LECTURE 17

0.53. **Cut locus.** We know that if M is complete then $\exp_p : T_pM \to M$ is surjective. We want to model M by part of T_pM. In particular, for S^n we have $\exp_p : D_\pi(0) \to S^n \setminus \{-p\}$ is diffeomorphism.

For a geodesic $\gamma(t)$ starting at p we know that restriction of γ to $[0, t]$ is minimizing geodesic for a small t. Define

$$I = \{t_0 \geq 0 | \gamma|_{[0,t_0]} is\ minimizing\ geodesic\}$$

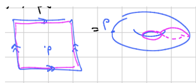
It means $t_0 = d(p, \gamma(t_0))$.

Then I is closed and if t_0 not in I and $t_1 > t_0$, then t_1 is not in I as well. Therefore, so $I = [o, T]$ for some $T > 0$ or $I = [0, \infty)$.

Definition 0.56. *The cut point of p along γ is $\gamma(T)$, if $I = [0, \infty)$, no cut point. The cut locus is $Cut(p) =$ cut points over all γ.*

Example:

- sphere: $Cut(p) = \{-p\}$
- $\mathbb{R}^2/\mathbb{Z}^2$, $Cut(p)$ is in this picture



- By Cartan-Hadamard Theorem we have that complete simply connected manifold with nonpositive curvature has empty cut locus.

Define $T(v) = T$ if $I = [0, T]$ and ∞ if $I = [0, \infty)$.

The preimage of the geodesic up to the cut point is the ray to $T(v)v$.

Define the set $U(p) = \{tv | v \in S^{n-1}, 0 \leq t \leq T(v)\} = \{t_0 v || v| = 1, \exp_p(tv)$ is minimizing past $t_0\}$

Some properties of $U(p)$: - $U(p)$ is star-shaped

- We can show that $T : S^{n-1} \to \mathbb{R}_+ \cup \{0\}$ is continuous and $U(p)$.

Proposition 0.36. *A complete M is the disjoint union $exp_pU(p) \sqcup Cut(p)$.*

Proof. For a point $q \in M$ there is a minimal geodesic from p to q by Hopf-Rinow theorem. Then either it stops being minimizing past q, then $q \in Cut(p)$, or not. Then $q \in \exp_p(U(p))$.

Now we need to prove that intersection is empty. Indeed, if $q \in \exp_p(U(p)) \cap Cut(p)$, then there is two minimizing geodesics from p to q, one of them is minimizing past q, another not. It is impossible by the following lemma.

Lemma 0.13. *If there exist two minimizing geodesics between p and q, then neither of them is minimizing past q.*

Proof. Let be γ_1, γ_2 those two geodesics with $L(\gamma_1) = L(\gamma_2)$. Extend γ_1 past q, then we can use γ_2 with cut off then to get a shorter path. □

□

Remark: This argument actually show that $\exp_p$ is an injection $U(p) \to M$ and a bijection $U(p) \to M \setminus Cut(p)$

Proposition 0.37. *$exp_p : U(p) \to M \setminus Cut(p)$ is a diffeomorphism.*

Proof. $v \in U(p) \subset T_pM \Rightarrow \gamma(t) = \exp_p(tv)$ is minimizing past $t = 1$. Then γ does not have conjugate points between p and $\exp_p((1+\epsilon)v)$, hence $\exp_p v$ isn't conjugate to p. It means that $\exp_p$ is a local diffeomorphism at v, and hence $\exp_p$ is a local diffeomorphism on $U(p)$, which is diffeomorphism since bijective. □

Recall that the injectivity radius of a Riemannian manifold (M, g) is

$$inj(M, g) = inf inj_p(M, g), p \in M,$$

where $inj_p(M, g)$ is the injectivity radius at p, defined by $inj_p(M, g) = sup\{r|\exp_p$ is diffeomorphism on $B_r(0) \subset T_pM\}$.

Now we know that $\exp_p$ is a diffeomorphism onto $M \setminus Cut(p)$, and $Cut(p)$ is closed in M. So $inj_p(M, g) = d(p, Cut(p))$ and $inj(M, g) = inf d(p, Cut(p))$.

Proposition 0.38. *Let $p \in M$, and $q \in Cut(p)$ so that $d(p, q) = d(p, Cut(p))$. Then one of the following assertions hold:*

(1) q is conjugate to p along a minimizing geodesic γ joining p to q,

(2) there exists exactly two normal minimizing geodesics γ, σ joining p to q. Moreover, in the second case, we must have $\gamma'(l) = -\sigma'(l)$, where $l = d(p, q)$.

