Lecture 10

Today we will prove Hopf-Rinow theorem and discus curvature tensor.

0.35. **Hopf-Rinow (continuation).** Now we are ready to study completeness.

Question: When is there always a minimal geodesic betwen two points?
Look first on a punctured plane.

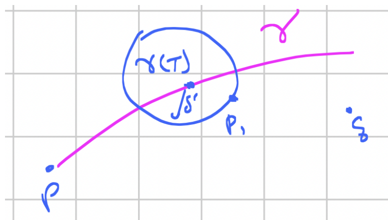
Proposition 0.19. *Let M be connected, $p \in M$. If exp_p is defined on all of T_pM, then $\forall q \in U$ there exists geodesic γ joining p to q with $l(\gamma) = d(p, q)$.*

Proof. Proof is based on a lemma that for any $p, q \in M$ there exists $p_0 \in S_\delta(p)$ for a sufficiently small δ with the property $d(p, p_0) + d(p_0, q) = d(p, q)$.

Indeed, let us choose $p_0 \in S_\delta(p)$ minimizing $d(p_0, q)$ (since the distance is continuous it exists). Therefore if γ joining p and q and $p' \in \gamma \cap S_\delta(p)$, we have $l(\gamma) = d(p, q) \geq d(p, p') + d(p', q) \geq d(p, p_0) + d(p_0, q) \geq d(p, q)$.

Suppose $d(p, q) = r$ and choose δ, p_0 as above, write $p_0 = \exp_p(\delta v)$. Let γ be the geodesic. Then $\gamma(r) = q$.

Define $I = \{t \in [0, r] | d(\gamma(t), q) = r - t\}$. Note that $\delta \in I$. Also I is closed, call $\delta \leq T = max\ I \leq r$, $T \in I$. If $T = r$ we are done. So $T < r$ and apply the lemma for $\gamma(T), q$ there exist $\delta', p_1 \in S_{\delta'}(\gamma(T))$ with $d(\gamma(T), p_1) = \delta'$, then $d(p_1, q) = r - T - \delta'$.



Therefore, $r = d(p, q) \leq d(p, p_1) + d(p_1, q)$. It means $d(p, p_1) \geq r - (r - T - \delta') = T + \delta'$. Now the path $p_1 = \gamma(T + \delta')$ has length $T + \delta'$ and T is not maximal. □

Theorem 0.19. ***(Hopf–Rinow Theorem).*** *Let (M, g) be a connected Riemannian manifold. The following are equivalent:*

(a) (M, g) is (geodesically) complete;

(b) exp_p is defined on all of T_pM for some $p \in M$;

(c) closed bounded subsets of M are compact;

(d) (M, d) is a complete metric space. Moreover, if (M, g) is complete then for all $p, q \in M$ there exists a geodesic γ from p to q such that $d(p, q) = L(\gamma)$.

Proof. (a) $\Rightarrow$ (d)

Suppose $\exp_p : T_pM \to M$ is globally defined, and let $q \in M$ be arbitrary. There is some $v \in T_pM$ with $|v| = 1$ so that $dist(p, \exp_p(sv)) + dist(\exp_p(sv), q) = dist(p, q)$ for some $s > 0$.

Let $\gamma : [0, \infty) \to M$ be the geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$,so that $\gamma(t) = \exp_p(tv)$. The set of t such that $dist(p, \gamma(t)) + dist(\gamma(t), q) = dist(p, q)$ is closed, so let t be maximal with this property. We claim $\gamma(t) = q$ and $|t| =$

$dist(p,q)$. For if not, there is some small r and some point $q' \in \partial B_r(\gamma(t))$ so that $dist(\gamma(t),q') + dist(q',q) = dist(\gamma(t),q)$.

Let $\sigma : [0,r] \to M$ be the unit speed geodesic with $\sigma(0) = \gamma(t)$ and $\sigma(r) = q'$. Then $dist(p,q') = length(\gamma([0,t]) \cap \sigma([0,r]))$, and therefore these two paths fit together at $\sigma(0)$ to form a smooth geodesic, contrary to the definition of t.

In particular, it follows that $\exp_p$ is surjective, and for every q there is a geodesic of length $dist(p,q)$ from p to q. Now if q_i is a Cauchy sequence, we can find $v_i \in T_pM$ with $\exp_p(v_i) = q_i$ and $|v_i| = dist(p,q_i)$. By compactness, the v_i have a subsequence converging to some v, and $\exp_p(v) = q$ is a limit of the q_i. This shows that *(b)* implies *(d)*.

Now suppose M is complete with respect to $dist$. We will deduce geodesically completeness *(a)*. Fix $v \in T_pM$. Then $\exp_p(sv) = \gamma_v(s)$ is defined on some maximal connected subset of $\mathbb{R}_+$ containing 0. This interval is open, irrespective of completeness.

