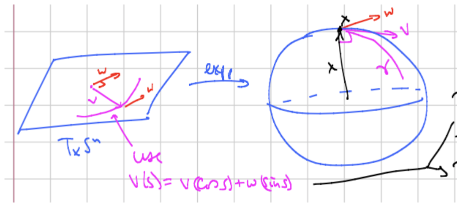
Lecture 14

Example:
We can use this to calculate section curvature for (S^n, round metric).



Fix a point x in a sphere, and two vectors $v, w \in T_x S^n$ which form an orthonormal basis. Let use pararametrization $v(s) = v\cos s + w \sin s$ for the curve in $T_x S^n$, and $\gamma(t) = x\cos t + v \sin t$ for the parametrization of the geodesic through x in the direction of v. Then $F(s,t) = \gamma_s(t) = x\cos t + (v\cos s + w \sin s)\sin t$ is the family of geodesics.

Then,

$$J(t) = \frac{\partial F}{\partial s}(0,t) = w \sin t$$

Therefore, $|J(t)|^2 = \sin^2 t = (t - \frac{t^3}{6} + ...)^2 = t^2 - \frac{t^4}{3} + ...$ It means that $K(\sigma(v,w)) = 1$. And for the sphere of radius R we have $K(\sigma(v,w)) = 1/R^2$.

Example:

Let us study Jacobi fields on manifolds of constant curvature., let J be a Jacobi field along the geodesic γ, normal to γ'. Using proposition 0.25 we have

$$g(R(\gamma', J)\gamma', W) = K\left(g(\gamma', W)g(J, \gamma') - g(\gamma', \gamma')g(J, W)\right) = -Kg(J, W) = -g(KJ, W)$$

for any vector field W.

Therefore, the equation of geodesics is

$$\frac{D^2 J}{dt^2} + KJ = 0$$

where we mean $\nabla_T \nabla_T J(T)$ as $\frac{D^2 J}{dt^2}$.

Now let $w(t)$ be a parallel field along γ with $g(\gamma', w(t)) = 0$, and $|w(t)| = 1$, then

$$J(t) = \begin{cases} \frac{\sin(t\sqrt{K}}{\sqrt{K}} w(t), & \text{if } K > 0, \\ t w(t), & \text{if } K = 0, \\ \frac{\sin(t\sqrt{-K}}{\sqrt{-K}} w(t), & \text{if } K < 0 \end{cases} \tag{1}$$

is a solution for Jacobi equation with initial conditions $J(0) = 0$ and $J'(0) = w(0)$. This can be easily verified, for example in the case $K > 0$ we have

$$\frac{D^2 J}{dt^2} + KJ(t) = \frac{D}{dt}\left(\cos(t\sqrt{K})w(t) + \frac{\sin(t\sqrt{K})}{\sqrt{K}} \frac{D}{dt} w(t)\right) + K\frac{\sin(t\sqrt{K})}{\sqrt{K}} w(t) =$$

$$= -\sqrt{K}\sin(t\sqrt{K})w(t) + \sqrt{K}\sin(t\sqrt{K})w(t) = 0$$

The red term is zero since $w(t)$ be a parallel field along γ.

Example: Let $F(s,t)$ $=\exp_p tv(s)$ be a parametrized surface (we assume everything is defined) and $v(s)$ is a curve in T_pM with $|v(s)| = 1,\ v(0) = v$ and $v'(0) = w,\ |w| = 1$.

First observation is that the rays $t \mapsto tv(s)$ starting from the origin $0 \in T_pM$ deviate from the ray $t \mapsto tv(0)$ with the velocity

$$|\frac{\partial}{\partial s}tv(s)(0)| = |t\frac{\partial}{\partial s}v(s)(0)| = |tv'(0)| = |tw| = t$$

On the other hand Equation (from Proposition 0.30 says that the geodesics $t \mapsto \exp_p(tv(s))$ deviate from the geodesic $\gamma(t)$ $=\exp_p tv(0)$ with a velocity that differs from t by a term of the third order of t given by $-\frac{1}{6}K(p,\sigma)t^3$.

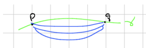
0.45. **Conjugate points and the Cartan-Hadamard Theorem.**

Definition 0.44. *(Conjugate points). Let $p \in M$, and let $v \in T_pM$. We say $q :=exp_p(v)$ is conjugate to p along the geodesic γ_v if $dexp_p(v) : T_vT_pM \to T_qM$ does not have full rank.*

It means that v is a critical point of $\exp_p$.

