Lecture 3

0.8. **One more viewpoint.** There is another definition of T_pM. Let $C_p^\infty(M)$ denote the subspace of functions vanishing at p, and let $C_p^\infty(M)^2$ consist of finite sums $\sum_i f_i g_i$ where $f_i, g_i \in C_p^\infty(M)$. Since any tangent vector $v : C^\infty(M) \to \mathbb{R}$ vanishes on constants, v is effectively a map $v : C_p^\infty(M) \to \mathbb{R}$. Since tangent vectors vanish on products, v vanishes on the subspace $C_p^\infty(M)^2 \subset C_p^\infty(M)$. Thus v descends to a linear map $C_p^\infty(M)/C_p^\infty(M)^2 \to \mathbb{R}$, i.e. an element of the dual space[10] $(C_p^\infty(M)/C_p^\infty(M)^2)^*$. The map $T_pM \to (C_p^\infty(M)/C_p^\infty(M)^2)^*$ just defined is an isomorphism, and can therefore be used as a definition of T_pM.

Remark. This may appear very fancy on first sight, but really just says that a tangent vector is a linear functional on $C^\infty(M)$ that *vanishes on constants and depends only on the first order Taylor expansion of the function at p.*

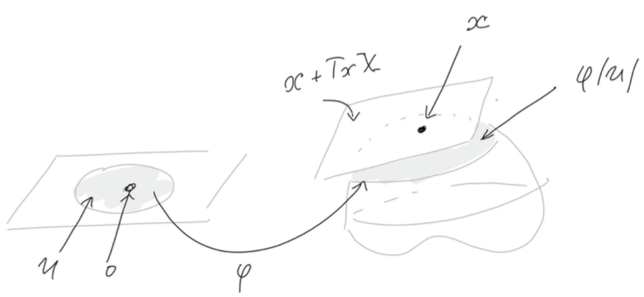
0.9. **Differential and local diffeomorphisms.** Recall from multivatiable calculus the following: If we have open $U \subset \mathbb{R}^n$ and $f : U \to \mathbb{R}^k$ be a smooth function (given by components $f_1(x_1, ..., x_n), ..., f_k(x_1, ..., x_n)$, where each f_i is $\mathbb{R}$-valued). **Differential** is a linear map $df_x : \mathbb{R}^n \to \mathbb{R}^k$ given for $h \in \mathbb{R}^n$ by $df_x(h) = \lim_{t\to 0} \frac{f(x+th)-f(x)}{t}$.

Remark: This linear map is given by the $n \times k$-matrix with ij-coefficient being equal to $\frac{\partial f_i}{\partial x_j}(x)$ (**Jacobian** as we know).

Remark: Differential df_x is the best linear approximation of f at $x \in \mathbb{R}^n$:

$$f(x+h) = f(x) + df_x(h) + \bar{o}(||h||)$$

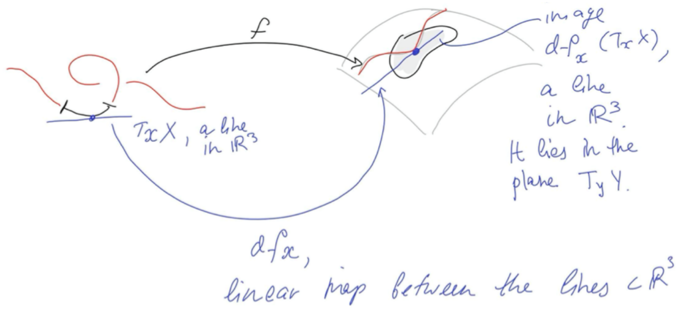
Remark: We can define the tangent space using differential as well. Let $\varphi : U \to X$ be a local parametrization near x such that $\varphi(0) = x$. Then T_xX – image of $d\varphi_0$ (it does not depend on a choice of local parametrization). Here the picture:



Definition 0.7. *Let $f : M \to N$ be a smooth map between manifolds. Let $X = \gamma'(0) \in T_pM$. Then $f \cdot \gamma$is a curve in N through $f(p)$. We define the differential of f at p, which is a linear map $df_p : T_pM \to T_{f(p)}N$, by $df_p(X) = (f \cdot \gamma)'(0)$.*

Geometrically in the case of manifolds in real vector spaces, differential is the following map: if x is moving with the velocity v, then $f(x)$ is moving with the velocity $df_x(v)$.

