Lecture 12

We will discuss properties of sectional curvature and spaces of constant curvature. After this we will start applying curvature to the deformations of geodesics.

0.40. **Sectional curvature (properties).** In more details the statement that full tensor R can be recovered from the sectional curvature is revealed by the proposition below.

Proposition 0.23. *The sectional curvature determines the Riemann curvature.*

Proof. Suppose that for all $p \in M$ and for all 2-dimensional subspaces $\sigma = Span\{X, Y\} \subset T_pM$ we have that $K = \tilde{K}$, we need to show that $R = \tilde{R}$. The condition $K = \tilde{K}$ is exactly another way to say $R(X, Y, Y, X) = \tilde{R}(X, Y, Y, X)$.

Then $R(X + Z, Y, Y, X + Z) = \tilde{R}(X + Z, Y, Y, X + Z)$ forX, Y, Z so

$$R(X, Y, Y, X)+R(Z, Y, Y, Z)+2R(X, Y, Y, Z) = \tilde{R}(X, Y, Y, X)+\tilde{R}(Z, Y, Y, Z)+2\tilde{R}(X, Y, Y, Z)$$

, using the fact that $\tilde{R}(Z, Y, Y, X) = \tilde{R}(Y, X, Z, Y) = \tilde{R}(X, Y, Y, Z)$ and the same is true for R. Thus

$$R(X, Y, Y, Z) = \tilde{R}(X, Y, Y, Z)$$

for all X, Y, Z.

Therefore, using $R(X, Y + W, Y + W, Z) = \tilde{R}(X, Y + W, Y + W, Z)$ we have

$$R(X, Y, W, Z) + R(X, W, Y, Z) = \tilde{R}(X, Y, W, Z) + \tilde{R}(X, W, Y, Z) \qquad (*)$$

and thus using the symmetry properties of $R, \tilde{R}$ we can see that $(*)$ is left invariant under the cyclic permutations of X, Y, Z.

So,

$$(R - \tilde{R})(X, Y, Z, W) = (R - \tilde{R})(Y, Z, X, W) = (R - \tilde{R})(Z, X, Y, W)$$

Then Bianchi identity means that $2(R - \tilde{R})(X, Y, Z, W) = (R - \tilde{R})(Y, Z, X, W) + (R - \tilde{R})(Z, X, Y, W) = -(R - \tilde{R})(X, Y, Z, W)$. Q.e.d $\square$

Examples:

1. For any flat manifold sectional curvature is 0.
2. For S^2, we that $T_pS^2 = Span\{X_1, X_2\}$ where $g(X_1, X_1) = 1$ and $g(X_2, X_2) = \sin^2\theta$ and $g(X_1, X_2) = 0$ so that $K(X_1, X_2) = 1$.
3. For $\mathbb{H}^2$ all sectional curvatures are -1.

Now let us connect what we are doing with the Curves and Surfaces course.

Proposition 0.24. *Let M be an oriented surface in $\mathbb{R}^3$. Then the sectional curvature $K(T_pM) = K(p)$, the Gaussian curvature of M at p.*

Recall that the eigenvalues of the second fundamental form are called *principal curvature* and Gaussian curvature is their product.

Example: For surfaces in $\mathbb{R}^3$, we see that $K(p) = K(T_pM) = Ric(E_1, E_1) = Ric(E_2, E_2)$ and thus $s(p) = Ric(E_1, E_1) + Ric(E_2, E_2) = 2K(p)$, i.e. the scalar curvature is just twice the sectional (or Gaussian) curvature.

Remark: If $X \in T_pM$ is unit vector, then *Ricci curvature* $Ric(X, X) := \sum_i R(X, E_i, X, E_i)$ is$n - 1$ times average sectional curvature of planes through X. Indeed, consider $X, E_1, ..., E_n$, where E_i are orthonormal basis, and denote $\sigma_i = \langle X, E_i\rangle$.

Then $K(\sigma_i) = R(X, e_i, X, e_i)$.

Let us parametrize planes through X by $v \in$ unit vectors perpendicular to X, what is S^{n-2}. With the usual measure on S^{n-2} we have the statement.