Lemma 0.8. *Let $\gamma : [0,1] \to M$ be a geodesic. The points $\gamma(0)$ and $\gamma(1)$ are conjugate along γ if and only if there exists a non-zero Jacobi field J along γ which vanishes at the endpoints.*

Remark: We can use this lemma 0.8 as a definition of conjugate points.



Proof. Let $w \in T_vT_pM$ be in the kernel of $d\exp_p(v)$, and by abuse of notation, use w also to denote the corresponding vector in T_pM. Define $F(s,t) :=\exp_p((v + sw)t)$. Then $dF(\partial_s) = d(\exp_p)_{tv}(tw)$ is a Jacobi field $J(t)$ along γ_v which vanishes at $p = \gamma_v(0)$ and $q = \gamma_v(v)$.

Conversely, suppose J is a nonzero Jacobi field along γ with $J(0) = J(1) = 0$. Then if we define $F(s,t) :=\exp_{\gamma(0)}((\gamma'(0) + sJ'(0))t)$,then $J = dF(\partial_s)$,and $d\exp_p(\gamma'(0))(J'(0)) = J(1) = 0$. □

Remark: It follows that the definition of conjugacy is symmetric in p and q.

Definition 0.45. *The dimension of the vector space $\{J(t_0) = J(t_1) = 0, J$ Jacobi vector field $\}$ is the multiplicity of the conjugate point.*

Example: On a sphere antipodal points are conjugate points along any geodesic with multiplicity $n-1$. Indeed, we know that the sphere has constant sectional curvature 1. As we have seen in the last example of the previous subsection the Jacobi equation is then of the form $\nabla_T\nabla_T J + J = 0$ and for every geodesic γ of S^n we know that $J(t) = (\sin t)w(t)$ with $w(t)$ being a parallel field along γ with $g(\gamma'(t), w(t)) = 0$ and $|w(t)| = 1$ is a Jacobi field along γ. Hence, we have

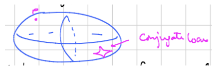
$$J(0) = (\sin 0)w(0) = 0 = (\sin\pi)w(\pi) = J(\pi)$$

As T_pS^n has dimension n we can choose $n-1$ linearly independent parallel fields $w(t)$ along γ satisfying the required conditions. Hence $\gamma(\pi)$ is a conjugate point of multiplicity $n-1$.

Definition 0.46. *The set of (first)*[31] *conjugate points to the point $p \in M$ for all geodesics that start at p is called the conjugate locus of p and is denoted by $C(p)$.*

Examples:

- The conjugate locus of $p \in S^n$ is $-p$.
- For the ellipsoid conjugate locus is bigger



- $\mathbb{R}^n$ has no conjugate points since Jacobi fields satisfy $J''(t) = 0$

Remark: The Jacobi equation and Lemma 0.8 together let us use curvature to control the existence and location of conjugate points (and vice versa). One important example of this interaction is the Cartan-Hadamard Theorem:

Theorem 0.24. *Let M be complete and connected, and suppose the sectional curvature satisfies $K \leq 0$ (nonpositive) everywhere. Then exp is nonsingular, and therefore $exp_p : T_pM \to M$ is a covering map. Hence (in particular), the universal cover of M is diffeomorphic to $\mathbb{R}^n$, and $\pi_i(M) = 0$ for all $i > 1$.*

Remark: We haven't defined homotopy groups so we will not go further deep on it. In the case of simply-connected manifolds we have the map $\exp_p : \mathbb{R}^n \to M$ to be isomorphism.

Definition 0.47. *A smooth map $f : M \to N$ is said to be a smooth covering map if for any $q \in N$, there is a neighborhood V of q in N and disjoint open subsets U_α of M so that $f^{-1}(V) = \cap_\alpha U_\alpha$, and for each $\alpha, f : U_\alpha \to V$ is a diffeomorphism.*

Remark: If f is a covering map, then $dimM = dimN$ and f is surjective.
Before we proceed to the proof, let us give one more definition.

Definition 0.48. *Let M be a topological space. A covering space of M is a topological space $\tilde{M}$ together with a continuous surjective map $\pi : \tilde{M} \to M$ such that for every $p \in M$, there exists an open neighborhood U of p, such that $\pi^{-1}(U)$ is a union of disjoint open sets in $\tilde{M}$, each of which is mapped homeomorphically onto U by π.*

The name is inspired by the following property: *universal cover (of the space M) covers any connected cover (of the space M).*
Moreover, universal cover exists for any manifold.
Example: $\mathbb{R}$ is a universal cover of S^1.