0.54. **Sphere theorems.** Intuitively, if K is small, then inj is large

Theorem 0.28. *If $0 < a \leq K \leq K_{max}$ for some a, K_{max}, then either there exists closed geodesic γ with $inj(M, g) = \frac{1}{2}L(\gamma)$ or $inj(M, g) \geq \pi/\sqrt{K_{max}}$.*

Remark: By Myers-Bonnet we know that M is compact in that case.

Theorem 0.29. ***(Klingenberg)***

Let (M, g) be a compact Riemannian manifold whose sectional curvature satisfies $K \leq C$ for some constant C. Then either $inj(M, g) \leq \sqrt{\pi}/C$ or there exists a closed geodesic γ in M whose length is minimum among all closed geodesics, such that

$$inj(M, g) = 1/2L(\gamma).$$

We are not going to prove Klingenberg theorem in this course, but it follows from the theorem above and Proposition 0.38.

In 1926 Hopf proved that any compact simply connected Riemannian manifold with constant curvature 1 must be the standard round sphere S^n. He conjectured that any compact simply connected Riemannian manifold whose sectional curvature is close to 1 must be homeomorphic to a sphere.

Theorem 0.30. ***(Differentiable Sphere Theorem)***

Let (M, g) be a compact simply connected n-dimensional Riemannian manifold with $1/4 < K \leq 1$. Then M is diffeomorphic to S^n.

Example: The complex projective space $\mathbb{C}P^n$ with the Fubini-Study metric (suitably scaled) satisfies $1/4 \leq K \leq 1$, and it is compact, orientable and therefore simply connected (by Synge's theorem) but is not diffeomorphic to S^{2n} for $n > 1$. So the bound is sharp.

Proof. First by Bonnet-Myers' theorem, M is compact. So there exists $k > 1/4$ so that $k \leq K \leq 1$. By Klingenberg's estimate,

$$l := diam(M, g) \geq inj(M, g) \geq \pi > \frac{\pi}{2\sqrt{k}}$$

Take $p, q \in M$ such that $d(p, q) = diam(M, g)$. Let $q_0 \in M$ be an arbitrary point such that

$$l_1 := d(p, q_0) > \frac{\pi}{2\sqrt{k}}$$

and let γ_1 be a minimizing normal geodesic connecting $p = \gamma_1(0)$ to $q_0 = \gamma_1(l_1)$.

Lemma 0.14. ***(Berger)***

Let (M, g) be a compact Riemannian manifold, $p, q \in M$ such that $d(p, q) = diam(M, g)$. Then for any $X_p \in T_pM$, there exists a minimizing geodesic γ connecting $p = \gamma(0)$ to q so that $g(\gamma'(0), X_p) \geq 0$.

By this lemma one can find a minimizing normal geodesic γ_2 from $p = \gamma_2(0)$ to $q = \gamma_2(l)$ such that $g(\gamma_1'(0), \gamma_2'(0)) \geq 0$.

Then by some results which we haven't studied (Toponogov comparison theorem) we have $M = B_r(p) \cup B_r(q)$, where $r = \frac{1}{2}\left(inj(M, g) + \frac{\pi}{2\sqrt{k}}\right)$.

On the other hand since $r < inj(M, g)$, both $B_r(p)$ and $B_r(q)$ are homeomorphic to R^n. Then the sphere theorem follows from the following theorem from topology:

Theorem 0.31. ***(Brown)***

Let M be a smooth compact manifold. If $M = U_1 \cup U_2$, where U_1, U_2 are open subsets in M that are homeomorphic to $\mathbb{R}^n$, then M is homeomorphic to the sphere S^n.

□

Remark: It is natural to ask if in the conditions of the sphere theorem manifold M is diffeomorphic to S^n?

Note that the problem is totally nontrivial since there exists exotic spheres, i.e. manifolds that are homeomorphic to a sphere but not diffeomorphic to a sphere. Whether an exotic sphere admits a Riemannian metric with $K > 0$ is still an open problem.

Remark: Sphere theorem in lower dimensions: For $n = 2$: let M be an oriented compact surface with $K > 0$, then by the Gauss-Bonnet formula

$$0 < \int_M K dA = 2\pi\chi(M)$$

Since the sphere is the only oriented smooth compact surface with positive Euler characteristic ($\chi(S^2) = 2$), we conclude that M is diffeomorphic to S^2.

For $n = 3$ Hamilton showed by introducing Ricci flow method that if (M, g) is a 3-dimensional compact Riemannian manifold with $Ric > 0$, then (M, g) is diffeomorphic to S^3.

For $n = 4$ there is Chang-Gursky-Yang (2003) theorem that M is diffeomorphic to S^4 or $\mathbb{R}P^4$.