Definition 0.40. *We say that (M, g) has constant curvature if $K(\sigma) = K$ for any point $p \in M$ and plane $\sigma \subset T_pM$.*

0.41. **Spaces of constant curvature.** We have seen that $\mathbb{R}^n$ is flat, sphere has positive curvature and hyperbolic space has negative. In fact, this gives all basic constant curvature spaces. More generally, if M is constant curvature space and G is the group of isometries, then the quotient M/G is constant curvature space.

Proposition 0.25. *The space has constant curvature K_0 iff*

$$R(X, Y, Z, W) = K_0 \cdot (g(X, W)g(Z, Y) - g(X, Z)g(W, Y))$$

Proof. It follows from the fact that the Riemann curvature is determined by the sectional. □

We can also describe the Ricci and scalar curvatures of Riemannian manifolds with constant sectional curvature.

Proposition 0.26. *If (M, g) has constant sectional curvature K_0 then $Ric = (n-1)K_0 g$ and $S = n(n-1)K_0$.*

Proof. By the Proposition 0.25 we see that in orthonormal basis

$$Ric(X, Y) = \sum_k R(X, E_k, E_k, Y) = K_0 \sum_k (g(X, Y)g(E_k, E_k) - g(X, E_k)g(Y, E_k)) = K(n-1)g(X, Y)$$

Hence,

$$s = \sum_{i,j} R_{ijji} = K_0 \sum_{ij} (g(E_i, E_i)g(E_j, E_j) - g(E_i, E_j)^2) = K_0(n^2 - n)$$

□

Remark: Riemannian manifolds with constant sectional curvature are Einstein manifolds and have constant scalar curvature.

Examples:

1. $\mathbb{R}^n$ has constant sectional curvature 0. The same is true of $\mathbb{R}^n/\mathbb{Z}^n \simeq T^n$. So their Ricci and scalar curvatures are also 0.
2. We saw that S^2 has constant sectional curvature 1. The same is also true of $\mathbb{R}P^2$. Their Ricci curvature tensors are $Ric = g$ and scalar curvature $s = 2$.
3. Hyperbolic space has curvature -1.

   Indeed, consider $\mathbb{H}^2$ as example. It is the set of points with $x_1^2 + x_2^2 - x_3^2 = -1, x_3 > 0$ with the Riemann metric given by the restriction g of $dx_1^2 + dx_2^2 - dx_3^2$. Let us parametrize $\mathbb{H}^2$ by $f(\theta, \varphi) = (\sinh\theta \cos\varphi, \sinh\theta \sin\varphi, \cosh\theta)$ and fix two fields $X_1 = f_*\partial_\theta,\ X_2 = f_*\partial\varphi$. Then

   $$g(\cdot, \cdot) = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\theta \end{pmatrix}$$

   For the Levi-Civita connection we have

$$\nabla_{X_1} X_1 = 0,\ \nabla_{X_2} X_2 = -\sinh\theta\cosh\theta X_1,\ \nabla_{X_2} X_1 = \nabla_{X_1} X_2 = \coth\theta X_2$$

Using $[X_1, X_2] = 0$ we have

$$R(X_1, X_2)X_2 = \nabla_{X_1}(-\sinh\theta\cosh\theta X_1) - \nabla_{X_2}(\coth\theta X_2) = (-\cosh^2\theta - \sinh^2\theta)X_1 + \cosh^2\theta X_1 = -\sinh^2\theta X_1$$

Therefore, $R_{1221} = -\sinh^2\theta$. With orthonormal basis we will have the curvature equal to -1.

Remark: If (M, g) has constant sectional curvature K, we can always rescale the metric so that $K \in \{1, 0, 1\}$.

In particular, if we multiply the metric by$t$ then the sectional curvature changes by a factor of t^{-1}. So, a 2-sphere of radius r has constant sectional curvature $1/r^2$.

We know that isometries of $\mathbb{R}^n$ are given by $O(n)$ together with translations. The question is how the isometries of S^n and $\mathbb{H}^n$ look like.

In the case of sphere the following holds

Theorem 0.20. *The unit sphere S^n with the induced metric g is complete, its geodesics are the great circles, it has constant sectional curvature 1, the set of isometries is the group $O(n+1)$.*

We know first by Hopf-Rinow theorem, as well as we described geodesics. We know that $O(n+1)$ defines isometries of $\mathbb{R}^{n+1}$, and these are the only linear maps in $\mathbb{R}^{n+1}$ that preserve S^n.

It extends in a natural way to the case of $K = -1$.

