ supported on U_α, such that $f_\alpha > 0$ on V_α,

(iii) One defines $f = \sum_\alpha f_\alpha > 0$, and sets $\chi_\alpha = f_\alpha/f$.

Theorem 0.10. *Every manifold M admits a Riemannian metric.*

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas of M. Passing to a refinement, we may assume that the atlas is locally finite.

Then choose a partition of unity (by theorem 0.9 χ_α for the cover $\{U_\alpha\}$. By definition of charts we have Riemannian metrics g_α on U_α induced from the standard Riemannian metrics on $\mathbb{R}^m$.

For all $p \in M$, the sum $g_p = \chi_\alpha(p)(g_\alpha)_p$ is well-defined. Since all $\chi_\alpha(p) \geq 0$, with at least one strictly positive, g_p is an inner product with clearly a smooth dependence on p. Thus g is a Riemannian metric on M. □

Remark. It is not true that every manifold admits a pseudo-Riemannian metric of given signature (d_+, d_-), where both $d_\pm \neq 0$.

0.22. **Length of curves.** Suppose (M, g) is a Riemannian manifold.

For any tangent vector $v \in T_pM$, recall that its length is $||v|| = g_p(v,v)^{1/2}$.

Definition 0.23. *Let $\gamma : [a, b] \to M$ be a smooth curve in M. One defines the length of γ to be the integral*

$$L(\gamma) = \int_a^b ||\gamma'(t)||dt$$

Remark. The length functional is invariant under reparametrizations of the curve γ.

Namely, we have

Proposition 0.15. *Let $\sigma : [a, b] \to \mathbb{R}$ be a smooth function, with the property that $\sigma(t_1) \geq \sigma(t_2)$ for $t_1 \geq t_2$. Let $\gamma : [a, b] \to M$ be a smooth curve of the form $\gamma = \tilde{\gamma} \cdot \sigma$. Then $L(\gamma) = L(\tilde{\gamma})$.*

The proof is by the direct computation.

Definition 0.24. ***(Distance function).*** *Let (M, g) be a connected Riemannian manifold. For $p, q \in M$, the distance $d(p, q)$ between any two points on M is infimum of $L(\gamma)$, as γ varies over all piecewise[23] smooth curves $\gamma : [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$.*

[23]There exists a subdivision of an interval $a = t_0, ..., b = t_N$ with each $\gamma|_{[t_i,t_{i+1}]}$.

Remark. For any manifold M the following are equivalent: (i) M is connected, (ii) any two points p, q can be joined by a continuous path/piecewise smooth path/smooth path. Therefore, distance is defined and finite for a connected manifold.

Theorem 0.11. *For any connected manifold M, the distance function d defines a metric on M. That is, $d(p,q) \geq 0$ with equality if and only if $p = q$, and for any three points p, q, r, one has the triangle inequality*

$$d(p,q) + d(q,r) \geq d(p,r).$$

Moreover, the following is true.

Theorem 0.12. *For any manifold M, the topology defined by the metric coincides with the manifold topology.*

0.23. **Connections and parallel transport.** First we are looking how to identify the fibers of vector bundle E over M for different points. In particular, how to transport a vector along the curve. There is no canonical way to identify the fibers of E over different points. A *connection* on E is a choice of such an identification, at least "infinitesimally".

Definition 0.25. *Let E be a smooth real vector bundle over M. A connection ∇ is a linear map satisfying the Leibniz rule*

$$\nabla : \Gamma(E) \to \Omega^1(M) \otimes \Gamma(E)$$
$$\nabla(fs) = df \otimes s + f\nabla(s)$$

for $f \in C^\infty(M)$ and $s \in \Gamma(E)$ (in the other words, element of $\Omega^1(M; End(E))$.

Such an operator ∇ is also called a *covariant derivative.* We denote $\nabla(s)(X) = \nabla_X(s)$ (we can plug vector field into) for $X \in \mathcal{X}(M)$.

Note that $\nabla_{fX}(s) = f\nabla_X s$ for a smooth function f.

Question: Does connection always exist on vector bundle?

Proposition 0.16. *Any smooth real vector bundle admits a connection. The space of connections on E is an affine space for $\Omega^1(M; End(E))$.*

Proof. First let us notice that there is a connection on the trivial bundle, given by $\nabla(s) = ds$, where we identify sections of E with n-tuples of smooth functions by using the trivialization. Using theorem 0.9 and second claim we can glue them together.

Now let us prove the second claim, let us ∇, ∇' be two connections, consider their difference:

$$(\nabla - \nabla')(fs) = f(\nabla - \nabla')(s)$$

Therefore, $(\nabla - \nabla')$ is a $C^\infty(M)$ homomorphism from $\Gamma(E)$ to $\Omega^1 \otimes \Gamma(M)$. $\square$

Remark: We can even extend the notion of connection to the operators on $\Gamma(\Lambda^p T^*M \otimes E) = \Omega^p(M; E) \to \Omega^{p+1}(M; E)$ via

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^p \alpha \wedge \nabla(s).$$

Facts: 1. A connection ∇ on E determines a connection on E^* implicitly by the Leibniz formula

$$X(\alpha(W)) = (\nabla^{E^*}_X \alpha)W + \alpha(\nabla^E_X W)$$

.

