Lecture 4

In this lecture we will continue discussion of vector bundles, and then discuss the Lie bracket, differential forms and Cartan's formula.

0.11. **Vector bundles.** Here we will introduce the notion of a vector bundle in a general setting. It will generalize the one which we just had above for the tangent bundle.

Definition 0.9. *A vector bundle of rank k over a manifold M is a manifold E, together with a smooth map $\pi : E \to M$, and a structure of a vector space on each fiber $E_p := \pi^{-1}(p)$, satisfying **the following local triviality condition**: Each point in M admits an open neighborhood U, and a smooth map $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, such that ψ restricts to linear isomorphisms $E_p \to \mathbb{R}^k$ for all $p \in U$.*

The map $\psi : E_U \equiv \pi^{-1}(U) \to U \times \mathbb{R}^k$ is **called a (local) trivialization** of E over U. In general, there need not be a trivialization over $U = M$.

Definition 0.10. *A vector bundle chart for a vector bundle $\pi : E \to M$ is a chart (U, φ) for M, together with a chart $(\pi^{-1}(U), \hat{\varphi})$ for $E_U = \pi^{-1}(U)$,such that $\hat{\varphi} : \pi^{-1}(U) \to \mathbb{R}^m \times \mathbb{R}^k$ restricts to linear isomorphisms from each fiber E_p onto $\varphi(p) \times \mathbb{R}^k$.*

Remark. Every vector bundle chart defines a local trivialization. Conversely, if $\psi : E|_U \to U \times \mathbb{R}^k$ is a trivialization of E_U, where U is the domain of a chart (U, φ), one obtains a vector bundle chart $(\pi^{-1}(U), \bar{\varphi})$ for E.

Example (Vector bundles over the Grassmannnian). Recall that Grassmanian $Gr(k, n)$ is the set of k-dimensional linear subspaces in $\mathbb{R}^n$[11] Then $Gr(1, n)$ is projective space $\mathbb{R}P^{n-1}$. For any $p \in Gr(k, n)$, let $E_p \subset \mathbb{R}^n$ be the k-plane it represents. Then $E = \cup_{p \in Gr(k,n)} E_p$ is a vector bundle over $Gr(k, n)$, called the *tautological vector bundle*.

Recall the definition of charts $\varphi_I : U_I \to L(R^I, R^{I'})$ for the Grassmannian, where any $p = \{E\} = U_I$ is identified with the linear map A having E as its graph. Let $\hat{\varphi}_I : \pi^{-1}(U_I) \to L(\mathbb{R}^I, \mathbb{R}^{I'}) \times \mathbb{R}^I$ be the map $\hat{\varphi}_I(v) = (\varphi(\pi(v)), \pi_I(v))$ where $\pi_I : \mathbb{R}^n \to \mathbb{R}^I$ is orthogonal projection. The $\hat{\varphi}$ serve as bundle charts for the tautological vector bundle.

There is another natural vector bundle E' over $Gr(k, n)$, with fiber $E'_p := E_p^\perp$ the orthogonal complement of E_p.

In the case of $\mathbb{R}P^{n-1}$ E is called the *tautological line bundle*, and E' the *hyperplane bundle*.

At this stage, we are mainly interested in tangent bundles of manifolds.

Theorem 0.3. *For any manifold M, the disjoint union $TM = \cup_{p \in M} T_pM$ carries the structure of a vector bundle over M, where π takes $v \in T_pM$ to the base point p.*

[11] Grassmanian is a manifold. A C^∞-atlas may be constructed as follows. For any subset $I \subset \{1, ..., n\}$ let $I' = \{1, ..., n\} \setminus I$ be its complement. Let $R^I \subset \mathbb{R}^n$ be the subspace consisting of all $x \in \mathbb{R}^n$ with $x_i = 0$ for $i \notin I$. If I has cardinality k, then $R^I \in Gr(k, n)$. Note that $\mathbb{R}^{I'} = (\mathbb{R}^I)^\perp$. Let $U_I = \{E \in Gr(k, n) | E \cap \mathbb{R}^{I'} = 0\}$. Each $E \in U_I$ is described as the graph of a unique linear map $A : \mathbb{R}^I \to \mathbb{R}^{I'}$, that $E = \{x + A(x) | x \in \mathbb{R}^I\}$. This gives a bijection $\varphi_I : U_I \to L(R^I, R^{I'}) \simeq \mathbb{R}^{k(n-k)}$.