Proof. The crucial observation is that the condition $K \leq 0$ implies that for J a Jacobi field along a geodesic γ, the length squared $g(J, J)$ is convex along γ.

Lemma 0.9. *Let M be a complete Riemannian with nonpositive sectional curvature. Then*

(1) *$C(p) = \emptyset$ for $\forall p \in M$.*
(2) *exp_p is a local diffeomorphism.*

Proof. Let us consider the Jacobi field $J(t)$ along γ with $\gamma(0) = p,\ \gamma(t_0) = q,\ J(0) = J(t_0) = 0$.

[31] First means no conjugate points before it.

Then

$$\frac{d}{dt}g(J(t),J(t)) = 2g(J',J)$$

$$\frac{d^2}{dt^2}g(J,J) = 2g(J',J') + 2g(J'',J) = 2|J'|^2 - 2g(R(J,T)T,J) =$$

$$= 2|J'|^2 - 2K(\sigma)|\gamma' \wedge J|^2 \geq 0$$

where σ is 2-plane generated by γ', J. However, if $J(t)$ has two zeros then $|J|^2$ has a max, where $\frac{d}{dt}g(J(t),J(t)) \leq 0$. That's a contradiction.

Moreover, it follows that $d\exp$ is nonsingular at every point, and $\exp_p : T_pM \to M$ is an immersion. $\square$

Lemma 0.10. *Let M, N be Riemannian manifolds, M (geodesically) complete and $f : M \to N$ surjective local isometry (in particular, local diffeomorphism). Then f is a covering map.*

Proof. Let $p \in N$ and $f^{-1}(p) = \{p_i\}$ be its preimages., and let $B_r(p)$ be a ball in N such that $\exp_p : V_r(0) \to B_r(p)$ is diffeomorphism.

Denote by $U := B_r(p)$ and $U_i = \exp_{p_i}(B_r(0)) \subset M$, they exist since M is complete.

We claim that $f^{-1}(U)$ is disjoint union of U_i and $f : U_i \to U$ is diffeomorphism. Indeed, first we prove that $f(U_i) \subset U$. For any point $q \in U_i$ there is geodesic γ which connects p_i and q with $L(\gamma) < r$. Since f is local isometry then $f \cdot \gamma$ is geodesic from p to $f(q)$, and $L(f \cdot \gamma) < r$, it means $f(q) \in U$, hence, $f(U_i) \subset U$. For the second assertion of a claim let us consider the following commuting diagram

$$\begin{array}{cccc} T_{p_i}M \supset & B_r(0) & \xrightarrow{\exp_{p_i}} & U_i \\ & \downarrow^{f_*} & & \downarrow^{f} \\ T_pN \supset & B_r(0) & \xrightarrow{\exp_p} & U \end{array}$$

Since f_* and $\exp_p$ are diffeomorphism, and $\exp_{p_i}$ is injective, therefore it is also bijective, which makes f bijective too. Hence, both $\exp_{p_i}$ and f are diffeomorphisms.

Now we will prove that $f^{-1}(U)$ is disjoint union of U_i. Suppose $\tilde{q} \in f^{-1}(U)$, $q = f(\tilde{q})$. Consider geodesic γ from p to q such that we have geodesic $\bar{\gamma}$ from q to p and let $v = \bar{\gamma}'(0)$. Then there exists geodesics $\tilde{\gamma}$ with $\tilde{\gamma}(0) = \tilde{q}$ and $\tilde{\gamma}'(0) = (f_*)^{-1}(v)$. Then $f \cdot \tilde{\gamma} = \bar{\gamma}$. Indeed, since $df_{\tilde{q}}T_qM \to T_qN$ is a linear isometry, one can find a unique $X_{\tilde{q}} \in T_{\tilde{q}}M$ so that $df_{\tilde{q}}(X_{\tilde{q}}) = \bar{\gamma}'(0)$. By completeness of M there exist geodesics $\tilde{\gamma}$ with required initial data. Then (We used completeness here!) Then $f \cdot \tilde{\gamma}$ is a geodesic in N with same initial conditions as $\bar{\gamma}$, so they coincide. Moreover, because $df_{\tilde{q}}T_qM \to T_qN$ is a linear isometry the lift $\tilde{\gamma}$ is unique.

Then the endpoint of geodesic $\tilde{\gamma}$ is p_i for some i. Hence, $\tilde{q} \in U_i$.

If there are two geodesics $\tilde{\gamma_1}, \tilde{\gamma_2}$ from point $\tilde{q}$ to some p_1, p_2, then they must project to the same geodesics from q to p by its uniqueness property.

