Lecture 18

0.55. **Lie groups and homogeneous spaces.**

Definition 0.57. *Let G admit both the structure of a group and a smooth manifold. G is a Lie group if the multiplication map $G \times G \to G$ and the inverse map $G \to G$ are smooth. The Lie algebra $\mathfrak{g}$ is the tangent space to G at the origin.*

Example (Myers–Steenrod) If M is any Riemannian manifold, Myers–Steenrod showed that the group of isometries $Isom(M)$ is a Lie group. One way to see this is to observe (e.g. by using the exponential map) that if M is connected, and ξ is any orthonormal frame at any point $p \in M$, an isometry of M is determined by the image of ξ. So if we fix ξ, we can identify $Isom(M)$ with a subset of the frame bundle of M, and see that this gives it the structure of a smooth manifold.

Remark: Most example of Lie groups are matrix groups. An example of a Lie group that is not isomorphic to a matrix Lie group is the double covering of SL(2, R).

We often denote the identity element of a Lie group by $e \in G$, so that $g = T_eG$. For every $g \in G$ there are diffeomorphisms $L_g : G \to G$ and $R_g : G \to G$ called (respectively) left and right multiplication, defined by $L_g(h) = gh$ and $R_g(h) = hg$ for $h \in G$. Note that $L_g^{-1} = L_{g^{-1}}$ and $R_g^{-1} = R_{g^{-1}}$. The maps $g \to L$ and $g \to R_{g^{-1}}$ are homomorphisms from G to $Diff(G)$.

A vector field X on G is said to be left invariant if $dL_g(X) = X$ for all $g \in G$. Since G acts transitively on itself with trivial stabilizer, the left invariant vector fields are in bijection with elements of the Lie algebra, where $X(e) \in g = T_eG$ determines a left- invariant vector field X by $X(g) = dL_gX(e)$ for all g, and conversely a left-invariant vector field restricts to a vector in T_eG. So we may (and frequently do) identify g with the space of left-invariant vector fields on G.

Remark: We can define the Lie bracket on $\mathfrak{g} = T_eG$ by its identification with left- invariant vector fields. A second Lie algebra structure on $\mathfrak{g}$ is defined by identifying T_eG with the space of right-invariant vector fields. How are the two brackets related? The answer is that they differ by sign.

If X and Y are left-invariant vector fields, then so is $[X, Y]$, since for any smooth map φ between manifolds, $d\varphi([X, Y]) = [d\varphi(X), d\varphi(Y)]$. Thus Lie bracket of vector fields on G induces a Lie bracket on $\mathfrak{g}$, satisfying the usual properties of the Lie bracket, in particular, Jacobi identity.

Definition 0.58. *A smooth map $\gamma : \mathbb{R} \to G$ is a 1-parameter subgroup if it is a homomorphism; i.e. if $\gamma(s + t) = \gamma(s)\gamma(t)$ for all $s, t \in \mathbb{R}$.*

Proposition 0.39. *1-parameter subgroups $\gamma : \mathbb{R} \to G$ are integral curves of left-invariant vector fields.*

Proof. Indeed, suppose we start with 1-parameter subgroup γ, and let $X(e) = \gamma'(0) \in T_eG = \mathfrak{g}$. If X is the corresponding left-invariant vector field on G.

Differentiating the defining equation of 1-parameter subgroup with respect to t at $t = 0$ we get

$$\gamma'(s) = d\gamma(s)(X(e)) = X(\gamma(s))$$

Therefore, γ is obtained as the integral curve through e of the left-invariant vector field X.

Conversely, if X is a left-invariant vector field, and γ is an integral curve of X through the origin, then γ is a 1-parameter subgroup. □

Similar computation shows that

Remark: Thus we see that every $X(e) \in \mathfrak{g}$ arises as the tangent vector at e to a unique 1-parameter subgroup. Moreover, every left-invariant vector field X on G is complete since the left multiplication permutes the integral curves.

Let $\gamma(t)$ denote the 1-parameter subgroup corresponding to $X \in \mathfrak{g}$. Then the flow of X is $\varphi_t(g) = g\gamma(t)$.