Proof. Consider an arbitrary chart $U, \varphi)$ for a manifold M, then we have $T_p\varphi : T_pM \to \mathbb{R}^m$ for all $p \in U$. Consider all these maps together, this gives a bijection,

$$T\varphi : \pi^{-1}(U) \to Y \times \mathbb{R}^m$$

Now we take the collection of $(\pi^{-1}, T\varphi)$ as vector bundle charts for TM. The thing that we need to show now is that the transition maps are smooth. To do so, consider another coordinate chart (V, ψ) with $U \cap V \neq \emptyset$, then the transition map for $\pi^{-1}(U \cap V)$ is given by

$$T\psi \cdot (T\varphi)^{-1} : (U \cap V) \times \mathbb{R}^m \to (U \cap V) \times \mathbb{R}^m$$

For any given point $p \in U \cap V$ the map $T_\psi \cdot (T_p\varphi)^{-1} = T_{\varphi(p)}(\psi \cdot \varphi^{-1})$ is just the Jacobian for the change of coordinates $\psi \cdot \varphi^{-1}$ and it depends smoothly on $\varphi(p)$. □

Example. For any vector bundle $E \to M$, $E^* = \cap_p E_p^*$ is again a vector bundle. It is called the dual bundle to E. In particular, one defines $T^*M := (TM)^*$, called *the cotangent bundle.* The sections of T^*M are called covector fields or "1-forms"[12].

Remark. Let $F \in C^\infty(M, N)$ be a smooth map. Then the collection of tangent maps $T_pF : T_pM \to T_{F(p)}N$ defines a map $TF : TM \to TN$ which is easily seen to be smooth. The map TF is an example of a vector bundle map: *It takes fibers to fibers, and the restriction to each fiber is a linear map.* For instance, local trivializations $\varphi : E|_U \to U \times \mathbb{R}^k$ are vector bundle maps.

0.12. **Vector fields.** Remember in multivariable calculus we often needed to compute the flow or flux of vector field along the curve/through something. Back in calculus vector field literally was just a vector attached to any point of some region. Now we are going to formalize it via the concept of vector bundle we just introduced.

Definition 0.11. *A (smooth) section of a vector bundle $\pi : E \to M$ is a smooth map $\sigma : M \to E$ with the property $\pi \cdot \sigma = id_M$. The space of sections of E is denoted $\Gamma^\infty(M, E)$.*

Therefore, a section is a family of vectors $\sigma_p \in E_p$ depending smoothly on p.

Definition 0.12. *A section of the tangent bundle TM is called a vector field on M. The space of vector fields is denoted $\mathcal{X}(M) = \Gamma^\infty(M, TM)$.*

Examples.

1. *Zero section.* Every vector bundle has it, namely $p \mapsto \sigma_p = 0$
2. *Trivial bundle.* Consider the bundle on M which is $M \times \mathbb{R}^k$[13]. A section of such bundle is equivalent to a smooth function from M to $\mathbb{R}^k$, locally in a trivialization chart $(U \times \mathbb{R}^k, \psi)$ it is a smooth function $\psi\sigma|_U : U \to \mathbb{R}^k$.
3. *Standard vector field on $\mathbb{R}^n$.* We have seen that $\partial_i = \frac{\partial}{\partial x_i}$ (we will use this notation from now on) are differential operators from $\mathbb{R}^n$ to $\mathbb{R}$.
4. *Restriction of a vector field from $\mathbb{R}^n$* Vector field on $M \subseteq \mathbb{R}^n$ is a restriction of a vector field on $\mathbb{R}^n$.

[12]We will differential forms later.

[13]Recall that not all bundles are like that.

5. Let $f : \mathbb{R}^n \to T^n$ be a map given by $f(\theta_1, ..., \theta_n) = (\cos\theta_1, \sin\theta_1, ..., \cos\theta_n, \sin\theta_n)$. The differential of this map is $df_{(\theta_1,\ldots,\theta_n)}\partial_{\theta_i} = -\sin\theta_i\partial_{2i-1} + \cos\theta_i\partial_{2i}$, so $X_i = -\sin\theta_i\partial_{2i-1} + \cos\theta_i\partial_{2i}$[14] are vector fields for T^n for all i.