[10]Here we use the well-known isomorphism between the dual space and homomorphisms from a given space.



We can also think of the differential in terms of a differential of a map between Euclidean spaces. Given a curve γ through p and a chart (U, φ) at p, we have the curve $a = \varphi \cdot \gamma$ in Euclidean space. The curve $f \cdot \gamma$ defines a curve $b = \psi \cdot f \cdot \gamma$ in Euclidean space where (V, ψ) is a chart at $f(p)$. The relationship between the tangent vectors between the curves a and b at 0 is:

$$b'(0) = (\psi \cdot f\gamma)'(0) = (\psi \cdot f \cdot \varphi^{-1} \cdot \gamma a)|_{\varphi(p)}|_{\varphi(p}(a'(0))\}$$

Hence the differential df_p may be viewed as $d(\psi \cdot f\varphi^{-1})|_{\varphi(p)}$ given the charts.

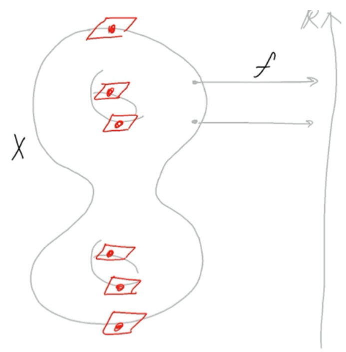
In particular,this means that if $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^m$ are manifolds and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth map such that $f(M) \subseteq N$ then $df_p : T_pM \to T_{f(p)}N$ is the restriction of the linear map $df_p : \mathbb{R}^n \to \mathbb{R}^m$.

General philosophy:

$$\begin{array}{ccc} X & \to^f & Y \\ \downarrow_\varphi & & \downarrow_\psi, \\ U & \dashrightarrow^h & V \end{array}$$

where $h = \psi \cdot f \cdot \varphi^{-1}$. Therefore, to study local properties of a smooth map f it is enough to study map h because φ, ψ are diffeomorphisms onto neighborhoods.

Remark: Differential df_x does not to be injective/surjective/isomorphism in general. For examplem, if $X \subset \mathbb{R}^3$ surface and $f : X \to \mathbb{R}$ the "height" (projection to vertical axis) function.



Here differential is either zero (horizonal) or onto (vertical).

Remark: Sometimes the differential for map $F \in C^\infty(M, N)$ is called the **tangent map**, and it could be defined as operator in the following way:

$$T_pF(v)(f) = v(f \cdot F)$$

It is easy to check that $T_pF(v)$ is a tangent vector indeed. And it is the same object as above.

Examples.

1. Let $f:\mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$ be given by $f(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, then

$$df_{(\theta,\varphi)} = \begin{pmatrix} \cos\theta\cos\varphi & -sin\theta\sin\varphi \\ \cos\theta\sin\varphi & \sin\theta\cos\varphi \\ -\sin\theta & 0 \end{pmatrix}$$

In terms of differential operators we therefore have that

$$df_{(\theta,\varphi)}(\partial_\theta) = \cos\theta\cos\varphi\partial_1 + \cos\theta\sin\varphi\partial_2 - \sin\theta\partial_3, \quad df_{(\theta,\varphi)}(partial_\varphi) = \sin\theta\varphi\partial_1 + \sin\theta\cos\varphi\partial_2$$

2. Let us calculate the differential of the map $f: S^2 \to \mathbb{R}P^2$ given by $f(x) = [x]$ at $(0,0,1) \in U_S$. Let $X \in T_{(0,0,1)}S^2$ be a vector in a tangent plane. Then $f(0,0,1) = [(0,0,1)] \in U_3 = \{[(y_1,y_2,y_3)] \in \mathbb{R}P^2 : y_3 \neq 0\}$. Now we want to calculate $df_{(0,0,1)}(X)$. We know that $\varphi_S(0,0,1) = (0,0)$ and for $(x_{1},_{2}) \in \mathbb{R}^2$ with $|x| < 1$,

$$\varphi_3 \cdot f \cdot \varphi_S^{-1}(x_1,x_2) = \varphi_3\left[\left(\frac{2x_1}{1+|x|^2}, \frac{2x_2}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2}\right)\right] = \left(\frac{2x_1}{1+|x|^2}, \frac{2x_2}{1+|x|^2},\right)$$