Definition 0.59. *For any Lie group G, with Lie algebra $\mathfrak{g}$, one defines the exponential map $e : \mathfrak{g} \to G, e(X) := \gamma(1)$.*

Example: Note that this generalizes the exponential map for matrices. Indeed, suppose $G \subseteq GL(n, \mathbb{R})$ is a matrix Lie group, with Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{R})$. Then the flow of the left-invariant vector field corresponding to $X \in \mathfrak{gl}(n, \mathbb{R})$ is just $\varphi_t(g) = ge^(tX)$ (using the exponential map for matrices).

Note that the derivative of this map at 0 is the identity map $\mathfrak{g} \to \mathfrak{g}$, and therefore exponentiation is a diffeomorphism from some neighborhood of 0 in $\mathfrak{g}$ to some neighborhood of e in G, although it is not typically globally surjective.

Remark: If G is given a Riemannian metric, then there is an exponential map $\exp_e : \mathfrak{g} \to G$ in the usual sense of Riemannian geometry. This map is closely related to exponentiation (as defined above), but the two maps are different in general, and we use different notation $\exp(X)$ and e^X to distinguish them. For matrix Lie groups, exp coincides with the exponential map for matrices (hence its name). The difference is that the geodesics through the identity will not be one-parameter subgroups for the general Lie group. We will see it later when study Riemannian metrics on Lie groups.

Example: Consider a nonabelian Lie group and any metric that is not bi-invariant. The simplest example is

$$G = \{\begin{vmatrix} a & b \\ 0 & a^{-1} \end{vmatrix} |a, b \in \mathbb{R}\}$$

This group acts simply transitively by isometries on the hyperbolic plane $\mathbb{H}^2$, and admits a (unique) left-invariant Riemannian metric making the orbit map an isometry. For that metric, the geodesics in G correspond to the geodesics in the hyperbolic plane.

The curve

$$c(t) = \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix} \cdot i = t + i$$

is a horocycle. So the one-parameter subgroups need not correspond to Riemannian geodesics.

Lie groups often arise as transformation groups, by some "action" on a manifold M.

Definition 0.60. *An action of a Lie group G on a manifold M is a group homomorphism $G \to Diff(M)$, $g \mapsto \Phi_g$ such that the action map $\rho : G \times M \to M$, $(g, p) \mapsto \rho_g(p)$ is smooth.*

For each $X \in \mathfrak{g}$ and associated 1-parameter subgroup $\gamma : \mathbb{R} \to G$ with $\gamma'(0) = X$ we get a 1-parameter family of diffeomorphisms $\varphi_t := \rho \cdot \gamma(t)$ on M.

Define $d\rho(X) \in \mathcal{X}(M)$ to be the vector field tangent to φ_t; i.e. $d\rho(X) := \frac{d}{dt}\varphi_t|_{t=0}$. Then $d\rho([X,Y]) = [d\rho(X), d\rho(Y)]$.

In the other words, the map $d\rho : \mathfrak{g} \to \mathcal{X}(M)$ is a homomorphism of Lie algebras. (it is also called the Lie algebra action on manifold). By exponentiating, we get the identity

$$\rho(e^X) = e^{d\rho(X)}.$$

Example:

1) Note that an action of the (additive) Lie group $G = \mathbb{R}$ is the same thing as a global flow, while an action of the Lie algebra $\mathfrak{g} = \mathbb{R}$ (with zero bracket) is the same thing as a vector field.
2) Every matrix Lie group $G \subset GL(n, \mathbb{R})$, and every matrix Lie algebra acts on $\mathbb{R}^n$ by multiplication.
3) The rotation action of $SO(n)$ on $\mathbb{R}^n$ restricts to an action on the sphere, $S^{n-1} \subset \mathbb{R}^n$.
4) Any Lie group G acts on itself by multiplication from the left, $L_a(g) = ag$, multiplication from the right $R_{a^{-1}}(g) = ga^{-1}$, and also by the adjoint (=conjugation) action (we will define it later).

