MATH0072: RIEMANNIAN GEOMETRY

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Lecture plan is (**VERY**) tentative:
Lecture 1, 04/10 - Manifolds: basics
Lecture 2, 07/10 - Tangent space and tangent bundle
Lecture 3, 11/10 - Differential forms, exterior algebra
Lecture 4, 14/10 - Lie bracket of vector fields; Riemannian metric, models of a hyperbolic space
Lecture 5, 18/10 - Riemannian manifolds, geodesics
Lecture 6, 21/10 - Levi-Civita connection, II fundamental form
Lecture 7, 25/10 - Geodesics, exponentional map
Lecture 8, 28/10 - Completeness, Hopf-Rinow theorem
Lecture 9, 01/11 - Curvature, sectional curvature
Lecture 10, 04/11 - Jacobi fields, conjugacy points, Cartan-Hadamard theorem
Lecture 11, 15/11 - Symplectic geometry of Jacobi fields
Lecture 12, 18/10 - Injectivity radius, Sphere theorem; spaces of constant curvature: hyperbolic space
Lecture 13, 22/11 - Variations of energy, Myers' theorem
Lecture 14, 25/11 - Lie groups and homogeneous spaces
Lecture 15, 29/11 - Characteristic classes
Lecture 16, 02/12 - Hodge Theory on Riemannian manifolds
Lecture 17, 06/12 - Bochner's method, Weitzenböck formulae
Lecture 18, 09/12 - Riemannian holonomy groups
Lecture 19, 13/12 - Cheeger-Gromoll splitting theorem
Lecture 20, 16/12

Lecture 1

I will introduce (and remind for ones who knows) the notion of smooth manifolds. The lectures 1 to 3 are about basic notions we need everywhere during the course.

Briefly Riemannian geometry is devoted to the study of smooth curved objects, which play a role in analysis, engineering (like imaging), group theory, number theory, physics (especially gravity) and topology. The smooth curved objects in question are called Riemannian manifolds and the basic examples come from surfaces. There are three key examples:

- the flat plane $\mathbb{R}^2$ (which is flat or zero curvature);
- the sphere S^2 (which is positively curved);
- the hyperbolic space $\mathbb{H}^2$ (which is negatively curved).

These three examples give the basic models for what objects with zero, positive and negative curvature look like even in higher dimensions. (Another way to think about areas of negative curvature is a saddle, like near points on the inner circle of a torus in $\mathbb{R}3$.)

0.1. **Manifolds.** We will start with definition and basic examples.

0.1.1. *Mechanical systems.* Consider the motion of a classic mechanical system - the pendulum. It has a fixed point connected by the straight rod to the free point with some mass which moves subject to gravity.

To specify the system at a given moment one must know:

- the position of endpoint, $p \in S^2 \in \mathbb{R}^3$
- the velocity, $q \in T_pS^2$ ("tangent plane")

The "configuration space" of the system is therefore the union of the tangent planes to the sphere.

This is an example of a *manifold*, and the evolution of the system is given by the flow of a *vector field* on it. We are going to discuss these notions in the first lectures.

In fact, the configuration space of the pendulum can be reduced to $M = \cup_{p\in S^2} S^1_p$, where S^1_p is the unit circle in the tangent plane.

It turns out that M is *diffeomorphic* to S^3, the 3-sphere. The vector field on 3-sphere defining the pendulum motion has two equilibria, ie points where the vector field is zero.

Question: Can there be a vector field on 3-sphere with no equlilibria?

Answer is yes. But for some manifolds , no! The answer to this question depends on the *topology* (global structure) of the manifold.

First guess: *A manifold is the natural notion of a smooth object.*

Although this definition is fake, it is useful in the sense that everything that you would imagine to be a smooth object (and thus a manifold) is manifold. Basically everything that you have encountered so far in geometry is a manifold: surfaces in $\mathbb{R}^3$ (from the course of Differential Geometry), like the sphere and torus, and manifolds in $\mathbb{R}^n$ (from Multivariable Analysis).

Moreover, the actual definition is not very enlightening. We need it, so that all of the theory makes sense, but once we have it we then very rarely need to use it.

0.1.2. *Basic examples.*

Example. $\mathbb{R}^2$ is a 2-dimensional manifold and in general $\mathbb{R}^n$ is an n-dimensional manifold. [1]

Example. The *upper-half plane* $\mathbb{H}^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ is a 2-dimensional manifold. Similarly, the n-dimensional upper half-space

$$\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}$$

is an n-dimensional manifold.

Example. The *unit disk*

$$B^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$$

is a 2-dimensional manifold. Similarly, the unit ball in $\mathbb{R}n$

$$B^n = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : |x^|2 = \sum_{i=1} x_i^2 < 1\}$$

is an n-dimensional manifold.

