Lecture 16

0.50. **Symplectic geometry of Jacobi fields.** Consider geodesic γ and a point p on it. For two Jaxobi fields U, V define the form

$$\omega(U,V) = g(U,V') - g(U',V)$$

evaluated at the point p. This is a symplectic structure[32] on Jacobi fields along γ.

The dependence on a point p in definition does not make this form enough nice to work with. Luckily, there is no dependence at all:

$$\frac{d\omega(U,V)}{dt} = g(U,V'') + g(U',V') - g(U',V') - g(U'',V) =$$
$$= g(U,R(T,V)T) - g(R(T,U)T,V)$$

The consequence of this symplectic structure is the fact that conjugate points are isolated.

Proposition 0.35. *Let γ be a geodesic with initial point p. The set of points that are conjugate along γ to p is discrete.*

The following Lemma shows that Jacobi fields are the "most efficient" variations with given boundary data, at least on geodesic segments without conjugate points.

Lemma 0.11. ***(Index inequality)*** *Let γ be a geodesic from p to q with no conjugate points along it, and let W be a section of the normal bundle along γ with $W(p) = 0$. Let V be the unique Jacobi field with $V(p) = W(p) = 0$ and $V(q) = W(q)$. Then $I(V,V) \le I(W,W)$, and equality holds iff $V = W$.*

Proof. For simplicity, let $p = \gamma(0)$ and $q = \gamma(1)$. Let V_i be a basis of Jacobi fields along γ vanishing at p. Since W also vanishes at p and since there are no conjugate points along γ (so that the V_i are a basis throughout the interior of γ) we can write $W = \sum_i f_i V_i$, and $V = \sum_i f_i(1)V_i$. Then

$$I(W,W) = \int_0^1 g(W',W') + g(R(T,W)T,W)dt = \int_0^1 Tg(W,W') - g(W,W'') + g(R(T,W)T,W)dt =$$
$$= \int_0^1 Tg(W,W') - g(W, \sum_i f_i'' V_i + 2\sum_i f_i' V_i') =$$
$$= \int_0^1 Tg(W,W') - Tg(W,\sum_i f_i' V_i) + g(W', \sum_i f_i' V_i) - g(W, \sum_i f_i' V_i)'$$

where we used the Jacobi equation $\sum f_i V_i'' = \sum f_i R(T,V_i)T$.

Also

$$\int_0^1 Tg(W, W' - \sum f_i' V_i)dt = g(W(1), \sum f_i V_i'(1)) = g(V(1), V'(1)) = I(V,V)$$

Also we have $g(V_i, V_j') = g(V_i', V_j)$ for any pair i, j. Hence,

$$g(W', \sum f_i' V_i) - g(W, \sum_i f_i' V_i') = g(\sum f_i' V_i + \sum f_i V_i', \sum f_i' V_i) -$$
$$-g(W, \sum f_i' V_i') = g(\sum f_i' V_i, \sum f_i' V_i) \ge 0$$

After integration we get $I(V,V) \le I(W,W)$ and equation holds iff $f_i' = 0$. □

[32] It is antisymmetric and nondegenerate

0.51. **Bonnet-Myers theorem.**

Theorem 0.25. ***(Bonnet-Myers)***
Let M be a complete Riemannian manifold. Suppose there is a positive constant H so that $Ric(v, v) \geq (n-1)H$ for all unit vectors v. Then every geodesic of length $\geq \pi/H$ has conjugate points. Hence the diameter of M is at most π/H ((i.e. $d(p, q) \leq \pi/H$).

We can reformulate the theorem in a nicer way: *If $Ric(v, v) \geq (n-1)\frac{1}{r^2} = Ric(S^n(r))$, then the diameter of $S^n(r)$.*

Remark: We see that the result is sharp in the sense that the n-sphere has $Ric = (n-1)g$ and $diam(S^n) = \pi$.

In fact, if equality holds in the diameter bound in the Bonnet–Myers theorem, then (M, g) is isometric to S^n with constant sectional curvature $1/r^2$ (which means radius r).

