Lecture 11

I will speak today about Ricci and sectional curvature.

Proposition 0.22. *The curvature tensor has the symmetry properties,*

$$R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z) = R(Z,W,X,Y)$$

and

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0.$$

Proof. The last identity is exactly Bianchi identity. Anti-symmetry of $R(X,Y,Z,W)$ in the first two entries X,Y is obvious from the definition. The second antisymmetry was shown in Proposition 0.20. And last symmetry left was shown in Proposition 0.21. □

Local expressions. C^∞-linearity of the curvature operator implies that in local charts, R is determined by its values on coordinate vector fields as we know from the discussion above. We can thus introduce components R^l_{ijk} of the curvature tensor, defined by

$$R(\partial_i, \partial_j)\partial_k = \sum_l R^l_{ijk}\partial_l$$

We can explicitly calculate, letting $\partial_l \Gamma^k_{ij} = X_l(\Gamma^k_{ij})$ that:

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ki} + \sum_m \Gamma^l_{im}\Gamma^m_{jk} - \sum_m \Gamma^l_{jm}\Gamma^m_{ki}$$

Remark: Recall that this complicated expression appeared in the proof of Gauss' theorem egregium in the curves and surfaces course, but it was somewhat unmotivated back then!

Therefore, we have that

$$R_{ijkl} = R(X_i, X_j, X_k, X_l) = g(R(X_i, X_j)X_k, X_l) = g(\sum_m R^m_{ijk} X_m, X_l) = \sum_m R^m_{ijk} g_{lm}$$

0.37. **Ricci curvature.** The full Riemann curvature tensor is difficult to work with directly; fortunately, there are simpler "curvature" tensors capturing some of the same information, that are easier to work with.

Definition 0.36. *The Ricci curvature tensor $Ric \in \Gamma(S^2T^*M)$ is the 2-tensor $Ric(X,Y) =$ trace of the map $Z \to R(Z,X)Y$*

Remark: A map $Z \to R(Z,X)Y$ only depends on the value of X,Y,Z at p, soit is a well-defined map from T_pM to T_pM, and thus may be viewed as a matrix once we choose a basis for T_pM.

Remark: If we choose an orthonormal basis e_i then $Ric(X,Y) = \sum_i g(R(e_i,X)Y,e_i)$. The symmetries of the Riemann curvature tensor imply that Ric is a symmetric bilinear form on T_pM at each point p.

Remark: The Ricci curvature also determines the Riemann curvature tensor in 3 dimensions, but in higher dimensions they are different.

Note that the Ricci curvature tensor is a symmetric (2,0)-tensor, and the same is true of the Riemannian metric so we can compare them.

Definition 0.37. *We say that (M, g) is Einstein if $Ric = \lambda g$ for some constant λ.*

The equation $Ric = \lambda g$ is the Riemannian version of Einstein's field equations from General Relativity in the absence of matter. If $\lambda = 0$, then M is called *Ricci-flat*. It is not the same as flat!

The object very related to Ricci curvature is scalar curvature.

Definition 0.38. *The scalar curvature s is the trace of Ric*

$$s = \sum_i Ric(e_i, e_i).$$

We can always construct trace-free Ricci tensor, which we denote as Ric_0, as we always do while constructing something trace-free:

$$Ric_0 = Ric - \frac{s}{n} g,$$

where g denotes metric.

Remark: Sometimes the scalar curvature is defined as s/n.

0.38. **Motivation to consider Ricci/scalar curvature.** This subsection is a bit unformal, we are not going to prove anything here at the moment.

Let us think of the curvature R as a section of $\otimes^4 T^*M$ by the formula $R(X, Y, Z, W) := g(R(X, Y)Z, W)$.

For each point p, the *automorphism* group of T_pM is isomorphic to the orthogonal group $O(n)$.

Fact: Moreover, T_pM and T_p^*M are isomorphic as $O(n)$-modules, and both isomorphic to the standard representation, which we denote E.