Theorem 0.21. *The unit sphere $\mathbb{H}^n$ with the induced metric g is complete, its geodesics are given by $\Pi \cap \mathbb{H}^n$ for 2-planes Π through the origin which meet $\mathbb{H}^n$*[30] *(they are called Lorentz planes), it has constant sectional curvature -1, the set of isometries is the group $O^+(n,1) = \{A \in M_{n+1}(\mathbb{R}) : A^T g A = g,\ a_{n+1,n+1} > 0\}$, where*

$$g = \begin{vmatrix} I & 0 \\ 0 & -1 \end{vmatrix}$$

Proof. The proof is very similar to the one for S^n. Clearly, the isometries are as stated because $O^+(n,1)$ preserves g on $\mathbb{R}^{n+1}$ by definition and preserves $\mathbb{H}^n$. Given $p = (0, ..., 0, 1) \in \mathbb{H}^n$ and $X \in T_p\mathbb{H}^n$, let γ be the unique geodesic through p with tangent vector X. If we define $\rho \in O(n,1)$ to be the reflection in the plane $\Pi = Span\{p, X\}$, since ρ is an isometry we see that $\rho \cdot \gamma$ is another geodesic with the same properties as γ.

Thus, by uniqueness of geodesics, we have $\gamma \in \Pi \cap \mathbb{H}^n$.

Concretely, for $p = (0, ..., 0, 1)$, if we take $X = (0, 0, ..., 1, 0)$ then $\gamma(t) = (0, ..., 0, \sinh t, \cosh t)$. Clearly, these geodesics are defined for all $t \in \mathbb{R}$ so $\mathbb{H}^n$ is complete and uniqueness implies that these are all the geodesics as claimed.

We can restrict to calculating the sectional curvature of $\mathbb{H}^2$, which we know is -1, so the result follows. □

There are several models of hyberbolic n-space. The model we have been using is called the *hyperboloid model.*

[30] In the case of sphere great circles are also given by analogous intersections.

Example: There is an isometry $f : (\mathbb{H}^n, g) \to (B^n, h)$, where B^n is the interior of a unit ball in $\mathbb{R}^n$ and

$$h = \sum_i \frac{4dy_i^2}{(1 - \sum_i y_i^2)^2}$$

given by

$$f(x_1, ..., x_{n+1}) = \frac{(x_1, ..., x_n)}{1 + x_{n+1}}$$

(B^n, h) is called the *Poincare disk model.*

Another model is given by isometry $f : (\mathbb{H}^n, g) \to (H^n, h)$, where H^n is the upper half-plane and

$$h = \sum_i \frac{dz_i^2}{z_n^2}$$

given by

$$f(x_1, ..., x_{n+1}) = \frac{(x_1, ..., x_{n-1}, 1)}{x_n + x_{n+1}}$$

This (H^n, h) is called *upper-half plane model* of hyperbolic n-space.

Remark: The metric on upper half-plane is not the standard Riemannian metric.

We finish this section with this easy-to-state-hard-to-prove result

Theorem 0.22. *Let (M, g) be a complete n-dimensional Riemannian manifold with constant sectional curvature $K \in$-1,0,1. Then there exists a discrete group G acting freely and properly discontinuous by isometries such that (M, g) is isometric to*

- *S^n/G if $K = 1$*
- *$\mathbb{R}^n/G$ if $K = 0$*
- *$\mathbb{H}^n/G$ if $K = -1$.*

As a corollary we have

Proposition 0.27. *Let M be a complete $2n$-dimensional Riemannian manifold with constant sectional curvature 1. Then M is isometric to S^{2n} or $\mathbb{R}P^{2n}$ with their standard Riemannian metrics.*

Question: What about the odd dimensions?

It fails. There are Lens spaces, given by the quotient of S^3 by any cyclic group $\mathbb{Z}_k$. Also tetrahedral and other more complicated subgroups of $O(4)$ act on S^3 in the appropriate way.

Let us proposition above.

Proof. By theorem 0.22 we have that it is S^{2n}/G for some G acting freely and properly discontinuous. If f_x is the isometry corresponding $x \in G$, then $det f_x = \pm 1$.

If it is 1, then all eigenvalues are 1 (they are real since non0real comes in pairs and we are in the case of $O(2n+1)$). Therefore, f_x has a fixed point, contradiction with the assumption the action is free, so $f_x = id$.