Suppose there is a finite t so that the maximal domain of definition is $[0,t)$. Then $\gamma_v(s)$ is a Cauchy sequence as $s \to t$, and therefore limits to some point q. By Lemma 0.4 there is an open neighborhood U of q and a positive ϵ so that for every point $x \in U$ the exponential map $\exp_x$ is defined on the ball of radius ϵ in T_xM.

Then for all s sufficiently close to t the point $\gamma_v(s)$ is in U and therefore we can find a geodesic δ of length ϵ beginning at $\gamma_v(s)$ and with initial tangent vector $\gamma_{v'}(s)$. Then δ fits together with $\gamma_v([0,s])$ to define $\gamma_v[0,s+\epsilon]$. But if we choose such an s with $|t-s| < \epsilon$ this violates the definition of t.

Therefore, by opposite it follows that $\exp_p$ is globally defined for any p. This shows that *(d)* implies *(a)*. The implication *(a)* implies *(b)* is obvious.

We can show that *(c)* implies *(d)*. Let (p_n) be a Cauchy sequence in M with respect to $dist$. Then (p_n) is bounded so $C = \overline{\{p_n : n \in \mathbb{N}\}}$ is closed and bounded and thus C is compact by assumption. We deduce by metric space theory that (p_n) has a convergent subsequence and thus $(M, dist)$ is complete by definition.

The last implication of *(b)* to *(c)* is left without a complete proof. The idea is the following: if $C \subseteq M$ is closed and bounded then $C \subseteq B^d_R(p) \subseteq \exp_p(\overline{B_{R'}(0)})$ for some $R, R' > 0$. Then we can connect p by a radial geodesic to any point $q \in C$ so that $dist(p,q)$ is the length of that geodesic. Since $\overline{B_{R'}(0)}$ is compact and $\exp_p$ is continuous we see that $\exp_p(\overline{B_{R'}(0)})$ is compact and thus C is compact as desired.

The final conclusion is obvious given that (b) implies the existence of a minimizing geodesic of length $dist(p,q)$ from p to any point q. □

Remark: The minimizing geodesic is not necessarily unique: if we take the North and South poles $N, S \in S^2$, then there are infinitely many minimizing geodesics between them given by the lines of longitude.

Moreover, we see that the upper half-space or the upper hemisphere has the property that there is a minimizing geodesic between any two points, but these manifolds are not complete.

Examples: Any compact M, any closed submanifold of a geodesically complete manifold, and any homogeneous manifold are geodesically complete.

0.36. **Riemann curvature.** The failure of holonomy transport along commuting vector fields to commute itself is measured by curvature. Informally, curvature measures the infinitesimal extent to which parallel transport depends on the path joining two endpoints.

Definition 0.34. *Let E be a smooth bundle with a connection ∇. The curvature (associated to ∇) is a trilinear map $R : \mathcal{X}(M) \times \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E)$ which we write $R(X,Y)Z \in \Gamma(E)$, defined by the formula*

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Operaror $R(X,Y)$ is called the Riemann curvature operator.

Remark: Although *a priori* it appears to depend on the second order variation of Z near each point, it turns out that the curvature is a tensor. The following proposition summarizes some elementary algebraic properties of R.

Proposition 0.20. *For any connection ∇ on a bundle E the curvature satisfies the following properties:*

(1) (tensor): $R(fX, gY)(hZ) = (fgh)R(X,Y)Z$ for any smooth functions f, g, h,
(2) (antisymmetry): $R(X,Y)Z = -R(Y,X)Z$,
(3) (metric): if ∇ is a metric connection, then $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$

Thus we can think of $R(\cdot,\cdot)$ as a section of $\Omega^2(M) \otimes \Gamma(End(E))$ with coefficients in the Lie algebra of the orthogonal group of the fibers.

Remark: We can informally think about this as pushing the vector Z around a parallelogram determined by the vector fields X and Y. The outcome of this procedure is a new tangent vector which may be different from Z. The limit of this procedure as the sides of the parallelogram goes to 0 is the operator $R(X,Y)$ (when $[X,Y] = 0$).

Proof. Antisymmetry follows directly from the definition.

Now let us prove the tensor property. We compute: $\nabla_{fX}\nabla_Y Z = f\nabla_X\nabla_Y Z$ whereas

$$\nabla_Y \nabla_{fX} Z = \nabla_Y(f\nabla_X Z) = f\nabla_Y\nabla_X Z + Y(f)\nabla_X Z$$

on the other hand $[fX, Y] = f[X,Y] - Y(f)X$ so

$$\nabla_{[fX,Y]}Z = f\nabla_{[X,Y]}Z - Y(f)\nabla_X Z$$

so R is tensorial in the first term. By antisymmetry it is tensorial in the second term as well.