Therefore,

$$d(\varphi_3 \cdot f \cdot \varphi_S^{-1})|_{(0,0)} = \frac{2}{(1-|x|^2)^2}\begin{pmatrix} 1+x_1^2-x_2^2 & 2x_1x_2 \\ 2x_1x_2 & 1-x_1^2+x_2^2 \end{pmatrix}|_{(0,0)} = 2I$$

Remark. We can use the differential of f at p to detect when f is a local diffeomorphism. For a given map f is difficult to know if it is a local diffeomorphism as it is nonlinear in general, but the differential is a linear map and so is easier to analyse.

Proposition 0.6. *A smooth map $f: M \to N$ is a local diffeomorphism at p if and only if $df_p : T_pM \to T_{f(p)}N$ is an isomorphism.*

Remark: Above, we of course must have equal dimensions.

Proof. $[\Rightarrow]$ Suppose that f is a local diffeomorphism at p. Then by definition there exist open $U \ni p$ and open $V \ni f(p)$ such that $f: U \to V$ is a diffeomorphism. Therefore, $d(f^{-1}\cdot f)_p = d(f^{-1})_{f(p)} \cdot d_{f(p)} = id$ and $df_p \cdot d($. Hence $df_p \cdot d(f^{-1})_{f(p)} = id$ is an isomorphism.

$[\Leftarrow]$ Now suppose that df_p is an isomorphism. Let (U,φ) and (V,ψ) be charts around points p and $f(p)$ respectively so that $f(U) \subseteq V$.

Again by the argument above $d\varphi^{-1}_{\varphi(p)} : \mathbb{R}^n \to T_pM$ and $d\psi_{f(p)} : T_{(f(p)}N \to \mathbb{R}^n$ are isomorphisms since $'\varphi^{-1}, \psi$ are local diffeomorphisms, so $d(\psi \cdot f \cdot \varphi^{-1})_{\varphi(p)} : \mathbb{R}^n! \to \mathbb{R}^n$ is a composition of isomorphisms by the chain rule and thus is an isomorphism: explicitly,

$$d(\psi \cdot f \cdot \varphi^{-1})_{\varphi(p)} = d\psi_{f(p)} \cdot df_p \cdot d(\varphi^{-1})_{\varphi(p)}$$

Now we can then use the Inverse Function Theorem to give open sets $U' \ni p$ and $V' \ni f(p)$ so that $\psi \cdot f \cdot \varphi^{-1} : \varphi(U') \to \psi(V')$ is a diffeomorphism (using the fact that φ, ψ are diffeomorphisms onto their images). Hence $f : U' \to V'$ is a diffeomorphism. $\square$

Examples.

1. The map $f : \mathbb{R}^2 \to S^2$ above has differential with full rank (and is therefore an isomorphism) except when $sin\theta = 0$. Hence f is not a local diffeomorphism, but it is one restricted to any region where $\sin\theta \neq 0$.

2. The map $f : \mathbb{R}^n \to T^n$ clearly has differential whose image is n-dimensional and thus is an isomorphism, and hence f is a local diffeomorphism.

3. The map $f : S^n \to \mathbb{R}P^n$ given by $f(x) = [x]$ has $df_p = d\varphi_3 \cdot 2id \cdot d(\varphi_S^{-1})_p$, so f is a local diffeomorphism. It is not a diffeomorphism because it is not a bijection.

Definition 0.8. *A smooth map $f : M \to N$ is an **immersion** if $df_p : T_pM \to T_{f(p)}N$ is injective for all $p \in M$ (so we obviously need $dim(N) = dim(M)$).*

*An injective immersion is called an **embedding**. If $f : M \to N$ is an embedding then $f(M)$ is a manifold and $f : M \to f(M)$ is a diffeomorphism.*

A smooth map $f : M \to N$ is a submersion if $df_p : T_pM \to T_{f(p)}N$ is surjective for all $p \in M$ (so we obviously need $dimN \leq dimM$).

A map which is both an immersion and a submersion is a local diffeomorphism.

Example.