From now on we will use the same notation for all connections induced by ∇ *on the tensor powers.*

2. In the special case, if $A\ inE_1^* \otimes .... \otimes E_n^*, \quad Y_i \in E_i$ then

$$X(A(Y_1, ..., A_n)) = (\nabla_X A)(Y_1, ..., Y_n) + \sum_i A(Y_1, ..., \nabla_X Y_i, ..., Y_n)$$

0.23.1. *Parallel transport.*

Definition 0.26. *A section* s *is parallel along a path* $\gamma : [0,1] \to M$ *if* $\nabla_{\gamma'(t)}(s) = 0$ *throughout* $[0,1]$.

Remark: There is a unique parallel section over any pathwith a given initial value (another consequence of the fundamental theorem of ODEs). Indeed, consider pullback bundle $\gamma^* E$ over $[0,1]$. By the theory of ODEs we have that for any $W_0 \in E_{\gamma(0)}$ there is a unique extension of W_0 to a parallel section W of $\gamma^* E$ over interval. Thus, a connection on E gives us a canonical way to identify fibers of E along a smooth path in M. If we let e_i be a basis for E locally, then we can express any W as $W = \sum_i w_i e_i$. Then by the properties of a connection, $0 = \nabla_{\gamma'} W = \sum_i \gamma'(w_i) e_i + \sum_i w_i \nabla_{\gamma'} e_i$ which is a system of first order linear ODEs in the variables w_i.

The isomorphism of $E_{\gamma(0)}$ and $E_{\gamma(1)}$ is called *parallel transport along* γ.

If $X, Y \in \mathcal{X}(M)$ then we can form $R(X,Y) := [\tilde{X}, \tilde{Y}] - [\widetilde{X,Y}] \in \mathcal{X}(E)$. This is a vertical vector field (i.e. it is tangent to the fibers of E).

For a vector space V, there is a canonical identification $T_v V = V$. R respects the structure of fibers and hence $R(X,Y)$ determines a linear endomorphism of each fiber E_x; i.e. $R(X,Y) \in \Gamma(End(E))$.

Then calculations lead to the following expression for R which is called the curvature of connection:

$$R(X,Y)(s) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

for any $s \in \Gamma(E)$.

Moreover, $R \in \Omega^2(M; End(E))$. In a lecture (or two) we will be back to discussing curvature in details.

0.24. **Levi-Civita connection.** Now we are looking for the connection associated with the metric. A bit changing notations from the above consider connection on bundle E, $\nabla : \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E)$ satisfying *tensor property* and *Leibniz property*, i.e. *affine connection.* Moreover, we want our special connection to be *metric.*

Definition 0.27. *If* E *is a bundle with a (fibrewise) inner product (i.e. a section* $q \in \Gamma(S^2 E^*)$*) a connection on* E *is metric if* $\nabla q = 0$*; equivalently, if*

$$d(q(U,V)) = q(\nabla U, V) + q(U, \nabla V)$$

Remark: d above is differential of scalar function, usually people write $X(g(Y,Z))$ for it.

Definition 0.28. *Let* ∇ *be a connection on* TM. *The torsion of* ∇ *is the expression*

$$Tor(V,W) := \nabla_V W - \nabla_W V - [V,W]$$

A connection on TM is torsion-free if $Tor = 0$.

Properties of connections give us that Tor is tensor.

In a different language being metric and being torsion-free means the following:

The condition that ∇ is a metric connection is exactly that the metric $g \in S^2T^*M$[24] is parallel.

The condition that ∇ is torsion-free says that for the induced connection on T^*M, the composition

$$\Gamma(TM) \to^{\nabla} \Gamma(TM \otimes TM) \to^{\pi} \Gamma(\Lambda TM)$$

is equal to exterior d, where π is the quotient map from tensors to antisymmetric tensors.

Remark: A Riemannian manifold usually admits many different metric connections. However, one connection is the special one and it uses the symmetry of the two copies of $\mathcal{X}(M)$ in the definition of ∇. The existence of it altogether with all properties is provided by the following theorem.

Theorem 0.13. ***(Fundamental Theorem of Riemannian Geometry)***. *There exists a unique map $\nabla : \Gamma(TM)\times\Gamma(TM) \to \Gamma(TM)$ denoted by $\nabla : (X, Y) to \nabla_X(Y)$ such that, if $X, Y, Z \in \Gamma(TM)$ and f, g are smooth functions on M then:*

(i) $\nabla_{fX+gY}Z = f\nabla_X Z + f\nabla_Y Z$,

(ii) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,

(iii) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$,

(iv) $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,

(v) $\nabla_X Y - \nabla_Y X = [X, Y]$.

We call $\nabla_X Y$ the covariant derivative of Y with respect to X and call ∇ the Levi-Civita connection of g.

Remark. Properties (i)-(iii) say ∇ is a connection (on TM). Property (iv) says that the connection is compatible with the Riemannian metric g. Property (v) says that the connection is torsion-free (or symmetric).

[24]Note that ∇ induces connections on T^*M and S^2T^*M.