Example. The n-dimensional sphere

$$S^n = \{x = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : |x|^2 = \sum_{i=1} x_i^2 < 1\}$$

.

Example. The torus in $\mathbb{R}^3$ is a 2-dimensional manifold.

Example. The n-dimensional torus $T^n \in \mathbb{R}^{2n}$ given by

$$T^n = \{(\cos\theta_1, \sin\theta_1, ..., \cos\theta_n, sin\theta_n) \in \mathbb{R}^{2n}; \theta_i \in \mathbb{R}\}$$

is an n-dimensional manifold.

The previous two examples give two possible realisations of the 2-dimensional torus: either in $\mathbb{R}^3$ or in $\mathbb{R}^4$. Are these the same? If not, how are they different? This is one of the questions we shall study, since it turns out they are the same manifold but different Riemannian manifolds: i.e. they have different curvature. is an n-dimensional manifold.

0.1.3. *About non-examples.* So what is a manifold? We have already seen that the simplest example of n-dimensional manifold is just $\mathbb{R}^n$ and this is the local model for all manifolds.

Second fake definition: *An n-dimensional manifold is something which locally "looks like" $\mathbb{R}^n$ (but globally can be much more interesting).*

Remark: What does it mean locally "looks like"? In particular, if you take a sphere in $\mathbb{R}^3$, it is clearly not just flat $\mathbb{R}^2$, but if you look near any given point you can define coordinates so it looks like a piece of $\mathbb{R}^2$.

The same trick can be done for all of the examples we have seen so far. With this second fake definition we may ask the question: what is not a manifold?

Example. A cube is not a manifold. It is not smooth at the edges and at the corners. It looks like $\mathbb{R}^2$ on the faces, but not at the edges or at the corners. Similarly, any polyhedron is not a manifold.

Example. The closed disk in $\mathbb{R}^2$ is not quite a 2-dimensional manifold because it looks like $\mathbb{R}^2$ in the interior where $|x| < 1$, but when $|x| = 1$ we have the circle S^1.

Remark. However, it is what is called a 2-dimensional manifold with boundary.

[1] We are using the word "dimension" which is going to be discussed later.

Example. The hyperboloid of one sheet

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^2 = 1\}$$

and the hyperboloid of two sheets

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^2 = -1\}$$

are 2-dimensional manifolds, but

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : h = 0\}$$

is a cone and so is not a manifold, because it is not smooth at 0, or it does not look like $\mathbb{R}^2$ there.

0.1.4. *Abstract examples.* We will see that examples below are manifolds later today.

Example. Let $M_n(\mathbb{R})$ be the $n \times n$ real matrices. Then the general linear group is $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : detA \neq 0\}$ and the special linear group is $SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : detA = 1\}$. These are indeed groups by multiplication as the determinant multiplicative. Then $GL(n, \mathbb{R})$ is an n^2-dimensional manifold and $SL(n, \mathbb{R})$ is an $n^2 - 1$-dimensional manifold.

Example. Let I be the identity matrix in $M_n(\mathbb{R})$. Then $O(n) = \{A \in M_n(\mathbb{R}) : A^T A = I\}$ and $SO(n) = \{A \in O(n) : det(A) = 1\}$ are the orthogonal and special orthogonal group. Then $O(n)$ and $SO(n)$ are $n(n-1)$-dimensional manifolds.

Example. Let $SU(2) = \{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1\}$. This is again a group and is a 3-dimensional manifold. In general, if we let $M_n(\mathbb{C})$ be the $n \times n$ complex matrices, then the special unitary group

$$SU(n) = \{A \in M_n(\mathbb{C}) : A^T A = I, detA = 1\}$$

is an $n^2 - 1$-dimensional manifold and the unitary group

$$U(n) = \{A \in M_n(\mathbb{C}) : A^T A = I$$

is an n^2-dimensional manifold.[2]

Remark. The examples just given in terms of matrices are all examples of manifolds which are groups: in fact, this is almost the definition of a **Lie group**, and these examples are all Lie groups.

Even more abstract examples.

1. Let $\mathbb{R}P^n$ be the set of straight lines in R^{n+1} through 0. Then $\mathbb{R}P^n$ is the real projective n-space and is an n-dimensional manifold. We can equivalently say that $\mathbb{R}P^n$ is the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalence relation $x \sim y$ if $x = \lambda y$ for some real number λ. So we usually denote points in $\mathbb{R}P^n$ by their equivalence classes.

2. In the analogous way one can define a $2n$-dimensional manifold, called complex projective n-space.

Both the real and complex projective spaces play an important role in the theory of Riemannian geometry.

[2]These are, sort of, complex analogues of the special orthogonal and orthogonal groups.