Let $C = \{(\cos\theta, \sin\theta, t) \in \mathbb{R}^3 : t \in \mathbb{R}\}$ be the cylinder. Let $f : S1 \to C$ be given by $f(\cos\theta, \sin\theta, t) = (\cos\theta, \sin\theta, 0)$. Then

$$T_{(\cos\theta, \sin\theta)}S^1 = \{\lambda(-\sin\theta\partial_1 + \cos\theta\partial_2) : \lambda \in \mathbb{R}\}$$

$$df_{(\cos\theta, \sin\theta)}((-\sin\theta\partial_1 + \cos\theta\partial_2)) = -\sin\theta\partial_1 + \cos\theta\partial_2$$

. So f is an immerssion. It is moreover injective and f is an isomorphism.

Now let $g : C \to S^1$ be a map given $f(\cos\theta, \sin\theta, t) = (\cos\theta, \sin\theta)$ then

$$T_{(\cos\theta, \sin\theta, t)}C = Span\{-\sin\theta\partial_1 + \cos\theta\partial_2, \partial_3\}$$

and

$$dg_{(\cos\theta, \sin\theta, t)}(-\sin\theta\partial_1 + \cos\theta\partial_2) = \sin\theta\partial_1 + \cos\theta\partial_2, \quad dg_{(\cos\theta, \sin\theta, t)}(\partial_3)$$

Hence g is a submersion.

2. Let $F : S^n \to \mathbb{R}$ given by $F(x_1, ..., x_{n+1}) = x_{n+1}$ is not a submersion because dF_p is the zero map at the North and South poles. However, $F : S^n \setminus \{N, S\}\mathbb{R}$ is a submersion. This shows the relationship between submersions and regular values.

Let us now discuss geometrical meaning of differential map.

0.10. **Tangent bundle.** Let M be a manifold of dimension m. If M is an embedded submanifold of $\mathbb{R}^n$, the tangent bundle TM is the subset of $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ given by

$$TM = \{(p, v) \in \mathbb{R}^n \times \mathbb{R}^n | p \in M, v \in T_pM\}$$

where each T_pM is identified as a vector subspace of $\mathbb{R}^n$. It is not hard to see that TM is, in fact, a smooth embedded submanifold of dimension $2m$. Moreover, the natural map $\pi : TM \to M, (p, v) \mapsto p$ is smooth, and its "fibers" $\pi^{-1}(p) = T_pM$ carry the structure of vector spaces.

Examples:

0. For $\mathbb{R}^n$: tangent space at point is $\mathbb{R}^n$ and tangent bundle is $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.

1. It is straightforward to see that points in TS^1 are given by $p = (\cos\theta, \sin\theta)$ and $q = \lambda(-\sin\theta, \cos\theta)$ since q must be orthogonal to p, for some $\lambda, \theta \in \mathbb{R}$.

   Hence, there is a natural diffeomorphism $f : S^1 \times \mathbb{R} \to TS^1$ given by
$$f : (\theta, \lambda) \mapsto \lambda(-\sin\theta, \cos\theta) \in T_{(\cos\theta, \sin\theta)}S^1$$

   It implies that TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$. Moreover, for fixed (θ, λ) the map $\lambda \to \lambda(-\sin\theta, \cos\theta) \in T_{(\cos\theta, \sin\theta)}S^1$ is an isomorphism of vector spaces.

Question: Is TM always isomorphic to $M \times \mathbb{R}^n$?
The answer is negative.

3. $TS^2 \neq S^2 \times \mathbb{R}^2$. It is the same as the set of all oriented straight lines in $\mathbb{R}^3$. Indeed, we know that points in TS^2 are given by $x \in S^2$ and $y \in \mathbb{R}^3$ orthogonal to x. Then x defines an oriented straight line through 0. Since y is orthogonal to x we can use it translate this straight line to get an oriented straight line through y in the direction x.

   Conversely, given an oriented straight line in $\mathbb{R}^3$, there is a unique closest point from the line to 0, which gives a vector $y \in \mathbb{R}^3$ orthogonal to the line. Translating by y gives an oriented straight line through 0, which is uniquely determined by some $x \in S^2$.

   Remark. The set of all straight lines in $\mathbb{R}^3$ is a 4-dimensional manifold, which is $T\mathbb{R}P^2$.
4. The same often happens in the higher dimensions.