Lecture 15

We will study the variation of Energy functional today, in particular, the second variation formula for the second derivative upon the variation.

0.47. **Variation of Energy.** While studying Jacobi fields we have learn that they correspond to variations, in particular, starting with Jacobi field we can construct a variation. We can do that in general too!

Proposition 0.31. *Let $V(t)$ be a smooth vector field along γ, then there is variation $F(s,t)$ such that V is its variational field. Moreover, if $V(a) = V(b) = 0$, then there exists a proper variation.*

Proof. 1. Using properties of exponentional map we have: for $\forall t \in [a,b]$ there exists $e(t)$ such that $s \mapsto \exp_{\gamma(t)}$ is defined for $|s| < e(t)$.
2. By compactness there exist an universal e such that $F(s,t) = \exp_{\gamma(t)}(sV(t))$ is defined on $(-e,e) \times [a,b]$
3. It is smooth since the geodesic flow is smooth.
4. $\frac{\partial F}{\partial s}(0,t) = V(t)$ since $F(\cdot,t)$ is geodesic through $\gamma(t)$ with initial velocity $V(t)$.
5. If $V(a) = 0$ then $F(s,a) = \gamma(a)$. $\square$

Our idea now: Consider the map $E : P(p,q) \to \mathbb{R}$, we want to find minimum. To get this we are going to calculate the first and second derivatives.

Proposition 0.32. ***(First variation formula)***
Let $F(s,t)$ be a variation of $\gamma : [a,b] \to M$ with variational field $V(t)$. Then

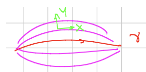
$$\frac{d}{ds}E(\gamma_s)|_{s=0} = g(V(t),\gamma'(t))|_{t=a}^{t=b} - \int_a^b g(V(t),\nabla_{\gamma'(t)}\gamma'(t))dt$$

Remark: We don't require γ_s to be geodesics so $\nabla_T T$ not necessarily zero then. If it is geodesic then the second term in the equation above vanishes.

Proof. Let us consider two vector fields:

$$Y(s,t) = \frac{\partial F}{\partial s}, \quad X(s,t) = \frac{\partial F}{\partial t}$$

such that $Y(0,t) \equiv V(t)$, $X(0,t) = \gamma'(t)$, $X(s,t) = \gamma_s'(t)$



Then by definition of energy functional we have $E(s) = 1/2\int_a^b g(X,X)dt$. Direct calculation would give $E'(s) = \frac{1}{2}\int_a^b \frac{\partial}{\partial s}g(X,X)dt = \int_a^b g(\nabla_Y X, X)dt = \int_a^b g(\nabla_X Y, X)dt$ where the last equation follows since X, Y commutes.

Continuing computation we have

$$E'(s) = \int_a^b \frac{\partial}{\partial t}g(Y,X) - g(Y,\nabla_X X) = g(Y,X)|_{t=a}^{t=b} - \int_a^b g(Y,\nabla_X X)dt$$

If we plug $s = 0$ we get the proposition. $\square$

Corollary 0.7. *For proper variations we have*

$$E'(0) = -\int_a^b g(V(t),\nabla_{\gamma'(t)}\gamma'(t))dt$$

First we have the following corollary.

Corollary 0.8. *γ is geodesic is equivalent $\forall$ proper variations of γ $E'(0) = 0$.*

Proof. The assertion from left to the right follows from the remark above and Proposition 0.32.

On contrary, if $E'(0) = 0$ then $\int_a^b g(V(t), \nabla_{\gamma'}\gamma')dt = 0$. Choose $V(t) = f(t)\gamma'(t)$ where $f(a) = f(b) = 0, f > 0$, then $\int_a^b f(t)|\nabla_{\gamma'}\gamma'|^2 = 0 \Rightarrow \nabla_{\gamma'}\gamma' = 0$, so γ is geodesic! $\square$

Now we know that geodesics are critical points of Energy functional E, to determine if they are minimizing we need second derivative.

0.48. **Second variation formula.**

Proposition 0.33. ***(Second variation formula)***
Let $F(s,t)$ be a variation of geodesic $\gamma : [a,b] \to M$. Then

$$E"(0) = g(\nabla_V V(t), \gamma')|_{t=a}^{t=b} + \int_a^b (g(\nabla_T V, \nabla_T V) - R(V,T,T,V))$$

Proof. As in the proof of the first variation:

$$\frac{1}{2}\frac{\partial}{\partial s} g(X,X) = g(\nabla_X Y, X)$$

Then

$$\frac{1}{2}\frac{\partial^2}{\partial s^2} g(X,X) = g(\nabla_Y\nabla_X Y, X) + g(\nabla_X Y, \nabla_X Y)$$

We have for the red term $\nabla_Y\nabla_X Y = \nabla_X\nabla_Y Y - R(X,Y)Y$ and for the blue one we have $\nabla_X Y = \nabla_Y X$. Hence,

$$\frac{1}{2}\frac{\partial^2}{\partial s^2} g(X,X) = g(\nabla_X\nabla_Y Y, X) - R(X,Y,Y,X) + g(\nabla_T V, \nabla_T V)$$

The red term is equal to $\frac{\partial}{\partial t} g(\nabla_Y Y, X) - g(\nabla_Y Y, \nabla_X X)$, the last term vanishes since γ is geodesic.