If $det f_x = -1$, then $f_x = \pm id$. Hence, S^{2n}/G is either sphere or the projective space. $\square$

0.42. **Mean curvature.** Let N be a smooth submanifold of M. It is instructive to compare the sectional curvature of a 2-plane σ contained in T_pN in N and in M. Choose (commuting) vector fields X and Y in $\mathcal{X}(N)$.

Evidently $||X \wedge Y||^2 = ||X||^2||Y||^2$ is the same whether computed in M or in N. We compute

$$K_M(\sigma) \cdot ||X \wedge Y||^2 = K_N(\sigma) \cdot ||X \wedge Y||^2 + g(\nabla_X \nabla_Y^\perp Y - \nabla_Y \nabla_X^\perp Y, X)$$

On the other hand, $g(\nabla_X^\perp Y, Z) = 0$ for any X, Y, Z and therefore $g(\nabla_X \nabla_Y^\perp Y, X) = -g(\nabla_Y^\perp Y, \nabla_X^\perp X)$. Similarly for the other term.

Using the symmetry of the second fundamental form, we obtain the so-called Gauss equation:

$$K_N(\sigma) \cdot ||X \wedge Y||^2 = K_M(\sigma) \cdot ||X \wedge Y||^2 + g(II(X,X), II(Y,Y)) - ||II(X,Y)||^2$$

In the special case that N is codimension one and co-orientable, the normal bundle μN may be identified with the trivial line bundle $\mathbb{R} \times N$ over N, and II may be thought of as an ordinary symmetric inner product on N. Using the metric inner product on N, we may express II as a symmetric matrix, by the formula $II(X,Y) = g(II(X), Y)$.

Definition 0.41. *Let N be a codimension one co-orientable submanifold of M. If we express II as a symmetric matrix by using the metric inner product, the eigenvalues of II are the principal curvatures, the eigenvectors of II are the directions of principal curvature, and the average of the eigenvectors (i.e. $1/dim(N)$ times the trace) is the mean curvature, and is denoted H.*

Definition 0.42. *Let N be a codimension 1 co-oriented smooth submanifold of $\mathbb{E}^n$. The Gauss map is the smooth map $g : N \to S^{n-1}$, the unit sphere in $\mathbb{E}^n$, determined uniquely by the property that the oriented tangent space T_pN and $T_{g(p)}S^{n-1}$ are parallel for each $p \in N$.*

Remark: For a surface S in $\mathbb{E}^3$, the sectional curvature can be derived in a straightforward way from the geometry of the Gauss map.

Remark: Another way to think of the Gauss map is in terms of unit normals. If N is codimension 1 and co-oriented, the normal bundle μN is canonically identified with $\mathbb{R} \times N$ and has a section whose value at every point is the positive unit normal. On the other hand, μN is a subbundle of $TE^n|_N$, and the fiber at every point is canonically identified with a line through the origin in E^n. So the unit normal section σ can be thought of as taking values in the unit sphere; the map taking a point on N to its unit normal (in S^{n-1}) is the Gauss map, so by abuse of notation we can write $\sigma = g$ (in Euclidean coordinates).

Lemma 0.7. *For vectors $u, v \in T_pN$ we have $II(u,v) = -\langle dg(u), v\rangle$.*

Proof. Let us first extend u, v to vector fields U, V near p. Then $\langle \sigma, V\rangle = 0$, where σ is the unit normal field.

Then

$$\langle \nabla_U \sigma, V\rangle + \langle \sigma, \nabla_U V\rangle = 0$$

Also, $\langle \sigma, \nabla_U V\rangle = \nabla_U^\perp V = II(U,V)$ (via identifying μN with $\mathbb{R} \times N$).

Furthemore, $\nabla_U \sigma = d\sigma(U) = dg(U)$. □

Corollary 0.4. *For a smooth surface S in $\mathbb{E}^3$ the form $K \cdot darea = g^* darea$; i.e. the pullback of the area form on S under g^* is K times the area form on S, where K is the sectional curvature (thought of as a function on S).*

Proof. At each point $p \in S$ we can choose an orthonormal basis e_1, e_2 for T_pS which are eigenvectors for II. If the eigenvalues (i.e. the *principal curvatures*) are k_1, k_2 then $dg(e_i) = -k_i e_i$ and therefore the Gauss equation implies that $K_S = k_1 k_2$ at each point. But this is the determinant of dg (thought of as a map from T_pS to $T_{g(p)}S^2 = T_pS$). □

0.43. **Jacobi fields.** To get a sense of the geometric meaning of curvature it is useful to evaluate our formulas in geodesic normal coordinates. It can then be seen that the curvature measures the second order deviation of the metric from Euclidean space.

