Lecture 2

In this lecture we will continue discussion of manifolds and will give two ways how to construct them - regular value theorem and group actions. Then we will discuss tangent vectors.

0.2. **Regular value theorem, cont.** Last time we stated the regulat value theorem 0.1.

Proof. (of 0.1) By the Implicit Function Theorem we have that for all $p \in F^{-1}(c)$ there exists a splitting of $R^{n+m} = \mathbb{R}^n \times \mathbb{R}^m = KerdF_p \times \mathbb{R}^m$ such that, if $p = (a, b)$ with respect to this splitting, then there exist open sets $a \in V_p \in \mathbb{R}^n$ and $b \in W_p \in \mathbb{R}^m$ and a smooth map $G_p : V_p \to W_p$ with $G_p(a) = b$ such that

$$F^{-1}(c) \cap (V_p \times W_p) = \{(q, G_p(q)) : q \in V_p\}$$

Let $U_p = F^{-1}(0) \cap (V_p \times W_p)$ which is an open set and $\cup_{p\in F^{-1}(0)} U_p = F^{-1}(0)$ (since $p \in U_p$).

Now we need to define maps, let's do like that: $\varphi_p(q, G_p(q)) = q$. Then map φ_p is homeomorphism. The last is check that the transition maps are smooth. Hence, $F^{-1}(c)$ satisfies the conditions of 0.1 and it is n-dimensional manifold. $\square$

In the course of Differential Geometry you in particular have studied surfaces. Simple check analogous to the proof above leads to the following

Proposition 0.1. *A surface in $\mathbb{R}^3$ is a 2-dimensional manifold.*

0.3. **Diffeomorphisms.** One of our goals was to establish differentation on manifolds. Recall that map $f : U \to \mathbb{R}^n$ from open subset $U \in \mathbb{R}^k$ is smooth if it is continuous and all its partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist and are continuous on U. Now if X is any subset in $\mathbb{R}^k$ if near each point $x \in X$ it extends to a smooth function on some open set containing x.

Definition 0.2. *Let M and N be manifolds of dimensions m and n respectively. A map $f : M \to N$ is smooth at p if for some coordinate charts (U, φ) at p and (V, ψ) at $f(p)$ with $f(U) \subseteq V$, the map*

$$\psi \cdot \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^m \to \psi(V) \subseteq \mathbb{R}^n$$

is smooth. We say f is smooth if it is smooth at all $p \in M$.

In a obvious way definition above generalizes the classical one. The only thing we need to check is that is well-defined, in particular if we take different open sets near point p. It follows from the smoothness of transition maps.

Examples (trivial): 1) Take U, φ as the coordinate chart for M, $(\mathbb{R}^n, id)$ as the chart for $\mathbb{R}^n$ and φ is smooth map then.

2) Identity map is smooth.

3) Restriction is the smooth map.

4) Multiplication map and the inversion map for matrices are smooth, it makes them Lie groups.

Definition 0.3. *A map $f : M \to N$ is a diffeomorphism if it is a smooth bijection with a smooth inverse. The manifolds M and N are then said to be diffeomorphic.*

A map $f : M \to N$ is a local diffeomorphism at p if there is open $U \ni p$, open $V \ni f(p)$ such that $f : U \to V$ is a diffeomorphism. We say f is a local diffeomorphism if it is a local diffeomorphism at all $p \in M$.

There are some simple examples and properties such as identity is diffeomorphism, composition of two diffeomorphisms is diffeomorphism, and the opposite is diffeomorphism. Therefore, the diffeomorphisms form a group which we write $Diff(M)$.

The maps in atlas for manifold are as well diffeomorphisms. In the discussed above case of matrix groups A defines a diffeomorphism on $\mathbb{R}^n$ iff $A \in GL_n(\mathbb{R})$.

Examples.

1) $X = \{x^2 + y^2 = 1, y = 0\} \subset \mathbb{R}^2_{x,y}, \quad Y = \{t \in (-1, 1)\} \subset \mathbb{R}_t$. Then X and Y are diffeomorphic. Indeed, take $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x$. It is smooth and bijection with f^{-1} given by the following map $t \mapsto (t, \sqrt{1 - t^2})$ which is smooth for $t \in (-1, 1)$.
2) Consider the closures of X and Y from the previous example, then the maps above do not work. So one needs to use a different f.