Remark. The space $\Gamma^\infty(M, E)$ is *(a)* a vector space itself, *(b)* C^∞-module under multiplication.

Proposition 0.7. *A vector bundle of rank m is trivial if and only if it has m linearly independent sections.*

Remark. The latter example above leads to the following: let $f : M \to N$ be a diffeomorphism. Then we define the pushforward $f_* : TM \to TN$ by $f_*(X)(f(p)) = df_p(X(p))$ for $\forall p \in M$. This clearly defines a vector field because f is a diffeomorphism. If f is not injective the potential pushforward vector field is not even well-defined, and if f is not surjective then the vector field is not defined on all of N.

This leads to the following:

Proposition 0.8. *Every vector field on $U \subseteq M$ can be given as*

$$(\varphi^{-1})_* \left(\sum_{i=1}^{n} a_i \partial_i \right)$$

via diffeomorphism $\varphi^{-1} : \varphi(U) \to U$.

Indeed, the local correspondence between vector fields on M and vector fields on $\mathbb{R}^n$ in a chart (U, φ) is exactly $X \mapsto \varphi_*(X)$ where we consider X restricted to U:

$$\varphi_*(X) = \sum_{i=1}^{n} a_i \partial_i$$

for some smooth functions $a_i : \varphi(U) \to \mathbb{R}$. The same way $\varphi^{-1} : \varphi(U) \to U$ is diffeomorphism so we can do pushforward for it too getting the statement above. We will use it a lot.

0.13. **Pollarizable manifolds.** Recall that (1) not all vector fields are linear combination of ∂_i. As we have seen for the sphere S^2, (2) the tangent bundle is trivial if it is isomorphic to $M \times \mathbb{R}^n$.

If TM is trivial we say that M is *parallelizable.*

As we have seen $\mathbb{R}^n$ and S^1 are parallelizable. And S^2 is not. Moreover,S^3 is parallelizable, but S^{2n} is not. This is not about odd or even as S^5 is not, for example.

By Proposition 0.7 we have a condition of being parallelizable.

For a 1-dimensional manifold, being parallelizable is the same as having a nowhere vanishing vector field. The field we constructed for S^1 is nowhere vanishing.

However, the **Hairy Ball Theorem** implies that every vector field on S^{2n} has at least one point where it vanishes,

[14]Here we transferred(="push forwarded") vector field from $\mathbb{R}^n$ to T^n.

0.14. **Lie brackets.** Let $X \in \mathcal{X}(M)$ be a vector field on M. Each $X_p \in T_pM$ defines a linear map $X_p : C^\infty(M) \to \mathbb{R}$. Letting p vary, this gives a linear map

$$X : C^\infty(M) \to C^\infty(M), (X(f))_p = X_p(f)$$

We need to show that the right hand side really does define a smooth function on M. Indeed, this follows from the expression in local coordinates (U, φ). In a local coordinates vector field is represented by the vector field which is the sum of ∂_i.

Moreover,

Theorem 0.4. *A linear map $X : C^\infty(M) \to C^\infty(M)$ is a vector field if and only if it is a derivation of the algebra $C^\infty(M)$:*

$$X(f_1 f_2) = f_2 X(f_1) + f_1 X(f_2), \quad \forall f_1, f_2 \in C^\infty$$

We omit the proof, the idea is to show that map $p \mapsto X_p$ defines a smooth section of TM. It is done via local computation, basically we show that coefficients in the decomposition with the respect with the standard basis ∂_i.

0.14.1. *Definition.* So a vector field allows us to differentiate functions. Now we would like to compose vector fields, just like you compose derivatives, but there is a problem – the composition $X \cdot Y$ is **not** a vector field. However, Lie bracket is.

Definition 0.13. *Given $X, Y \in \Gamma(TM)$ we define the Lie bracket of X, Y to be $[X, Y]]XY - YX$, i.e. for a smooth function f:*

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

Then $[X, Y] \in \Gamma(TM)$.

Remark. Notice that $[Y, X] = -[X, Y]$ so $[X, X] = 0$.

To definition make sense we need to check correctness. Indeed, it is easily checked that the right hand side defines a derivation, and by 0.4 we know it is the vector field.