Exponentiation satisfies the formula $e^{sX} = \gamma(s)$ so that $e^{sX}e^{tX} = e^{(s+t)X}$ for any $s, t \in \mathbb{R}$. Moreover, if $[X,Y] = 0$ then e^{sX} and e^{tY} commute for any s and t, by Frobenius' theorem, and $e^{X+Y} = e^X e^Y = e^Y e^X$ in this case. We have already observed that exponentiation defines a diffeomorphism from a neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of e in G; we denote the inverse by log. And using some explicit calculation one will have:

Theorem 0.32. ***(Campbell-Baker-Hausdorff Formula)***

For $X, Y \in \mathfrak{g}$ sufficiently close to 0, if we define $e^X e^Y = e^Z$ then there is a convergent series expansion for Z:

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] - \frac{1}{24}[Y,[X,[X,Y]]] - ...$$

Remark: An explicit closed formula for the terms involving n-fold brackets was obtained by Dynkin. Note that if $\mathfrak{g}$ is a nilpotent Lie algebra — i.e. if there is a uniform n for which any n-fold bracket vanishes — then the CBH formula becomes a polynomial, which converges everywhere. The CBH formula shows that the group operation of a Lie group can be reconstructed at least on a neighborhood of the identity from its Lie algebra.

Definition 0.61. *The group G acts on itself by conjugation; i.e. there is a map $G \to Aut(G)$ sending $g \to L_g \cdot R_{g^{-1}}$. Conjugation fixes e. The adjoint action of G on $\mathfrak{g}$ is the derivative of the conjugation automorphism at e; i.e. the map $Ad : G \to Aut(\mathfrak{g})$ defined by $Ad(\mathfrak{g})(Y) = d(L_g \cdot R_{g^{-1}})Y$.*

The adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$ is the map $ad : \mathfrak{g} \to End(\mathfrak{g})$ defined by $ad(X)(Y) = dAd(e^{tX})(Y)|_{t=0}$.

Remark: If we think of $\mathfrak{g}$ as a smooth manifold, the adjoint action is a homomorphism $Ad : G \to Diff(g)$ and its derivative is a homomorphism of Lie algebras $ad : \mathfrak{g} \to X(\mathfrak{g})$. Thus we obtain the identity $e^{ad(X)} = Ad(e^X)$. Since all maps and manifolds under consideration are real analytic, this identity makes sense when interpreted as power series expansions of operators.

Remark: The adjoint representation $Ad : G \to Aut(\mathfrak{g})$ is an example of a linear representation.

Example: $(SL(2,\mathbb{R}))$

Unlike the exponential map on complete Riemannian manifolds, exponentiation is not typically surjective for noncompact Lie groups. The upper half-space model of hyperbolic 2-space $\mathbb{H}^2$ consists of the subset of $z \in \mathbb{C}$ with $Im(z) > 0$. The group $SL(2,\mathbb{R})$ acts on hyperbolic 2-space:

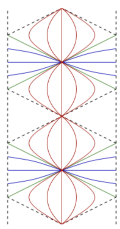
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

This action is called Möbius transformation, the kernel consists of $\pm Id$ and the image is the group $PSL(2,\mathbb{R})$ which acts transitively and faithfully on $\mathbb{H}^2$ by isometries. There are three kinds of 1-parameter families of isometries of $\mathbb{H}^2$:

1. Elliptic subgroups, which are conjugate to $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. These elements fixes $i \in \mathbb{C}$ anc act by rotation through angle 2θ.
2. Parabolic subgroups consisting of elements conjugate to $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. It fixes ∞ and acts by translation by t.
3. Hyperbolic subgroups. These are conjugate to $\begin{pmatrix} e^t & 0 \\ -0 & e^{-t} \end{pmatrix}$, it fixes 0 and ∞, action is by dilation e^{2t}.

Note that $SL(2,\mathbb{R})$ double-covers $PSL(2,\mathbb{R})$, hence, they have isomorphic Lie algebras, and there is a bijection between 1- parameter subgroups. In particular, any matrix in $SL(2,\mathbb{R})$ with trace in $(-\infty,-2)$ is not in the image of exponentiation.

Identifying $PSL(2,\mathbb{R})$ with the unit tangent bundle of $\mathbb{H}^2$ shows that it is diffeomorphic to an open $S^1 \times D^2$, and $SL(2,\mathbb{R})$ is isomorphic to it too.



