Lecture 5

We will continue the Lie derivative, differential forms, and Cartan formula.

0.16. Lie derivative (continuation).

Definition 0.16. *Any complete vector field $X \in \mathcal{X}(M)$ with flow φ_t gives rise to a family of maps $\varphi_t^* : \mathcal{X}(M) \to \mathcal{X}(M)$. One defines the **Lie derivative** L_X of a vector field $Y \in \mathcal{X}(M)$ by*

$$L_X(Y) = \frac{d}{dt}|_{t=0} \varphi_t^* Y \in \mathcal{X}(M)$$

.

Remark. The definition of Lie derivative also works for incomplete vector fields, since the definition only involves derivatives at $t = 0$.

Remark. The Lie derivative of vector field is also a vector field.

Question: What is the geometric meaning of the Lie derivative? Let $X, Y \in \Gamma(TM), p \in M$ and consider the flow φ_t^X of X near p so that we can look at how Y "changes" along the flow of X. First we can look at Y at time t along the integral curve γ_p, which is the tangent vector $Y\varphi_t^X(p) \in Y_{\varphi_t^X(p)}M$ (because $\varphi_t^X(p) = \gamma_p(t)$. Second, $\varphi_{-1}^x(\varphi_t^X(p)) = p$)so

$$d(\varphi_{-t}^X)_{\varphi_t^X(p)} : T_{\varphi_t^X(p)}M \to T_pM$$

Analogously we can map back to T_pM the tangent vector $Y(\varphi_t^X(p)$ via $(\varphi_{-t}^X)_*$. And now we can compare this translated vector with $Y(p)$.

This gives $L_x(Y)(p) = \lim_{t\to 0} \frac{(\varphi_{-t}^X)_*(Y(\varphi_t^X(p))-Y(p)}{t}$, which exactly the same Lie derivative as defined above.

Example. Let $Y = \sum_j b_j \partial_j$ be a vector field on $\mathbb{R}^n$. We know that flows are given by $\varphi_t^{\partial_i} = p + te_i$. Therefore, $(\varphi_{-t}^{\partial_i})_* = id$[17]

Then we have by definition

$$L_{\partial_i} Y(p) = \lim_{t\to 0} \frac{\varphi_{-t}^X(Y(\varphi_t^X(p)) - Y(p)}{t} = \lim_{t\to 0} \frac{\sum_j b_j(p+te_i)(\varphi_{-t}^{\partial_i})_*\partial_j - b_j(p)\partial_j}{t}$$
$$= \sum_j \lim_{t\to 0} \frac{b_j(p+te_i) - b_j(p)}{t}\partial_j = \sum_j \frac{\partial b_j}{\partial_j}(p)\partial_j$$

In particular, $L_{\partial_i}\partial_j = 0$.

And if $X = x_1\partial_2 - x_2\partial_1$, then $L_{\partial_1}X = \partial_2, L_{\partial_2}X = -\partial_1, L_{\partial_3}X = 0$.

Proposition 0.14. *The Lie derivative $L_XY = [X, Y]$. Moreover, $[L_x, L_Y] = L_{[X,Y]}$*

Proof. Let φ_t be the flow of X. For any $f \in C^\infty(M)$ we calculate,

$$(L_XY)(f) = \frac{d}{dt}|_{t=0} (\varphi_t^*Y)(f) = \frac{d}{dt}|_{t=0}\varphi_t^*\left(Y\left(\varphi_{-t}^*(f)\right)\right) = \frac{d}{dt}|_{t=0}\varphi_t^*(Y(f)) - Y(\varphi_t^*(f)) = X(Y(f)) - Y(X(f)) =$$

The identity $[L_X, L_Y] = L_{[X,Y]}$ just rephrases the Jacobi identity for the Lie bracket. □

[17] Here we immediately identify tangent spaces $T_{p+te_i}\mathbb{R}^n$ and $T_p\mathbb{R}^n$.

0.17. **Differential forms.** Recall that on $\mathbb{R}^n$ we have 1-forms $dx_1, ..., dx_n$ so every 1-form can be written as a combination of these 1-forms $\sum_{i=1}^n a_i dx_i$ for functions a_i. These 1-forms are linear maps defined by their action on tangent vectors:

$$dx_i(\partial_j) = \delta_{ij}$$

Remark. The cotangent bundle T^*M is defined to be the dual bundle to TM; i.e. the bundle whose fiber at each point is the dual to the corresponding fiber of TM. Sections of T^*M are 1-forms (covectors).

Remark. Recall the following construction: let V be a (real) vector space, and let V^* denote its dual. Then $(V^*)^{\otimes n} = (V^{\otimes n})^*$. For $u_i \in V^*$ and $v_i \in\in V$ the pairing is given by

$$u_1 \otimes ... \otimes u_n(v_1 \otimes ... \otimes v_n) = u_1(v_1) \cdots u_n(v_n)$$

We denote by $T(V)$ the graded algebra $T(V) := \bigoplus_{n=0}^{\infty}(V^{\otimes n})$ and call it **the tensor algebra** of V. Let $I(V) \subset T(V)$ be the 2-sided ideal generated by elements of the form $v \otimes v$ for $v \in V$. This is a graded ideal and the quotient inherits a grading.