Now let us show that R is tensorial in the third term too:

$$\nabla_X\nabla_Y(fZ) = \nabla_X f\nabla_Y Z + \nabla_X Y(f)Z = f\nabla_X\nabla_Y Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + X(Y(f))Z$$

and there is an analogous formula for $\nabla_Y\nabla_X(fZ)$ with X and Y reversed, whereas

$$\nabla_{[X,Y]}(fZ) = f\nabla_{[X,Y]}Z + (X(Y(f)) - Y(X(f)))Z$$

Therefore, we conclude that R is tensorial in the third term too.

To prove the metric identity, first replace X and Y by commuting vector fields with the same value at some given point. We are able to do that because of tensoriality of R. Then,

$$\langle \nabla_X\nabla_Y Z, W\rangle = X\langle \nabla_Y Z, W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle = X(Y\langle Z, W\rangle) - X\langle Z, \nabla_Y W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle =$$
$$= X(Y\langle Z,W\rangle) - \langle \nabla_X Z, \nabla_Y W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle - \langle Z, \nabla_X\nabla_Y W\rangle$$

Subtracting off $\langle \nabla_X\nabla_Y Z, W\rangle$ expanded similarly, the first terms cancel because X, Y commute, the second and third terms cancel identically. The only thing left is $-\langle Z, R(X,Y)W\rangle$. ☐

As usual we pay most of attention to TM as the bundle E.

Corollary 0.3. *The curvature R is bilinear in its arguments, $R(X,Y)$ is a linear operator and $R(X,Y)Z(p) \in T_pM$ only depends on $X(p), Y(p), Z(p) \in T_pM$.*

Proof. If we let $X_1, ..., X_n$ be a coordinate frame field in a chart (U, φ) at p and write decompositions for X, Y, Z with coefficients a_i, b_i, c_i respectively then a direct computation shows that

$$R(X,Y)Z = \sum_{i,j,k=1}^{n} a_i b_j c_k R(X_i, X_j) X_k$$

and since $R(X_i, X_j)X_k$ is independent of X, Y, Z, this shows that $R(X,Y)Z(p)$ only depends on $X(p), Y(p)$ and $Z(p)$. □

Examples:

1. For $\mathbb{R}^n$ with the Euclidean metric, we know that $[\partial_i, \partial_j] = 0$ and $\nabla_{\partial_i}\partial_j = 0$ so $R(\partial_i, \partial_j)\partial_k = 0$.
2. Since $\nabla_{X_i} X_j = [X_i, X_j] = 0$ for the standard vector fields on $T^n \subset \mathbb{R}^{2n}$ we see that $R = 0$.

Definition 0.35. *We call Riemannian manifolds for which $R = 0$ flat.*

Example: On S^2 we have $R(X_1, X_2)X_1 = -X_2$, $R(X_1, X_2)X_2 = \sin^2\theta X_1$. Therefore, $R(X_1, X_2, X_1, X_1) = 0, R(X_1, X_2, X_1, X_2) = -\sin^2\theta, R(X_1, X_2, X_2, X_1) = \sin^2\theta, R(X_1, X_2, X_2, X_2) = 0$.

If we let $E_1 = X_1, E_2 = X_2/\sin^2\theta$, then E_1, E_2 are orthonormal and we have $R(E_1, E_2, E_1, E_2) = 1$, this is something we expected.

Proposition 0.21. *If $E = TM$ and ∇ is torsion-free, then it satisfies the so-called Jacobi identity:*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

Consequently, the Levi-Civita connection on TM satisfies the following symmetry:

$$\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$$

Proof. We again use tensoriality to reduce to the case of commuting vector fields. Then the term $\nabla_X\nabla_Y Z$ in $R(X,Y)Z$ can be rewritten as $\nabla_X\nabla_Z Y$ which cancels a term in $R(Z,X)Y$ and so on:

$$\begin{aligned}R(X,Y)Z + R(Y,Z)X + R(Z,X)Y &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z + \\ +\nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X &+ \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y = \\ = \nabla_X[Y,Z] - \nabla_{[Y,Z]}X + \nabla_Y[Z,X] &- \nabla_{[Z,X]}Y + \nabla_Z[X,Y] - \nabla_{[X,Y]}Z = \\ &= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0\end{aligned}$$

The last symmetry (under interchanging (X,Y) with (Z,W)) follows formally from the metric property, the antisymmetry of R under interchanging X and Y, and the Jacobi identity. □

Remark: The symmetry/antisymmetry identities, and the fact that R is a tensor, means that if we define $R(X,Y,Z,W) := \langle R(X,Y)Z, W\rangle$ then $R \in \Gamma(S^2\Lambda^2(T^*M))$. Such R is well-defined because at $p \in M$ it only depends on g_p and the values of X, Y, Z, W at p. We call R the *Riemann curvature tensor*.

To sum up, we have the following identities for the Riemann curvature tensor.