Therefore,

$$\frac{1}{2}\frac{\partial}{\partial s} g(X,X)|_{s=0} = \frac{\partial}{\partial t} g(\nabla_Y Y, \gamma') - R(\gamma', V, V, \gamma') + g(\nabla_T V, \nabla_T V)$$

and integrate from a to b. $\square$

Remark: If we elaborate $\frac{1}{2}\frac{\partial^2}{\partial s^2} g(X,X) = g(\nabla_X\nabla_Y Y, X) - R(X,Y,Y,X) + g(\nabla_X Y, \nabla_X Y) = \frac{\partial}{\partial t} g(\nabla_Y Y, X) - R(Y,X,X,Y) + -g(\nabla_X\nabla_X Y, Y) + \frac{\partial}{\partial t} g(\nabla_X Y, Y)$ we have for $s = 0$

$$E''(0) = (g(\nabla_V V, \gamma') + g(\nabla_T V, V))|_{t=a}^{t=b} - \int_a^b (R(V,\gamma',\gamma',V) + g(\nabla_T\nabla_T V, V))dt$$

Corollary 0.9. *If F is proper variation then*

$$E''(0) = \int_a^b (g(\nabla_T V, \nabla_T V) - R(V,T,T,V))\, dt$$

Remark: If γ is a local minimum for E then $E''(0) \geq 0$ for any V.

We can prove the Proposition 0.33 for the 2-parameter variation $F(v,w,t)$. Let $\gamma_{v,w}$ be a restriction of $F(v,w,t)$ to the

Proposition 0.34. *For $|v|, |w|$ small, let $\gamma_{v,w}$ be a 2-parameter variation of geodesic $\gamma : [a, b] \to M$. We denote $\gamma'_{v,w}$ by T and let be V, W two vector fields tangent to the variations. Then there is a formula*

$$\frac{d^2}{dvdw}E(\gamma_{v,w})|_{v=w=0} = g(\nabla_W V, T)|_a^b + \int_z^b g(\nabla_T V, \nabla_T W) - g(R(W, T)T, V) - Tg(V, T)Tg(W, T)dt$$

The proof is analogous to the one we had above. If either V or W vanishes at endpoints (any variation with endpoints fixed can be reparameterized to be perpendicular) then the first term drops. The resulting formula is the same as in Corollary 0.9:

$$\frac{d^2}{dvdw}E(0) = \int_a^b (g(\nabla_T V, \nabla_T W) - R(W, T, T, V))\, dt$$

0.49. **Index form and conjugate points.**

Definition 0.54. *(Index form). Let $\mathcal{V} := V(\gamma)$ denote the space of smooth vector fields along γ which are everywhere perpendicular to γ and $\mathcal{V}_0$ the subspace of perpendicular vector fields along γ that vanish at the endpoints. The index form is the symmetric bilinear form I on $\mathcal{V}$ is defined by*

$$I(V, W) := \int_a^b (g(\nabla_T V, \nabla_T W) - R(W, T, T, V))\, dt$$

Remark: The index form is symmetric and as in the remark after Proposition 0.33 we have

$$I(V, W) = g(\nabla_T V, W)|_a^b - \int_a^b g(\nabla_T \nabla_T V - R(T, V)T, W)$$

From the Second variation formula we have the following

Corollary 0.10. *Suppose I is positive definite on $\mathcal{V}_0$. Then γ is a unique local minimum for length among smooth curves joining p to q. More generally, the null space of I on $\mathcal{V}_0$ is exactly the set of Jacobi fields along γ which vanish at the endpoints.*

Proof. It follows from the Second variation formula using Jacobi equation: γ is a unique local minimum for length among smooth curves joining p to q.

Let V, W both vanish at the endpoints, then V is in the null space of I iff V is a Jacobi vector field. □

Remark: If V is Jacobi vector field with $V(a) = V(b) = 0$ then points $\gamma(a), \gamma(b)$ are conjugate along γ. In opposite if the endpoints are conjugate then I has a non-trivial null space. The dimension of it is the dimension of the null space of $d\exp$ at the relevant point.