Remark: The paraboloid (for example $z = x^2 + y^2 \subset \mathbb{R}^3$) has $K > 0$ and therefore $Ric > 0$ but is certainly not compact. This is example when we dom't have positive constant H.

Example: The fundamental group of torus T^n is $\mathbb{Z}^n$, which is not finite, we see that T^n cannot have a Riemannian metric with $Ric > 0$ by Bonnet-Myers Theorem. This is new information for $n \geq 3$.

Proof. It suffices to show that $\forall p, q \in M$ and γ geodesic between p, q we have $L(\gamma) \leq \pi/\sqrt{H}$.

Let $\gamma : [0, l] \to M$ be a unit-speed geodesic[33], and e_i an orthonormal basis of perpendicular parallel fields along γ. Define vector fields $W_i := \sin(\pi t/l)e_i$ along M.

Then we compute

$$\sum_i I(W_i, W_i) = -\int_0^l g(W_i, W_i'' + R(W_i, T)T)dt =$$

$$= \int_0^l \left(\sin(\pi t/l)\right)^2 \left((n-1)\pi^2/l^2 - Ric(T, T)\right) dt$$

So if $Ric(T, T) > (n-1)H$ and $l \geq \pi/H$ then $\sum_i I(W_i, W_i) < 0$.

In particular, there is i such that $I(W_i, W_i) < 0$. Suppose γ had no conjugate points on $[0, l + \epsilon]$ then by Lemma 0.11 we could find a non-zero Jacobi field V_ϵ with conditions

$$V_\epsilon(0) = 0, \quad V_\epsilon(l + \epsilon) = W + \epsilon, \quad I(V_\epsilon, V_\epsilon) < 0$$

. Taking the limit as $\epsilon \to 0$ we obtain a Jacobi field V with $I(V, V) < 0$ and $V(0) = V(l) = 0$, which is absurd because they sit in the null space of index form.

Therefore γ has a conjugate point on $[0, l]$ if $l \geq \pi/\sqrt{H}$. Hence, the diameter is at most $\pi/\sqrt{H}$. □

Corollary 0.11. *If M satisfies the conditions of Bonnet-Myers theorem, then it is compact and $\pi_1(M)$ is finite.*

[33]That means $L(\gamma) = l$ and $|\gamma'| = 1$

Proof. Since M is bounded by Bonnet-Myers theorem and complete, then it is compact.

If $\pi : \tilde{M} \to M$ is the universal cover of M, then pull back the metric to $\tilde{M}$, and the map π is local isometry. Therefore, $\tilde{M}$ is complete too.

Passing to the universal cover does not affect the uniform lower bound on Ric, so we deduce that the universal cover is compact too, and with the same diameter bound. Hence, number of sheets is finite, and $\pi_1(M)$ is finite. □

Example: If M, N are compact, $\pi_1(M)$ is infinite, then $M \times N$ admits no Riemannian metric of positive Ricci curvature.

0.52. **Synge theorem.**

Theorem 0.26. *Let (M, g) be a compact n-dimensional Riemannian manifold with $K > 0$.*

(a) *If M is orientable and n is even, then M is simply connected.*
(b) *If n is odd, then M is orientable.*

Example: The real projective plane $\mathbb{R}P^2$ is even-dimensional and has $K > 0$ but is not simply connected (and it is not orientable), and $\mathbb{R}P^3$ is odd-dimensional and has $K > 0$ but is not simply connected.

Since $\mathbb{R}P^n$ has a metric with constant sectional curvature 1, we see that Synge's theorem tell us that $\mathbb{R}P^n$ is orientable for n odd.

Definition 0.55. *A geodesic $\gamma : [0, a] toM$ is closed (or periodic) if $\gamma(0) = \gamma(a)$ and $\gamma'(0) = \gamma'(a)$*

Proof. Let us start with *(a)*. Key to the proof is closed geodesics.

Lemma 0.12. *If M isn't simply-connected then there is a nontrivial minimizing closed geodesics (in any homotopy class*[34] *of free loops).*

Let γ be a geodesic provided by lemma. Say $\gamma(0) = \gamma(a) = p$. Parallel transport along γ gives the map $P : T_pM \to T_pM$ orientation preserving, and $P(\gamma'(0) = \gamma'(0)$.