Alternatively, the calculation can be carried out in local coordinates (U, φ): if X_U is represented by $\sum_i a_i \partial_i$ and Y_U by $\sum_i b_i \partial_i$, then $[X, Y]_U$ is represented by

$$\sum_i a_i \partial_i \left(\sum_j b_j \partial_j \right) - \sum_i b_i \partial_i \left(\sum_j a_j \partial_j \right) = \sum_j \left(\sum_i a_i \frac{\partial b_j}{\partial x_j} - b_i \frac{\partial a_j}{\partial x_j} \right) \partial_j$$

Example.

1. For standard vector fields ∂_i, ∂_j on $\mathbb{R}^n$ we have $[\partial_i, \partial_j] = 0$.
2. Let $E_1 = x_3\partial_2 - x_2\partial_3, E_2 = x_1\partial_3 - x_3\partial_1, E_3 = x_2\partial_1 - x_1\partial_2$ be three vector fields on $\mathbb{R}^3$[15]. Then $[E_1, E_2] = E_3, [E_2, E_3] = E_1, [E_3, E_1] = E_2$.

Proposition 0.9. *Let $f : M \to N$ be a diffeomorphism. Then $f_*[X, Y] = [f_*X, f_*Y]$.*

[15] We may recall that this fields are connected with circls in the coordinate planes.

Proof. If we choose local charts (U,φ) and (V,ψ) for M and N such that $f: U \to V$ is a diffeomorphism and $\psi \cdot f = \varphi$ (we can do this because f is a diffeomorphism so we can define the charts on N using the charts on M in this way), then $\psi_* \cdot f = \varphi_*$ so it follows immediately from the fact that $\varphi_*[X,Y] = [\varphi_* X, \varphi_* Y]$ (just local computation). ☐

Coordinate vector fields. Recall that ∂_i are the standard vector fields on $\mathbb{R}^n$, then we call fields $X_i = (\varphi^{-1})_*(\partial_i)$ on the chart (U,φ) coordinate vector fields.

Remark. The Lie bracket transforms canonically under the diffeomorphism φ. Now if we have a straightening φ_U of the vector field U (such that $\varphi_U^*(U)$ is constant in our coordinate system), then $[U,V]$ is just the derivative of V along (the constant direction) U in that coordinate system.

The main property of the Lie bracket is given by the following

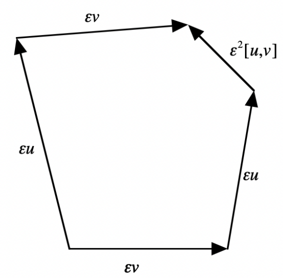
Proposition 0.10. *The Lie bracket satisfies the Jacobi identity: i.e. for $X, Y, Z \in \Gamma(TM)$*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Proof is by local computation.

0.14.2. *Geometric meaning of a Lie bracket.* Suppose that we are given two vector fields u and v on a manifold M. Flowing with v and then u is usually different from flowing with u and then v.

Example. $u = \partial_1$ and $v = -y\partial_1 + x\partial_2$



This difference is described by the Lie bracket (also known commutator) of u and v. This is another vector field, denoted $[u,v]$.

We have seen the pushfofward before, there is also a pullback which we will discuss later though.

If $f \in C^\infty(N)$ and $F \in C^\infty(M,N)$ we define the *pull-back* $F^*(f) = f \cdot F \in C^\infty(M)$. Thus pull-back is a linear map, $F^* : C^\infty(N) \to C^\infty(M)$.

Remark. Using pull-backs, the definition of a *tangent map* reads

$$T_pF(v) = v \cdot F^* : C^\infty(N) \to \mathbb{R}.$$

For any vector field $X \in \mathcal{X}(M)$ and any diffeomorphism $F \in C^\infty(M,N)$ we have $F^*X(F^*f) = F^*(X(f))$. That is, $F^*X = F^* \cdot X \cdot (F^*)^{-1}$.

In particular,

Proposition 0.11. *If X, Y are vector fields on N, $F^*[X,Y] = [F^*X, F^*Y]$.*

0.15. **Flow and Lie derivative.** Given a curve $\gamma : (-\epsilon, \epsilon) \to M$, we can define $\gamma'(t) \in T_{\gamma(t)}M$ by $\gamma'(t) = \gamma_t'(0)$ where $\gamma_t(s) = \gamma(s+t)$.