Elliptic subgroups are indicated in red, parabolic subgroups in green, and hyperbolic subgroups in blue. The dotted vertical lines are "at infinity". The white gaps are matrices with trace ¡ -2 and the slanted dotted lines are matrices with trace -2 which are not in the image of exponentiation.

0.56. **Homogeneous spaces.** The left and right actions of G on itself induce actions on the various tensor bundles associated to G as a smooth manifold, so it makes sense to say that a volume form, or a metric (or some other structure) is left-invariant, right-invariant, or bi-invariant.

Remark: Since G acts on itself transitively with trivial point stabilizers, a left-invariant tensor field is determined by its value at e, and conversely any value of the field at e can be transported around by the G action to produce a unique left-invariant field with the given value, the same with right-invariant tensor fields. The left and right actions commute, giving an action of $G \times G$ on itself; but now, the point stabilizers are conjugates of the (anti-)diagonal copy of G, acting by the adjoint representation. Therefore, the bi-invariant tensor fields are in bijection with the tensors at e fixed by the adjoint representation.

Definition 0.62. *A Riemannian metric on G is left-invariant if L_h is an isometry for $\forall h \in G$:*

$$\forall h \in G,\ \forall v, w \in T_g G, \quad g(v,w)_g = g((L_h)_* v, (L_h)_* w)_{hg}$$

Similar definition for right-invariant and if both holds then biinvariant.

It is easy to construct such metric following remark above: given an inner product on $\mathfrak{g} = T_e G$ define

$$g(v,w)_g = g((L_{g^{-1}})_* v, (L_{g^{-1}})_* w)_e$$

Fact: Any compact Lie froup has biinvariant metric(see later).

The natural extension of these concepts to k–forms is that a k- form $\omega \in \Omega^k(G)$ is left–invariant if it coincides with its pullback by left translations, i.e., $L_g^* \omega = \omega$ for all $g \in G$. Right-invariant and bi-invariant forms are analogously defined.

Once more, given any $\omega_e \in \Lambda^k T_e G$, it is possible to define a left-invariant k-form $\omega \in \Omega(G)$ by setting for all $g \in G$ and $X_i \in T_g G$,

$$\omega_g(X_1, ..., X_k) = \omega_e(d(L_{g^{-1}})_g X_1, ..., d(L_{g^{-1}})_g X_k),$$

and the right-invariant case is once more analogous.

Proposition 0.40. *There is a bijective correspondence between left-invariant (resp. right invariant) metrics on a Lie group G, and inner products on the Lie algebra $\mathfrak{g}$ of G.*

Example: An example of a bi-invariant metric on the group of orthogonal matrices with positive determinant $SO(n)$ is that inherited from $\mathbb{R}^{n \times n}$, namely the canonical metric $g(X,Y) = tr(X^T Y)$. The same happens in the unitary case, but changing the transpose for a conjugate transpose $g(X,Y) = tr(\bar{X}^T Y)$.

Remark: If metric is bi-invariant, exponentiation agrees with the exponential map and is surjective.

Definition 0.63. *A smooth manifold M admitting a transitive (smooth) G action for some Lie group G is said to be a homogeneous space for G.*

If G has a left-invariant (resp. right-invariant) metric, since left-invariant (resp. right-invariant) translations are isometries and act transitively on G, the space G is a homogeneous Riemannian manifold.

If we pick a basepoint $p \in M$ the orbit map $G \to M$ sending $g \to g(p)$ is a fibration of G over M with fibers the conjugates of the point stabilizers, which are closed subgroups H. Hence, homogeneous spaces for G are simply spaces of the form G/H for closed Lie subgroups H of G.

Definition 0.64. *An action of G on a homogeneous space $M = G/H$ is effective if the map $G \to Diff(G/H)$ has trivial kernel.*

It is immediate from the definition that the kernel is precisely equal to the normal subgroup $H_0 := \cap_g gHg^{-1}$, which may be characterized as the biggest normal subgroup of G contained in H. If H_0 is nontrivial, then we may define $G' = G/H_0$ and $H' = H/H_0$, and then $G/H = G'/H'$ is a homogeneous space for G', and the action of G on G/H factors through an action of G'. Thus when considering homogeneous spaces one may always restrict attention to homogeneous spaces with effective actions.