Lemma 0.1. *If $X \to Y$ is a diffeomorphism then f is also homeomorphism between x and Y considered as topological spaces.*

Proof. (Sketch)

The lemma follows from the following claim: if f is a smooth map between subsets in $\mathbb{R}^n$ then f is continuous (ie preimages of open sets are open).

□

Non-example. Consider X the circle and Y an interval. Then there is no diffeomorphism $f : X \to Y$. Indeed, fix $p \in X$, then f would give a homeomorphism $X \setminus \{p\} \to Y \setminus \{f(p)\}$. However one is connected while the another has 2 connected components.

This non-example shows that we can not have one chart for the circle as the manifold.

Non-example 2. Let X be "angle" on the plane and Y be an interval. There is no diffeomorphism in this case too.

Idea: Let f be such diffeomorphism, then the inverse $g = f^{-1}$ would have two components since $X \subset \mathbb{R}^2$. We can assume $g_1(0) = g_2(0) = 0$. We claim that either $g_1'(0) \neq 0$ or $g_2'(0) \neq 0$. This gives a contradiction (**exercise**).

0.4. **Quotient construction.** There is an important way how to construct manifolds.

Definition 0.4. *We say that a group G acts on M by diffeomorphisms if there is a homomorphism $G \to Diff(M)$; i.e. for all $g \in G$ there exists a diffeomorphism f_g of M so that $f_e = id$ and $f_{gh} = f_g \cdot f_h$ (any g, h).*

Definition 0.5. *Let G be a discrete group [4] acting on M by diffeomorphisms as above. We say that G acts* ***freely and properly discontinuously*** *if*

- $\forall p \in M \exists$ *open* $V \ni p$ *with* $V \cap f_g(V) = \emptyset \forall g \neq e$[5].
- $\forall p, q \in M$ *with* $p \neq f_g(q) \forall g \in G \exists$ *open* $V \ni p$ *and* $W \ni q$ *with* $V \cap f_g(W) = \emptyset \forall g \in G$

Now we are ready to construct more manifolds using these actions!

[4]i.e. a finite group or $\mathbb{Z}^n$ or some other countable group

[5]It essentially says that f_g has no fixed points if $g \neq e$.

Theorem 0.2. *Let M be an n-dimensional manifold and let G be a discrete group acting freely and properly discontinuosly on M by diffeomorphisms. Define an equivalence relation $\sim$ on M by $p \simeq q \Leftrightarrow q = f_g(p)$ for some $g \in G$. Then the quotient space $M/\sim = M/G$ is an n-dimensional manifold.*

Proof. Let $\{(V_i, \psi_i) : i \in I\}$ be an atlas for M such that $V_i \cap f_g(V_i) = \emptyset$ (by properly discontinuous action). Let π bn the projection, and call by $U_i = \pi(V_i)$ and $\cup_i U_i = M/G$. Since $\pi_i = \pi|_{V_i}$ is diffeomorphism (by above) then we can define $\varphi_i = \psi_i \cdot \pi_i^{-1} : U_i \to \psi_i(V_i) \subseteq \mathbb{R}^n$ which is homeomorphism.

Now we need to check that everything works on the intersections.

If $U_i \cap U_j \neq \emptyset$ then by definition of φ_i:

$$\varphi(U_i \cap U_j) = \psi \cdot \pi_i^{-1}(U_i \cap U_j) = \psi_i(V_i \cap \pi^{-1}(U_j)) = \pi_i(V_i \cap \cup f_g(V_j))$$

which is a disjoint union of open sets and clearly $\varphi_j \cdot \varphi_i^{-1}$ is a homeomorphism, so it is enough to show that it and its inverse are smooth.

Let $p \in \varphi_i(U_i \cap U_j)$, then there exists unique $g \in G$ such that $p \in W = \psi_i(V_i \cup f_g(V_j))$, so $\varphi_j \cdot \varphi_i^{-1}|_W = \psi_j \cdot \pi_j^{-1} \cdot \pi_i \cdot \psi_i^{-1}|_W$, therefore it is enough to show that $\pi_j^{-1} \cdot \pi_i$ is smooth on $V_i \cup f_g(V_j)$. If $q \in V_i \cup f_g(V_j)$ and $q' = \pi_j^{-1} \cdot \pi_i(q) \in V_j$ then $\pi_j(q') = \pi_i(q)$ so there exists $g_q \in G$ such that $f_{g_q}(q') = q$. Therefore $q \in f_{g_q}(V_j) \cap f_g(V_j)$ so

$g_q = g$ and hence $\pi_j^{-1} \cdot \pi_i = f_{g^{-1}}|_V$ which is smooth. So $\varphi_j \cdot \varphi_i^{-1}$ is smooth with the same argument for the inverse. □

Examples.