□

The map $\exp_p$ is well-defined, surjective and by the first lemma it is local diffeomorphism. Then the Riemannian metric on M pulls back to a Riemannian metric on T_pM, i.e. $\exp_p$ is a local isometry. So radial lines tv through the origin are geodesics since they map to geodesics. Thus, by the Hopf-Rinow Theorem 0.19 the metric on T_pM is complete, and therefore $\exp_p$ is a covering map by the second lemma. □

Remark: The second lemma is known as *Ambrose theorem.*

Corollary 0.6. *Let (M, g) be a complete simply connected flat manifold. Then (M, g) is isometric to $(\mathbb{R}^n, g_0)$.*

Proof. Choose any p. Identify T_pM with $\mathbb{R}^n$ as usual and let $\bar{g} = \exp_p^* g$ on $\mathbb{R}^n$. We have already proved that the map $\exp_p : (\mathbb{R}^n, \bar{g}) \to (M, g)$ is both a diffeomorphism and a local isometry. So it is a global isometry. Since g is flat, $\bar{g}$ is then a flat metric on $\mathbb{R}^n$. But two flat metrics on $\mathbb{R}^n$ differ only by a linear isomorphism So the conclusion follows. □

Definition 0.49. *A complete simply-connected Riemannian manifold with non-positive curvature is called a Cartan-Hadamard manifold, or an Hadamard manifold.*

0.46. **Energy and variations (recap).** First recall about the energy.

Consider $p, q \in M$ and let $P(p, q)$ be piecewise differentiable paths from p to q, length gives a map $L : P(p, q) \to \mathbb{R}_{\geq 0}$ given by

$$L(\gamma) = \int_a^b g(\gamma'(t), \gamma'(t))^{1/2} dt$$

Minimizing geodesic is a global minimum of L, i.e. critical point of map L.

Let us recall and also summarized few thing we have proved before:

- if γ is length-minimizing curve from p to q then $L(\gamma) \leq L(\tilde{\gamma}), \forall \tilde{\gamma} \in P(p, q)$, and equation holds only if γ is reparametrization of a geodesic.
- We can introduce another functional:
$$E(\gamma) = \frac{1}{2} \int_a^b g(\gamma', \gamma') dt$$
as we did in the Lecture 8 (subsection 0.31).
- Proposition 0.17 says that for length-minimizing geodesic γ $E(\gamma) \leq E(\tilde{\gamma})$ for any $\tilde{\gamma}$ and equation holds iff $\tilde{\gamma}$ is length-minimizing (i.e. $L(\gamma) = L(\tilde{\gamma})$ and $\tilde{\gamma}$ has constant speed).

Types of variations:

- *2-parameter variation*

  Definition 0.50. *A 2-parameter variation of $\gamma : [a, b] \to M$ is a smooth map $F(v, w, t) : (-\epsilon, \epsilon) \times (-\delta, \delta) \times [a, b] \to M$ for some $\epsilon > 0$ with $F(0, 0, t) = \gamma(t)$.*

- *Variation*

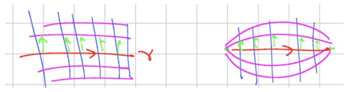
Definition 0.51. *A variation of* $\gamma : [a, b] \to M$ *is a smooth map* $F : (-\epsilon, \epsilon) \times [a, b] \to M$ *for some* $\epsilon > 0$ *with* $F(0, t) = \gamma(t)$.

- *Geodesic variation*

  Definition 0.52. *A geodesic variation* $F(s, t)$ *is a variation such that* $\gamma_s(t) := F(s, t)$ *is a geodesics for any* s.

- *Proper variation*

  Definition 0.53. *A proper variation is a variation with* $F(s, a) = \gamma(a), F(s, b) = \gamma(b)$



For any given variation we have associated vector field:

$$F(s, t) \rightsquigarrow V(t) = \frac{\partial F}{\partial s}(0, t) \equiv dF(\partial s)$$

which is called *variational field*.

We have another field which is tangential to γ: $T := dF(\partial_t)$. Both are vector fields near γ.

Remark: If we have 2-parameter variation $F(v, w, t)$ of γ then we denote $T := dF(\partial_t), V := dF(\partial_v), W := dF(\partial_w)$ (we have named coordinates here accordingly to vector fields).

We will usually work with variation $F(s, t)$.

Remark: Proper variation means that variational field has $V(a) = V(b) = 0$.

Remark: Jacobi field is a variation field for the geodesic variation.