Proposition 0.41. ***(Invariant metrics on homogeneous spaces)***
Let G be a Lie group and let H be a closed Lie subgroup with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively.

(1) *The G-invariant tensors on the homogeneous space G/H are naturally isomorphic with the $Ad(H)$ invariant tensors on $\mathfrak{g}/\mathfrak{h}$.*
(2) *Suppose G acts effectively on G/H. Then G/H admits a G-invariant metric if and only if the closure of $Ad(H)$ in $Aut(\mathfrak{g})$ is compact.*
(3) *If G/H admits a G-invariant metric, and G acts effectively on G/H, then G admits a left-invariant metric which is also right-invariant under H, and its restriction to H is bi-invariant.*
(4) *If G is compact, then G admits a bi-invariant metric.*

For the proof of this proposition we will use the differential forms discussed in the beginning of the course.

Proof. (1) Any G-invariant tensor on G/H may be restricted to $T_H G/H = \mathfrak{g}/\mathfrak{h}$ whose stabilizer is H acting by a suitable representation of $Ad(H)$. Conversely, any $Ad(H)$- invariant tensor on $\mathfrak{g}/\mathfrak{h}$ can be transported around G/H by the left G action by choosing coset representatives.

(2) If G acts effectively on G/H, then for any left-invariant metric on G the group G embeds into the isometry group G^* and H embeds into the the isotropy group H^*, the subgroup of G^* fixing the basepoint $H \in G/H$. By example of Myers-Steenrod (see above) G^* and H^* are Lie groups, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and since G is effective, the natural maps $\mathfrak{g} \to \mathfrak{g}^*$ and $\mathfrak{h} \to \mathfrak{h}^*$ are inclusions. Since H^* is a closed subgroup of an orthogonal group of some dimension, it is compact, and therefore so is its image $Ad(H^*) \in Aut(\mathfrak{g}^*)$.

A right-invariant metric on compact group gives rise to a right-invariant volume form[36] which can be rescaled to have total volume 1. Denote by ω such a right-invariant volume form on $Ad(H^*)$. For any inner product $g(\cdot,\cdot)$ on $\mathfrak{g}^*$ define a new inner product $\tilde{g}(\cdot,\cdot)$ by

$$\tilde{g}(X,Y) := \int_{Ad(H^*)} g(Ad(h)(X), Ad(h)Y)\omega(h)$$

Note that $\tilde{g}$ is positive definite if g is so.

Then for any $z \in H^*$ we have

$$\tilde{g}(Ad(z)X, Ad(z)Y) = \int_{Ad(H^*)} g(Ad(hz)(X), Ad(hz)Y)\omega(h) =$$
$$= \int_{Ad(H^*)} g(Ad(h)(X), Ad(h)Y)R^*_{z^{-1}}\omega(hz^{-1}) = \tilde{g}(X,Y)$$

[36] i.e., $\omega \in \Omega^n(G)$ is a non zero n-form

since ω is right-invariant.

Therefore $Ad(H^*)$ (and hence $Ad(H)$) acts by isometries on $\mathfrak{g}^*$ for some positive definite inner-product. Hence the restriction of $Ad(H)$ preserves a positive definite inner-product on $\mathfrak{g}$. It imples that $Ad(H)$ is contained in the orthogonal group of this inner-product, which is compact, and therefore the closure of $Ad(H)$ is compact.

Conversely, if the closure of $Ad(H)$ is compact, by averaging any metric under a right- invariant volume form on $Ad(H)$ as above we obtain an $Ad(H)$-invariant metric on $\mathfrak{g}$. Let p be the orthogonal complement $p = \mathfrak{h}^{\perp}$ of $\mathfrak{h}$ in this $Ad(H)$-invariant metric. Then $Ad(H)$ fixes p and preserves its inner metric. Identifying $p = \mathfrak{g}/\mathfrak{h}$ we get an $Ad(H)$-invariant metric on $mathfrakg/\mathfrak{h}$ and a G-invariant metric on G/H.