However, the symmetries of R mean that it is actually contained in $S^2\Lambda^2 E$ (two symmetries and two antisymmetries). Furthermore the Bianchi identity shows that R is in the kernel of the $O(n)$-equivariant map $b : S^2\Lambda^2 E \to S^2\Lambda^2 E$ defined by the formula

$$b(T)(X, Y, Z, W) = \frac{1}{3}(T(X, Y, Z, W) + T(Y, Z, X, W) + T(Z, X, Y, W))$$

The image of b is $\Lambda^4 E$ and there is the decomposition $S^2\Lambda^2 E = Imb \oplus Kerb$.

Fact: As an $O(n)$-module $\Lambda^4 E$ is irreducible ($n > 2$), but $Kerb$ is not. We call $Kerb$ the space of curvature tensors on E.

Taking a trace over the second and fourth indices gives us a contraction $S^2\Lambda^2 E \to S^2 E$, for which $R \mapsto Ric$.

As an $O(n)$-module $S^2 E$ is not irreducible, since it contains the invariant inner product on E (and its scalar multiples), it is an $O(n)$-invariant vector. Therefore, $S^2 E = S^2_0 E \oplus \mathbb{R}$, and the trace $S^2 E \to \mathbb{R}$ takes Ric to s, and Ric_0 is the part in $S^2_0 E$.

The part of R in the kernel of the contraction $S^2\Lambda^2 E \to S^2 E$ is called the *Weyl curvature tensor*, W.

$$Kerb = \mathbb{R} \oplus S_0^2 E \oplus W(E)$$

For $n \neq 4$ these factors are all irreducible, for $n < 4$ some of them vanish, in particular $W(E) = 0$ (R is determined by Ric!). But for $n = 4$ there is a further decomposition of W into "self dual" and "anti-self dual" parts coming from the exceptional isomorphism $\mathfrak{o}(4) = \mathfrak{o}(3) \oplus \mathfrak{o}(3)$.

0.39. **Sectional curvature (definition).** Recall that in the dimension two curvature is given by one number – R_{1212}.

Definition 0.39. *Let $\sigma = Span\{X, Y\} \subset T_pM$ be a 2-plane. The sectional curvature of σ is given by*

$$K(\sigma) = K(X, Y) = \frac{R(X, Y, Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2}.$$

First let us note that $K(\sigma)$ is independent of the choice of basis of σ (therefore, well-defined).

Any other basis is of the form $\{aX + bY, cX + dY\}$ where $(ad - bc)^2 \neq 0$ so that the vectors are linearly independent. By the properties of R we have

$$R(aX + bY, cX + dY, cX + dY, aX + bY) = (ad - bc)^2 R(X, Y, Y, X)$$

For the denominator we have

$$g(aX+bY, aX+bY)g(cX+dY, cX+dY)-g(aX+by, cX+dY) = (ad-bc)^2 \left(g(X, X)g(Y, Y) - g(X, Y)^2\right)$$

The common non-zero factors cancels and hence $K(\sigma)$ is independent on the choice of basis.

Remark: The Riemannian metric on M induces a (positive-definite) symmetric inner product on the fibers of $\Lambda^p(TM)$ for every p. For $p = 2$ we have a formula

$$||X \wedge Y||^2 = g(X, X)g(Y, Y) - g(X, Y)^2$$

Geometrically, $||X \wedge Y||$ is the area of the parallelogram spanned by X and Y in T^pM. As observed above, the curvature R also induces a symmetric inner product on each $\Lambda^2(T_pM)$, and the ratio (which is exactly scalar curvature defined above) of the two inner products is a well-defined function on the space of rays $\mathbb{P}(\Lambda^2 T_pM)$ (it means we can rescale).

Note that since both R and $|| \cdot ||$ are symmetric inner products on $\Lambda^2 T_pM$, the definition is independent of the choice of basis.

Note that a symmetric inner product on a vector space can be recovered from the length function it induces on vectors, it follows that the full tensor R can be recovered from its ratio with the Riemannian inner product as a function on $\mathbb{P}(\Lambda^2 T_pM)$. Since the Grassmannian of 2-planes in V is an irreducible subvariety of $\mathbb{P}(\Lambda^2 V)$ it follows that the full tensor R can be recovered from the values of the sectional curvature on all 2-planes in TM.