1. $\mathbb{Z}_2$ act on $\mathbb{R}^n$ with ± 1. Clearly $-id$ is a diffeomorphism of $\mathbb{R}^n$ but it is not a free action because 0 is fixed. However, if we take any point $x \neq 0$ in $\mathbb{R}^n$ then there exists some coordinate $x_i \neq = 0$. Overall, $\mathbb{Z}$ acts freely and properly discontinuously by diffeomorphisms on $\mathbb{R}^n \setminus \{0\}$. Hence it acts freely and properly by diffeomorphisms on any manifold $M \subset \mathbb{R}^n \setminus \{0\}$ such that $-M = M$

   In particular, if $M = S^n$, then $S^n/\mathbb{Z}$ is $\mathbb{R}P^n$. If we have cylinder, $C = \{(x, y, z) : x^2 + y^2 = 1, -1 < z < 1\}$ with $-C = C$. Hence, $C/\mathbb{Z}$ is called Mobius band.

Proposition 0.2. *If a discrete group G acts freely and properly discontinuously on M then the projection $\pi : M \to M/G$ is a surjective local diffeomorphism.*

0.5. **Tangent vectors: idea.** Let M be a smooth manifold of dimension n. Suppose that we understand the tangent bundle of $\mathbb{R}^n$, and then we can define the tangent bundle TM by locally pulling back $T\mathbb{R}^n$ on coordinate charts, and using the derivative of the transition function to glue the bundle together on overlaps.

If f is a smooth function on $\mathbb{R}^n$ and v is a vector, then it makes sense to take the partial derivative of f in the direction v. If we fix coordinates x_i on $\mathbb{R}^n$, then we can write $v = \sum v_i \frac{\partial}{\partial x_i}$ and then $v(f) = \sum v_i \frac{\partial f}{\partial x_i}$.

So the latter gives the idea of vector field as the derivation.

Using coordinate functions on open charts gives us a way to take the derivative of a smooth function f on M along a vector field X on M. Note that at each point p the vector X_p acts as a derivation of the ring of germs of smooth functions at p; that is, $X_p(fg) = X_p(f)g + fX_p(g)$ (Leibnitz rule). In fact, *a vector at a point can* *be defined as a linear derivation of the ring of germs of smooth functions at that point,* and a vector field can be defined as a smoothly varying family of derivations.

We denote the space of smooth vector fields on M by $\mathcal{X}(M)$. A smooth map $g : N \to M$ induces a smooth map $dg : TN \to TM$ satisfying $dg(X)(f) = X(f \cdot g)$ for any vector field X on N and smooth function f on M. Note that the composition $f \cdot g$ is also written $g^* f$ and called the pullback of f by g.

Okay, this was the outline for what we are going to talk about today. To give more details first we start with the definition of tangent vector.

0.6. **Tangent vector as the derivative.** For a curve in the plane $\gamma : \mathbb{R} \to \mathbb{R}^2$ (or into $\mathbb{R}^n$), it is just the line tangent to the curve, which we can calculate by writing and computing the derivative

$$\gamma'(t) = (\gamma_1(t), \gamma_2(t))$$

so the tangent vector at $\gamma(0) = p$ say is

$$\gamma'(0) = (\gamma_1(0), \gamma_2(0))$$

Let M be an n-dimensional manifold in $\mathbb{R}^{n+m}$. *If we look at curves in M through p then the tangent vectors will form a vector space of dimension n* [6] *which we denote by T_pM.* In particular, we can calculate the tangent space to M at p for manifolds given by the regular value theorem.

Proposition 0.3. *Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a smooth map and let c be a regular value of F, so that $M = F^{-1}(c)$ is an n-dimensional manifold. Then for all $p \in M, T_pM \simeq KerdF_p$.*

Proof. Let $p \in M = F^{-1}(c)$ and let γ be a curve in M through p. Then $F(\gamma(t)) = c$ for all t since $\gamma(t) \in F^{-1}(c)$ for all t. Differentiating both sides we see that $dF(\gamma(t))/dt = 0$.