(3) If G acts effectively on G/H and G/H admits a G-invariant metric, then by (2), $Ad(H)$ has compact closure in $Aut(\mathfrak{g})$, and preserves a positive-definite inner product on $\mathfrak{g}$. This inner product defines a left-invariant Riemannian metric on G as in (1), and its restriction to H is $Ad(H)$-invariant, and is therefore bi-invariant, since the stabilizer of a point in H under the $H \times H$ action coming from left- and right- multiplication is $Ad(H)$.

(4) Since G is compact, so is $Ad(G)$. Thus $Ad(G)$ admits a right-invariant volume form, and by averaging any positive-definite inner product on $\mathfrak{g}$ under the $Ad(G)$ action (with respect to this volume form) we get an $Ad(G)$-invariant metric on $\mathfrak{g}$, and a bi-invariant metric on G. □

Remark: Let us present the direct computation of (4) to make it more clear: Let ω be a right-invariant form on G and $g(\cdot,\cdot)$ a right-invariant metric. Define for all $X, Y \in T_xG$,

$$\tilde{g}(X,Y)_x = \int_G g(dL_gX, dL_gY)_{gx}\omega$$

Now we check that $\tilde{g}$ is both left- and right-invariant. Indeed,

$$\tilde{g}(dL_hX, dL_hY)_{hx} = \int_G g(dL_g(dL_hX), dL_g(dL_hY))_{g(hx)}\omega = \int_G g(dL_{gh}X, dL_{gh}Y)_{(gh)x}\omega =$$

$$= \int_G R_h^*(g(dL_gX, dL_gY)_{gx}\omega) = \int_G g(dL_gX, dL_gY)_{gx}\omega = \tilde{g}(X,Y)$$

where we used the right-invariance of the volume form ω. So we have $\tilde{g}$ to be left-invariant.

It is also right-invariant:

$$\tilde{g}(dR_hX, dR_hY)_{xh} = \int_G g(dL_g(dR_hX), dL_g(dR_hY))_{g(xh)}\omega =$$

$$= \int_G g(dR_h dL_gX, dR_h dL_gY)_{(gx)h}\omega = \int_G g(dL_gX, dL_gY)_{gx}\omega = \tilde{g}(X,Y)_x$$

Remark: For n-form ω on $\mathfrak{g}$ let us consider form $\omega_g := R^*_{g^{-1}}\omega_e$. This makes it into a right-invariant n-form on the manifold, which we call the *right Haar measure.*

We have this particular case of (1) of the Proposition above, which we will need soon

Corollary 0.13. *If an inner product on $\mathfrak{g}$ is Ad-invariant then $g(Y, adX(Z)) = -(adX(Y), Z)$, $\forall X, Y, Z \in \mathfrak{g}$. In other words, the adjoint ad^*X of the map adX with respect to the inner product is $-adX$ (it is often called skew-adjoint).*

Proof. We have that, by definition

$$adX(Y) = \frac{d}{dt}(Adexp(tX)(Y))|_0$$

so, deriving the equation

$$g(Adexp(tX)(Y), Adexp(tX)(Z)) = g(Y, Z)$$

with respect to t we get the result.

□

Remark: If the group is abelian, the construction of metric above is still valid without the need of the averaging trick, since Ad is the identity map, so every inner product is automatically Ad-invariant.

Example (Killing form):
The Killing form is the 2-form β on $\mathfrak{g}$ defined by

$$\mathfrak{g}(X, Y) = tr(ad(X)ad(Y))$$

Since the trace of a product is invariant under cyclic permutation of the factors, β is symmetric. Furthermore, for any Z, we have
$(ad(Z)(X), Y) = tr(ad([Z, X])ad(Y)) = tr([ad(Z), ad(X)]ad(Y)) = tr(ad(Z)ad(X)ad(Y) - ad(X)ad(Z)ad(Y)) = -tr(ad(X)ad(Z)ad(Y) - ad(X)ad(Y)ad(Z)) = -tr(ad(X)[ad(Z), ad(Y)]) = -tr(ad(X), ad([Z, Y])) = -\beta(X, ad(Z)(Y))$.