Lecture 13

I will continue to talk about Jacobi fields, in particular we will see that we can construct variation of geodesic starting from Jacobi vector field.

0.44. **Jacobi fields as deformations.** Coming from the last example of the previous lecture one can wonder if $J(t) = t^2\gamma'$ is a Jacobi field. It is not since

$$\nabla_T \nabla_T (t^2\gamma') = \nabla_T(2t\gamma') = 2\gamma' \neq 0.$$

Proposition 0.29. *The following two are equivalent:*

- $J(0), J'(0) \perp \gamma'(0)$
- $J(t) \perp \gamma'(t),\ \forall t$

Proof. Let us write $|\gamma'(t)| = a$, then $e_1(t) = \frac{\gamma'(t)}{a}$.

Then $\langle J(t), \gamma'(t)\rangle = af'(t),\ R(\gamma', e_j)\gamma', e_i) = 0$. Hence $\frac{d^2 f'}{dt^2} = 0$. Indeed,

$$\frac{d^2}{dt^2} g(\gamma'(t), J) = \frac{d}{dt} g(\gamma', J') = g(\gamma', J'') = g(\gamma', R(\gamma', J)\gamma') = 0$$

It means $\langle J(t), \gamma'(t)\rangle = At + B$.

When $t = 0$ we have $B = \langle J(0), \gamma'(0)\rangle,\ A = \frac{d}{dt}|_{t=0}\langle J(t), \gamma'(t)\rangle = \langle J'(0), \gamma'(0)\rangle$. □

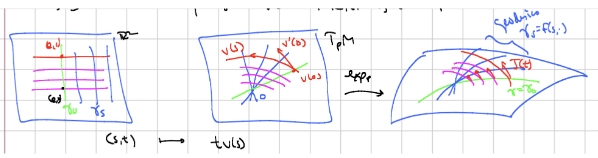
This shows that the tangential part of a Jacobi field is of the form $(At+B)\gamma'$ for some constants aA, B and therefore one may as well restrict attention to normal Jacobi fields. In particular, we deduce that if a Jacobi field J vanishes at two points on γ (or more), then J and J' are everywhere perpendicular to γ.

Remark: We have the following supspaces of Jacobi vector fields:

dim	without restrictions	J is orthogonal to γ
$J(0)$ arbitrary	$2n$	$2n-2$
$J(0) = 0$	n	$n-1$

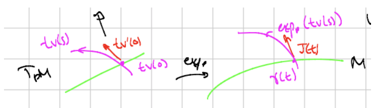
where J_1, J_2 are not orthogonal to γ and $J(0) = 0$ corresponds to the situation when all geodesics fo through p.

We can easily see why the space of Jacobi fields with $J(0) = 0$ is n dimensional. Let us consider a family $\gamma_s(t)$ with $\gamma_s(0) = p$. Each γ_s is determined by $\gamma_s' = v(s)$ in the following way $F(s,t) = \gamma_s(t) = \exp_p(tv(s))$.



Then $J(t) = \frac{\partial F}{\partial s}(0,t) = (d(\exp_p))_{tv}(tw)$, where $v = v(0) = \gamma'(0)$ and $w = v'(0) \in T_pM$

Remark: The equation $\frac{\partial F}{\partial s}(0,t) = (d(\exp_p))_{tv}(tw)$ appeared in the proof of Gauss' Lemma 0.5.



We can compute
$\frac{d}{dt}J(0) = \frac{d}{dt}|_{t=0}(td(\exp_p))_{tv}(w) = (d\exp_p)_0(w) = w$. So, $J(0) = 0$ and $J'(0) = w$. Hence, the dimension is n.

Remark: We have $F(s,t) = \exp_p(t(T + sJ'))$.

Theorem 0.23. *The variation field of a family of geodesics is a Jacobi field. Conversely, for every Jacobi field there is a family of geodesics whose variation field is the Jacobi field.*

Proof. The first part is already known.

Consider $F(s,t) = \exp_p(tv(s)$ for a path $v(s) \in T_p(M$ with $v(0) = \gamma'(0), v'(0) = \gamma'(0)$. Then $\frac{\partial F}{\partial s}(0,t)$ is the Jacobi field which coincide with the given one because the initial conditions are the same. □

Proposition 0.30. *Let σ be plane spanned by $v = \gamma'(0)$ and $w = J'(0) = v'(0)$. We assume $J(0) = 0$. Then*

$$|J(t)|^2 = t^2 - \frac{1}{3}K(\sigma)t^4 + o(t^4)$$

Thus: in 2-planes with positive sectional curvature, radial geodesic diverge slower than in Euclidean space, whereas in 2-planes with negative sectional curvature, radial geodesics diverge faster than in Euclidean space. And that for small t the value $K(p,\sigma)t^3$ furnishes and approximation for the extent of this spread with an error of order t^3.

Proof. We compute the first few terms in the Taylor series for the function $t \to \langle J(\gamma(t)), J(\gamma(t))\rangle$ at $t = 0$ (note that $J'(t) := J'(\gamma(t)) = (\nabla_T J)(\gamma(t))$ and so on).

$$\langle J, J\rangle|_{t=0} = 0$$
$$\langle J, J\rangle'|_{t=0} = 2\langle J, J'\rangle|_{t=0} = 0$$
$$\langle J, J\rangle''|_{t=0} = 2\langle J', J'\rangle|_{t=0} + 2\langle J'', J\rangle|_{t=0} = 2||J'||^2 = 2$$
$$\langle J, J\rangle'''|_{t=0} = \langle J'', J'\rangle|_{t=0} + 2\langle J''', J\rangle|_{t=0} = 0$$

where we use the Jacobi equation to write $J'' = R(T,J)T$ which vanishes at $t = 0$ (since it is tensorial, and J vanishes at $t = 0$).

On the other hand,

$$J'''|_{t=0} = \nabla_T(R(T,J)T)|_{t=0} = (\nabla_T R)(T,J)T|_{t=0} + R(T,J')T|t = 0 = R(T,J')T|t = 0$$

where we used the Leibniz formula for covariant derivative of the contraction of the tensor R with T, J, T,and the fact that $\nabla_T T = 0$ and $J_{t=0} = 0$.

Hence

$$\langle J, J\rangle''''|_{t=0} = 8\langle J''', J'\rangle|_{t=0} + 6\langle J'', J''\rangle|_{t=0} + 2\langle J'''', J\rangle|_{t=0} = -8\langle R(v,w)v, w\rangle = -8K(\sigma)$$

where σ is the 2-plane spanned by v and w. In other words,

$$||J(t)||^2 = t^2 - \frac{1}{3}K(\sigma)t^4 + O(t^5)$$

□