Applying the Chain rule at $t = 0$, we see that

$$dF_{\gamma(0)}(\gamma'(0)) = dF_p(\gamma'(0)) = 0.$$

Hence the derivative $\gamma'(0)$ sits in $KerdF_p$.

We just constructed the (linear) map from tangent space in We thus have a linear map to $KerdF_p$. This map is clearly injective. Since c is a regular value, we know by the rank-nullity theorem that $dimKerdF_p = n + m - m = n$, so since T_pM is also n-dimensional the map must be surjective.

□

Examples

1. Consider $F(x) = \sum_{i=1}^{n+1} x^2 - 1$, then $S^n = F^{-1}(0)$. Then $dF_x = (2x_1, ..., 2x_{n+1})$ so $KerdF_x = \{y \in \mathbb{R}^{n+1} : \langle y, x\langle = 0\} = \langle x\langle^{\perp}$, the orthogonal complement of the line through x. Thus $T_xS^n \simeq \langle x\langle^{\perp}$, which is geometrically clear.
2. Consider the set $SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : det(A) = 1\}$ is $F^{-1}(0)$ where $F : M_n(\mathbb{R}) \to \mathbb{R}$ is $F(A) = det(A) - 1$. Now if A is invertible, then

$$F(A+B)-F(A) = det(A+B)-det(A) = det(A(I+A^{-1}B))-det(A) = det(A)(det(I+A^{-1}B)-1)$$

and by expanding one sees that
$det(I + A^{-1}B)) = 1 + tr(A^{-1}B) + O(|B|^2)$.

[6]Dimension follows from the same argument of linear independence of derivative of "directional" curves.

Hence, $dF_A(B) = tr(A^{-1}B))$ for $A \in SL_n(\mathbb{R})$, which is not zero since $dF_A(A) = n$. Hence by 0.1 we have that $SL(n, \mathbb{R})$ is an (n^2-1)-dimensional manifold. Moreover,

$$T_A SL(n, \mathbb{R}) = \{B \in M_n(\mathbb{R}) : tr(A^{-1}B) = 0\}) \Rightarrow T_I SL(n, \mathbb{R}) = B \in M_n(\mathbb{R}) : tr(B) = 0$$

This says that the **Lie algebra** $\mathfrak{sl}(n, \mathbb{R}) = T_I SL(n, \mathbb{R})$ of the **Lie group** $SL(n, \mathbb{R})$ is the trace-free matrices (as a vector space). *In fact the bracket operation on the Lie algebra is just the matrix commutator*[7]*, which is true of all matrix Lie groups with matrix Lie algebras.*

0.7. **Tangent vector as the directional derivative/operator.** For embedded submanifolds $M \subset \mathbb{R}^n$, the tangent space T_pM at $p \in M$ can be defined as the set of all velocity vectors $v = \gamma'(0)$, where $\gamma : \mathbb{R} \to M$ is a smooth curve with $\gamma(0) = p$. Thus T_pM becomes a vector subspace of $\mathbb{R}^n$. To extend this idea to general manifolds, note that the vector $v = \gamma(0)$ defines a "directional derivative" $C^\infty(M) \to \mathbb{R}$:

$$v : f \mapsto \frac{d}{dt}|_{t=0} f(\gamma(t)).$$

Here we formally define T_pM as a set of directional derivatives.

Definition 0.6. *Let M be a manifold, $p \in M$. The tangent space T_pM is the space of all linear maps $v : C^\infty(M) \to \mathbb{R}$ of the form $v(f) = d|_{t=0} f(\gamma(t))$ for some smooth curve $\gamma \in C^\infty(M, \mathbb{R})$ with $\gamma(0) = p$.*

Here $\partial/\partial x_i = (0, ..., 1, ...0)$ form a basis of tangent space.

Proposition 0.4. *Let (U, φ) be a coordinate chart around p, with $\varphi(p) = 0$. A linear map $v : C^\infty(M) \to \mathbb{R}$ is in T_pM if and only if it has the form,*

$$v(f) = \sum_{i=1}^{n} a_i \frac{\partial f \cdot \varphi^{-1}}{\partial x_i}|_{x=0}$$

for some $a = (a_1, ..., a_m) \in R^n$

Proof. Given a linear map v of this form, let $\gamma(t)$ be any smooth curve with $\varphi(\gamma(t)) = ta$ for $|t|$ sufficiently small. Then