Consider $T_p^{\perp}M$, orthogonal complement to $\gamma'(0)$ in T_pM. Since P is preserving $\gamma'(0)$ then it is acting there, so $P \in SO(n-1)$[35]. Hence, there is a vector v with $P(v) = v$ because if the dimension of T_pM is even then the dimension of $T_p^{\perp}M$ is odd.

Now let $V(t)$ be the parallel vector field along γ with $V(0) = V(a) = v$.

Thus for the variation γ_s of $\gamma = \gamma_0$ whose variation field is V, we have

$$E''(0) = g(\nabla_V V, T) - \int_0^a (g(\nabla_T V, \nabla_T V) - R(V, T, T, V)) = -\int_0^a (-R(V, T, T, V)) < 0$$

where $T = \gamma'$ and the red term vanishes since γ is periodic. The last inequality follows from the fact that γ has strictly positive sectional curvature.

This contradicts with the fact that γ is minimum in its homotopy class.

Let us prove *(b)* now. Suppose M is not orientable, then there is a nontrivial free homotopy class $\mathcal{C}$ so that for any closed curve $\gamma : [0, 1] \to M$ in $\mathcal{C}$, $detP^{\gamma}_{0,1} = -1$. We will take γ to be the one with minimal length in this class. Since $P(\gamma'(0)) = \gamma'(0)$,

[34] I don't give the precise definition here, literally it is meant that there is a variation map (called homotopy) which deform one curve into another

[35] We need P to have positive determinant since we already know that $P \in O(n)$

we have $detP^{\gamma}_{0,1}|_E = -1$, where $E = (\gamma'(0))^{\perp}$, orthogonal complement of $\gamma'(0)$ in T_pM. Since E is even dimensional, again we conclude that there exists $X_p \in E$ so that $P^{\gamma}(X_p) = X_p$.

Now by the same argument of the proof of Synge theorem we conclude that γ is not minimum in its homotopy class, a contradiction. □

Corollary 0.12. *If $((M, g)$ is a compact even dimensional Riemannian manifold of positive sectional curvature, and M is not orientable, then $\pi_1(M) = \mathbb{Z}_2$.*

Proof. Let $\bar{M}$ be the orientable double covering of M, endowed with the induced pull-back metric. Then $\bar{M}$ is orientable and satisfies all the conditions of Synge theorem. Then it follows that $\bar{M}$ is simply connected and therefore $\pi_1(M) = \mathbb{Z}_2$. □

Example: $\mathbb{R}P^2 \times \mathbb{R}P^2$ admits no metric of positive sectional curvature since $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2) = (\mathbb{Z}/2)^2$.

Remark: Recall that it is still unknown whether $S^2 \times S^2$ admits a positive sectional curvature metric: that is the *Hopf's conjecture*.

Remark: In the odd dimensional case we cannot say too much of its fundamental group. In fact, it is well-known that S^{2n+1} can be the universal covering space of a lot of spaces of constant curvature 1.

We can also study the fundamental group of negative curved manifolds.

Theorem 0.27. ***(Preissman)***. *Let (M, g) be a compact Riemannian manifold with negative sectional curvature, and let $\{1\} \neq H \subset \pi_1(M)$ be a nontrivial abelian subgroup of the fundamental group. Then H is infinite cyclic.*

Remarks:

(1) Recall: a cyclic group is a group generated by one element.
(2) An an immediate consequence, we see that manifolds like $T^n, \mathbb{R}P^n$ admits no metric of negative sectional curvature.
(3) The theorem was strengthened by Byers to: Under the same assumption, any nontrivial solvable subgroup of $\pi_1(M)$ is infinite cyclic.
(4) For any closed surface M_g of genus $g \geq 2$, there is Riemannian metric of constant negative sectional curvature.

    The fundamental group of M_g is

$$\langle a_1, b_1, ..., a_g, b_g | a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1} = e \rangle$$

    This group is not abelian, while all its abelian subgroups are isomorphic to $\mathbb{Z}$.