The quotient $T(V)/I(V)$ is denoted $Lambda(V)$, it is a graded algebra which is called exterior algebra of V. The part of $\Lambda(V)$ in dimension j is denoted $\Lambda_j(V)$. If V is finite dimensional, and has a basis $v_1, ..., v_n$ then a basis for $\Lambda^j(V)$ is given by the image of j-fold "ordered" products $v_{i_1} \otimes ... \otimes v_{i_j}$ with $i_k < i_l$ for $k < l$. In particular, the dimension of $Lambda^j(V)$ is $n!/j!(n-j)!$, and the total dimension of $\Lambda(V)$ is 2^n.

We define Ω^m to be the space of smooth sections of the bundle $\Lambda^m T^*M$, whose fiber at each point p is equal to $\Lambda^m T_p^*M$. An element of Ω^m is called a (smooth) m-form.

We also define $\Omega^0 = C^\infty(M)$, the space of smooth functions on M[18]. An m-form can be expressed in local coordinates as a sum

$$\omega = \sum_J a_J dx_J,$$

where J denotes a multi-index of length m, so that dx_J stands for an expression of the form $dx_J := dx_{j_1} \wedge ... \wedge dx_{j_m}$ for some $j_1 < j_2 < ... < j_m$.

In the language of vector bundles the above means that k-forms are secrions of vector bundle $\Lambda^k T^*M = \cup_{p \in M} \Lambda^k T_p^*M$.

Example: Consider the form $\omega = \frac{x_1 dx_2 - x_2 dx_1}{x_1^2 + x_2^2}$ on $\mathbb{R}^1$ $\{0\}$, then we can evaluate this form ω on the vector field $x_1\partial_2 - x_2\partial_1$. It gives

$$\omega(x_1\partial_2 - x_2\partial_1) = \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2} = 1$$

Definition 0.17. . *There is a linear operator $d : \Omega^m \to \Omega^{m+1}$ defined in local coordinates by*

$$d(a_J dx_J) = \sum_i (\partial_i a_J) dx_i \wedge dx_J$$

Exterior derivative satisfies a Leibniz rule

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{deg(\alpha)} \alpha \wedge d\beta$$

[18]Via "identity" $\Lambda^0 V = \mathbb{R}$ for a real vector space V.

Remark. We have seen that for function we have $df = \sum \frac{\partial_i f}{\partial_i} dx_i$.

Remark. Exterior derivative satisfies $d(d\omega) = 0$.[19].

Remark. If TM is trivial we have n linearly independent vector fields $X_1, ..., X_n$, so we can define $\omega_i(X_j) = \delta_{ij}$. Therefore, by the Proposition 0.7 T^*M is also trivial.

Question: We see that we can relate vector fields and 1-forms when TM is trivial, but what about in general? Well, if there is a *Riemannian metric*, then we can. This is the natural duality between a inner product space and its dual.

Definition 0.18. *Let $f : M \to N$ be a smooth map. If $\omega \in \Gamma(\Lambda^k T^*N)$ we can define the pullback $f^*\omega \in \Gamma(\Lambda^k T^*M)$ by*

$$(f^*\omega)(p)(X_1, ..., X_k) = \omega(f(p))(df_p(X_1), ..., df_p(X_k))$$

for all $p \in M$ and $X \in T^pM$.

Remark. Notice that $(f \cdot g)^* = f^* \cdot f^*$.

Example: For the 1-form above field $X = x_1\partial_2 - x_2\partial_1$, and a map $f : \mathbb{R} \to \mathbb{R}^2$ given by polar coordinates. Then $f_*(\partial_\theta) = -\sin\theta\partial_1 + \cos\theta\partial_2$ is the field on S^1. Hence, $f^*\omega(\partial_\theta) = \omega(f_*\partial\theta) = \omega(X) = 1$.

Since all computations local we can define exterior derivative locally for pullback to $\mathbb{R}^n$ and then pullback it back..

0.18. **Cartan's formula.** Now we can define Lie derivative for forms also!

Definition 0.19. *Given $X \in \Gamma(TM)$ and $\omega \in \Gamma(\Lambda^k T^*M)$, the Lie derivative of ω with respect to X:*

$$L_X(\omega)(p) = \lim_{t\to 0} \frac{(\varphi_t^X)^*(\omega(\varphi_t^X(p)) - \omega(p)}{t}$$

It is also a form of the same degree. In particular, for function the Lie derivative is $X(f)$, a directional derivative of function.

Definition 0.20. *We define the interior product of X with k-form ω , to be the $(k-1)$-form defined by*

$$i_x\omega(Y_1, ..Y_{k-1}) = \omega(X, Y_1, ..., Y_{k-1}$$

Interior product satisfies $i_X i_Y \omega = -i_X i_Y \omega$ and the Leibniz rule

$$iX(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^p \alpha \wedge (i_X\beta)$$

whenever $\alpha \in \Omega^p$.

Theorem 0.6. *Let X be a vector field and ω a k-form on M. Then Cartan's formula holds*

$$L_X\omega = d(i_X\omega) + i_X(d\omega).$$

The proof itself isn't worth much interest, since it is purely calculation.

We can easily apply it for functions, indeed, since $i_X f = 0$ for a function f we have $L_X f = i_X df = df(X) = X(f)$.

[19]It follows that d makes Ω^* into a chain complex of real vector spaces, whose homology is **the de Rham cohomology** of M, and is denoted $H^*_{dR}(M)$. Explicitly, an m-form ω is said to be closed if $d\omega = 0$, and to be exact if there is some $m-1$-form α with $d\alpha = \omega$. Then $H^m(M)$ is defined to be the quotient of the vector space of closed m-forms by the vector subspace of exact m-forms.