Hence, we have a map $t \mapsto \gamma'(t)$ from $(-\epsilon, \epsilon)$ into TM is smooth, so defines a vector field γ' along curve γ.

Proposition 0.12. *Let $X \in \Gamma(TM)$ and $p \in M$. There exists a unique curve γ_p through p such that $\gamma_p'(t) = X(\gamma_p(t))$ for $\forall t \in (-\epsilon, \epsilon)$.*

Indeed, we can consider local chart (U, φ), and then write $\varphi \cdot \gamma(t) = (x_1(t), ..., x_n(t))$ and $\varphi_*(X) = \sum_i a_i \partial_i$. Therefore, $\varphi_*(\gamma_p'(t)) = (\varphi \cdot \gamma_p)'(t) = \sum_i x_i'(t)\partial_i$.

It gives us a system of ODEs $x_i'(t) = a_i(x_1(t), ...x_n(t))$ with initial condition $\varphi(p) = (x_1(0), ..., x_n(0))$.

Remark. The vector field $X_{\gamma(t)}(f)$ is $\sum_i (x(t)) \frac{\partial}{\partial x_i}(f \cdot \varphi^{-1})|_{x(t)}$.

Definition 0.14. *Given $X \in \Gamma(TM)$ and $p \in M$, then there is a open set V such that for all points $q \in V$ there are curves $\gamma_q(t) \in V$ centered at q with $\gamma_q'(t) = X(\gamma_q(t))$.*

*These curves are called the **integral curves** (or **solution**[16]) of vector field X.*

Remark. Uniqueness and existence of solution is the a very important and fundamental result from the theory of ODEs. Also the solution depends smoothly on initial conditions.

Examples. 1) If $V = (0, 1) \subset \mathbb{R}$ and $a(x) = 1$, the solution curves to $x' = a(x(t)) = 1$ with initial condition $x_0 \in V$ are $x(t) = x_0 + t$, defined for $-x_0 < t < 1 - x_0$.

2) $a(x) = x^2$ has solution curves, $x(t) = \frac{-1}{t-c}$, these escape to infinity for $t \to c$.

3) Let $X = x_1\partial_2 - x_2\partial_1$ and let $(a_1, a_2, a_3) \in \mathbb{R}^3$. The integral curve $\gamma(t) = (x_1(t), x_2(t), x_3(t))$ of X through x satisfies

$$x_1'\partial_1 + x_2'\partial_2 + x_3'\partial_3 = x_1\partial_2 - x_2\partial_1$$

Therefore, we have three equations which we can solve since $x_1''(t) = -x_1(t)$ forces $x_1 = A\cos t + B\sin t, x_2 = -B\cos t + A\sin t$ then it means we have

$$x_1 = a_1\cos t - a_2 \sin t, x_2 = a_2\cos t + a_1 \sin t, x_3 = a_3$$

Therefore the integral curves of X are circles in a plane $x_3 = a_3$.

Remark. Note that the uniqueness part uses the Hausdorff property in the definition of manifolds. Indeed, the uniqueness part may fail for non-Hausdorff manifolds. A counter-example is the non-Hausdorff manifold $Y = \mathbb{R} \times \{1\} \cup \mathbb{R} \times \{1\}/\sim$, where $\sim$ glues two copies of the real line along the strictly negative real axis. Let $U_\pm$ denote the charts obtained as images of $\mathbb{R} \times \{\pm\}$. Let X be the vector field on Y, given by $\frac{\partial}{\partial x}$ in both charts. It is well-defined, since the transition map is just the identity map. Then $\gamma_+(t) = \pi(t, 1)$ and $\gamma_-(t) = \pi(t, -1)$ are both solution curves, and they agree for negative t but not for positive t.

Definition 0.15. *Let $X \in \Gamma(TM)$ and $p \in M$. Let $V \ni p$ be an open set such that we have integral curves $\gamma_q : (-\epsilon, \epsilon) \to M$ of X through q for all $q \in V$. We define*

[16] Because to find them we solved ODE...

the flow of X on V as the family of smooth maps

$$\{\varphi_t^X : V \to M, t \in (-\epsilon, \epsilon)\}$$

given by $\varphi_t^X(q) = \gamma_q(t)$. Notice that φ_0^X is the identity on V.