Fix some point $p \in M$ and let v, w be two vectors in T_pM. Consider for small s the 1-parameter family of rays through the origin in T_pM defined by

$$\rho_s(t) = (v + sw)t$$

Note that $\exp \cdot \rho_s$ is a geodesic through p with tangent vector at zero equal to $v + sw$. We denote it $\gamma_s(t)$.

It gives us a 1-parameter family of geodesics $\gamma_s(t),\ s \in (-\epsilon, \epsilon)$.

Let us define $F(s,t) := \gamma_s(t)$, the map from $[0,1] \times (-\epsilon, \epsilon) \to M$.

Remark: We could have started with the 1-parameter of geodesics $\gamma_s(t)$, not necessary intersecting at a point p.

Note that the infinitesimal change in geodesics is exactly $\frac{\partial}{\partial s}F(s,t)$, at $s = 0$ this is a vector field along γ_0.

Let us define T and J (at least locally in M) to be $dF(\partial_t)$ and $dF(\partial_s)$ respectively, thought of as vector fields along (the image of) F. The vector field $J(t) = \frac{\partial}{\partial s}(0,t)$ measures the "spread" of the geodesics.

For each fixed s the image $F : [0,1] \times s \to M$ is a radial geodesic through p, so $\nabla_T T = 0$ throughout the image.

Since T and J commute, we have $[T,J] = 0$ and $\nabla_T J = \nabla_J T$ and we obtain the identity $R(T,J)T = \nabla_T \nabla_V T = \nabla_T \nabla_T J$.

Definition 0.43. *(Jacobi equation). Let J be a vector field along a geodesic γ, and let $\gamma' = T$ along γ. The Jacobi equation is the equation*

$$R(T,J)T = \nabla_T \nabla_T J \quad (\textit{Jacobi equation})$$

for J , and a solution is called a Jacobi field.

The existence of Jacobi fields with fixed initial data follows from the theory of ODEs (as it often happens).

Proposition 0.28. *If $\gamma : [0,a] \to M$ is geodesic. Then there exists an unique Jacobi field for sufficiently small a and specified initial conditions $J(0), J'(0) = \frac{\partial}{\partial t}J(0) \in T_p(M)$.*

Proof. If we let e_i be a parallel orthonormal frame along a geodesic γ with tangent field T, and let t parametrize γ proportional to arclength, and $J = \sum_i v_i e_i$, then

$$\nabla_T \nabla_T J = \sum_i v_i'' e_i$$

while $R(T, J)T = \sum_j v_j R(T, e_j)T$ so the Jacobi equations may be expressed as a system of second order linear ODEs:

$$v_i'' = \sum_j v_j \langle R(T, e_i, T, e_i \rangle$$

and therefore there is a unique Jacobi field J along T with a given value of $J(0)$ and $J'(0) := \nabla_T J|_{t=0}$. □

Corollary 0.5. *The vector space of Jacobi vector fields along geodesics is* $2n$*-dimensional.*

Example: There are some natural Jacobi fields:

- $J_1(t) = \gamma'(t)$, then $\nabla_T \gamma' = 0$, $J_1(0) = \gamma'(0)$, $J_1'(0) = 0$. This corresponds to the variation $F(s,t) = \gamma_s(t) = \gamma(s+t)$.
- $J_2(t) = t\gamma'(t)$, then $\nabla_T \nabla_T (J_2(t)) = 0$ with $J_2(0) = 0$, $J_2'(0) = \gamma'(0)$. This field corresponds to $F(s,t) = \gamma_s(t) = \gamma((s+1)t)$.

Note: There is map from Jacobi fields along $\gamma(t) \to \mathbb{R}^2$, given by $J(t) \mapsto (\langle J(0), \gamma'(0) \rangle, \langle J'(0), \gamma'(0) \rangle)$. This map is surjective linear map because of J_1, J_2. The kernel is Jacobi fields with $J(0), J'(0) \perp \gamma'(0)$, that is $(2n-2)$-dimensional vector space.