$$\frac{d}{dt}|_{t=0} f(\gamma(t)) = \frac{d}{dt}|_{t=0} (f \cdot \varphi^{-1})(ta) = \sum_{i=1}^{n} a_i \frac{\partial f \cdot \varphi^{-1}}{\partial x_i}|_{x=0}$$

by the chain rule. Conversely, given any curve γ with $\gamma(0) = p$, let $\tilde{\gamma} = \varphi \cdot \gamma$ be the corresponding curve in the image $\varphi(U)$ (defined for small $|t|$). Then

$$\frac{d}{dt}|_{t=0} f(\gamma(t)) = \frac{\partial}{\partial t}|_{t=0} (f \cdot \varphi^{-1})(\tilde{\gamma}(t)) = \sum_{i=1}^{n} a_i \frac{\partial f \cdot \varphi^{-1}}{\partial x_i}|_{x=0}$$

where $a = \frac{d\tilde{\gamma}}{dt}|_{t=0}$. Once again by the chain rule.

□

[7]We will discuss commutator later.

Remark. Hence, using the $\frac{\partial}{\partial x_i}|_{x=0}$ as a basis, we can identify the tangent vector to the curve $\tilde{\gamma}$ in $\mathbb{R}^n$ at $\varphi(p) = 0$ with the differential operator $\sum_{i=1}^n a_i \frac{}{\partial x_i}|_{x=0}$ acting on the function $f \cdot \varphi^{-1}$ (which is how we identify functions on M locally with functions on $\mathbb{R}^n$). Notice that $\frac{\partial}{\partial x_i}|_{x=0}$ is the tangent vector to $t \mapsto \varphi^{-1}(0, ..., 0, t, 0, ..., 0)$, which is the image of a straight line, and forms a local basis for the tangent vectors to curves by the above calculation.

So, in this definition, tangent vectors are essentially differential operators on locally defined functions on M. We can also think of it as a vector in $\mathbb{R}^n$, using the given chart (U, φ) as described above.

To sum up we have the following corollary

Corollary 0.1. *If $U \subset \mathbb{R}^n$ is an open[8] subset, the tangent space T_pU is canonically identified with R^n.*

Question: How to get rid off charts?

We now describe a third definition of T_pM which characterizes "directional derivatives" in a coordinate-free way, without reference to curves γ. As we have already seen every tangent vector $v \in T_pM$ satisfies a product rule since we have it in local coordinates (U, φ),

$$v(f_1f_2) = f_1(p)v(f_2) + v(f_1)f_2(p)$$

for all $f_j \in C^\infty(M)$.

Proposition 0.5. *A linear map $v : C^\infty(M) \to \mathbb{R}$ is a tangent vector if and only if it satisfies the product rule above.*

Proof. Let $v : C^\infty(M) \to \mathbb{R}$ be a linear map satisfying the product rule. Let us show that $v \in T_pM$. We use the second definition of T_pM in terms of local coordinates.

First, note that by the product rule applied to the constant function $1 = 1 \cdot 1$ we have $v(1) = 0$[9]. Hence v vanishes on constants.

Now show that $v(f) = 0$ if $f = 0$ near p. Choose $\chi \in C^\infty(M)$ with $\chi(p) = 1$, zero outside a small neighborhood of p so that $f\chi = 0$. The product rule tells us that

$$0 = v(f\chi) = v(f)\chi(p) + v(\chi)f(p) = v(f).$$

Thus $v(f)$ depends only on the behavior of f in an arbitrarily small neighborhood of p.

In particular, taking (U, φ) as a coordinate chart around p such that $\varphi(p) = 0$, we may assume that $supp(f) \subset U$. Consider the Taylor expansion of $\tilde{f} = nf \cdot \varphi^{-1}$ near $x = 0$:

$$\tilde{f}(x) = \tilde{f}(0) + \sum x_i \frac{\partial}{\partial x_i}|_{x=0}\tilde{f} + r(x),$$

where the remainder vanishes at $x = 0$ with its first derivatives. This means that it can be written (non-uniquely) in the form $r(x) = \sum_i x_i r_i(x)$, where r_i are smooth functions that vanish at 0. (**prove the existence of such decomposition**).

[8]For closed ones don't forget about the boundary.

[9]It is the standart type argument which you might have seen in the course of Groups.

By the product rule, v vanishes on $r \cdot \varphi^{-1}$ (since it is a sum of products of functions that vanish at p). It also vanishes on the constant $\tilde{f}(0) = f(p)$. Thus applying v to the above we are done.

$\square$