0.1.5. *Formal definition.* As we have seen the formal definition should look in a way so lots of object are somehow "the same" and some are conidered as "bad" (with corners, angles, cusps) and these objects should be smooth enough to measure distance and define differentation.

Definition 0.1. *An n-dimensional* ***manifold*** *is a (separable*[3]*/second countable) metric space M such that there exists a family $\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}$ where:*

- *U_i is an open set in M and $\cup_{i\in I} U_i = M$;*
- *$\varphi_i : U_i \to \mathbb{R}^n$ is a continuous bijection onto an open set $\varphi_i(U_i)$ with continuous inverse (i.e. a* ***homeomorphism****);*
- *whenever $U_i \cap U_j \neq \emptyset$, the transition map $\varphi_j \cdot \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a smooth (infinitely differentiable) bijection with smooth inverse (i.e. a* ***diffeomorphism****).*

This family $\mathcal{A}$ is called *atlas* and each pair (U_i, φ_i) is called *chart.*
Now we can re-consider examples above in respect with this definition:

1. $\mathbb{R}^n$ (and any open subset): take $U = \mathbb{R}^n, \varphi = id$.
2. S^n: take $U_S = S^n \setminus \{S\}$, where $S = (0, ..., 0, -1)$ is the South pole. The same way we define U_N. Functions $\varphi_k : U_k \to \mathbb{R}^n, k = N, S$ are given by
$$\varphi_N(x) = \frac{(x_1, ..., x_n)}{1 - x_{n+1}}, \varphi_S(x) = \frac{(x_1, ..., x_n)}{1 + x_{n+1}}$$
We have explicit inverse maps **(exercise!)**, maps φ_S, φ_N are called stereographic projections. The gluing of charts is given by inversion map $\varphi_S \cdot \varphi_N^{-1}(y) = \frac{y}{|y|^2}$, which is a diffeomorphism because it is smooth as $y \neq 0$ and it is its own inverse.
3. $\mathbb{R}P^n$: For $i = 1, ..., n+1$ we let $U_i = \{(x_1, ..., x_{n+1})] \in \mathbb{R}P^n : x_i \neq 0\}$, and define $\varphi_i : U_i \to \mathbb{R}^n$ by $\varphi_i([x]) = (\frac{x_1}{x_i}, ..., \frac{x_{n+1}}{x_i})$.
Absolutely the same works for the $\mathbb{C}P^n$.

Remark. We could have chosen different atlases which could give different manifold structures.However, if atlases are equivalent they give the same manifold structure, and an equivalence class of atlases is called a smooth structure. *Two atlases are equivalent the union of the two atlases is still an atlas.*

0.1.6. *Regular value theorem.* Now we give a general technique for constructing manifolds which is very helpful.

Theorem 0.1. *(Regular value theorem) Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a smooth map and suppose that for all $p \in F^{-1}(c)$, where $F^{-1}(c) \neq \emptyset$ the derivative $dF_p : R^{n+m} \to \mathbb{R}^m$ is surjective. Then $F^{-1}(c)$ is an n-dimensional manifold.*

. Before we give the proof, let us consider some applications of 0.1.

1. Let $F(x_1, ..., x_{2n}) = (x_1^2 + x_2^2 - 1, ..., x_{2n-1}^2 + x_{2n}^2 - 1$. Simple check gives that the matrix of dF_x has rank n. Therefore $F^{-1}(0) = T^n$ is n-dimensional manifold.
2. Let $F : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $F(A) = A^T A - I$, the image of F is in symmetric matrices since $F(A)^T = F(A)$. As the **exercise** compute the the derivative of smooth map F. It is $B^T A + A^T B$. If $C \in Sym_n(\mathbb{R})$ and

[3]Separable means there is a countable dense subset: for $\mathbb{R}^n$ just take $\mathbb{Q}^n$ as the countable dense subset

$A \in F^{-1}(0)$ then $dF_A(\frac{1}{2}AC) = C$ so dF_A is surjective. By the 0.1 we see that $O(n) = F^{-1}(0)$ is an $\frac{1}{2}n(n-1)$-dimensional manifold.

Remark. Being a regular value of a function is a sufficient condition to ensure that the level set is a manifold, it is not necessary. In particular, let $F(x, y) = x^3 - y^3$. Then $dF(x, y) = (3^2, 3y^2)$ so 0 is not a regular value of F because $dF(0, 0) = 0$. However $F^{-1}(0) = \{(x, y) \in \mathbb{R}^2 : x^3 = y^3\} = \{(x, x) \in \mathbb{R}^2\}$ which is a 1-dimensional manifold (just a diagonal line in the plane).

Remark. Moreover, in general, if you look at the zero set of a system of polynomials you will get a manifold if the system has a root. This is the entrance to the world of Algebraic Geometry.