Geometrically speaking, the flow says how points on M move by the vector field X. As much as we did it in $\mathbb{R}^n$ we can now think of X as a family of "arrows" on M which point in the direction of the flow (the integral curves). Despite being defined locally, quite often the flow is globally defined.

Examples.

1. For ∂_i on $\mathbb{R}^n$ we saw that $\gamma_q(t) = q+te_i$ so $\varphi_t^{\partial_i}$ so the flow is just translation in the direction of e_i.
2. The flow of the vector field X_j on T^n is $\varphi_t^{X_j} = (\cos\theta_1, \sin\theta_1, ..., \cos\theta_n, \sin\theta_n) = (\cos\theta_1, \sin\theta_1, ..., \cos(\theta_i + t), \sin(\theta_i + t), ..., \cos\theta_n, \sin\theta_n)$.

Proposition 0.13. *Let $p \in M$ and let $\{\varphi_t^X :\to M, t \in (-\epsilon, \epsilon)\}$ be the flow of $X \in \Gamma(TM)$ on $V \ni p$. Then $\varphi_{t_1}^X \cdot \varphi_{t_2}^X = \varphi_{t_1+t_2}^X$ if both sides are well-defined and φ_t^X is a local diffeomorphism at p.*

Proof. Consider the two curves

$$\gamma(t) = \varphi_t(\varphi_{t_2}(p)), \quad \lambda(t) = \varphi_{t+t_2}(p).$$

By definition of φ, the curve γ is a solution curve with initial value $\gamma(0) = \varphi_{t_2}(p)$ (assuming it is in the small neighborhood of point p).

We claim that λ is also a solution curve, hence coincides with γ by uniqueness of solution curves. We calculate

$$\frac{d}{dt}\lambda(t) = \frac{d}{dt}\varphi_{t+t_2} = \frac{d}{du}|_{u=t+t_2}\varphi_u(p) = X_{\varphi_u(p)}|_{u=t+t_2} = X_{\lambda(t)}$$

□

Remark. A vector field $X \in \mathcal{X}(M)$ is called complete if the domain of definition of its flow φ^X is $M \times \mathbb{R}$.

Here is the most important result of this section

Theorem 0.5. *If X is a complete vector field, the flow φ_t defines a 1-parameter group of diffeomorphisms. Namely each φ_t is a diffeomorphism and*

$$\varphi_0 = id_M, \quad \varphi_{t_1} \cdot \varphi_{t_2} = \varphi_{t_1+t_2}.$$

Conversely, if φ_t is a 1-parameter group of diffeomorphisms such that the map $(t,p) \mapsto \varphi_t(p)$ is smooth, the equation

$$X_p(f) = \frac{d}{dt}|_{t=0} f(\varphi_t(p))$$

defines a smooth vector field on M, with flow φ_t.

Proof. By previous proposition the first part is done.

Clearly, X_p is a tangent vector at $p \in M$. Using local coordinates, one can show that X_p depends smoothly on p, hence it defines a vector field. Given $p \in M$ we have to show that $\gamma(t) = \varphi_t(p)$ is an integral curve of X. Indeed,

$$\frac{d}{dt}\varphi_t(p) = \frac{d}{ds}|_{s=0}\varphi_{t+s}(p) = \frac{d}{ds}|_{s=0}\varphi_s(\varphi_t(p)) = X_{\varphi_t(p)}$$

□

Example. Given $A \in M_m(\mathbb{R})$ let $\varphi_t : \mathbb{R}^m \to \mathbb{R}^m$ be multiplication by the matrix $e^{tA} = \sum_j \frac{t^j}{j!} A^j$ (i.e. exponential map of matrices).

Since $e^{(t_1+t_2)A} = e^{t_1 A} e^{t_2 A}$, and since $(t, x) \mapsto e^{tA} x$ is a smooth map, φ_t defines a flow. What is the corresponding vector field X? For any function $f \in C^\infty(\mathbb{R}^m)$ we calculate

$$X_x(f) = \frac{d}{dt}|_{t=0} f(e^{tA} x) = \sum_j \frac{\partial f}{\partial x_j} (Ax)_j = \sum_{jk} A_{jk} x_k \frac{\partial f}{\partial x_j}$$