Nikon Kurnosov
University College London

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outline of course

1. Legendre families, period map
2. Hodge structure of curves and abelian varieties
3. Hodge decomposition, Kähler manifolds
4. Period domains, mixed Hodge structures
5. Deformation theory intro and variations of the Hodge structure, p-adic Hodge structures

Before

- Elliptic curves = genus 1 Riemann surfaces, parametrized by $\lambda \in \mathbb{C} \setminus \{0, 1\}$
- $H^1(E_\lambda) = \mathbb{C}[\omega = \frac{dx}{y}] \oplus \mathbb{C}[\overline{\omega}]$ - Hodge structure of weight one
- local period map $P : \mathbb{P}^1 \setminus \{0, 1, \infty\} \to \mathbb{H}$ given by ratio of periods (integrals of ω on the basis of cycles)
- monodromy representation $\rho : \pi_1((\mathbb{P}^1 \setminus \{0, 1, \infty\}), \lambda_0) \to Sl_2(\mathbb{Z})$ (bc we can change the basis)
- global period map $\tilde{P} : \mathbb{P}^1 \setminus \{0, 1, \infty\} \to img\,\rho \backslash \mathbb{H}$
- A real (rational, integer) Hodge structure of weight k is a real vector space $H_\mathbb{R}$ ($H_\mathbb{Q}$, free $\mathbb{Z}$-module $H_\mathbb{Z}$) together with a decomposition:

  $$H_\mathbb{C} := H_\mathbb{R} \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$$

  for $\mathbb{C}$-subspaces $H^{p,q} \subset H_\mathbb{C}$ st $H^{p,q} = \overline{H}^{q,p}$
- Hodge structure might be defined by **filtration** **(Lecture 2)** or as **real algebraic representation** **(Lecture 3)**

- There is notion of polarization (which is generalization of the intersection form for elliptic curve). it is a quadratic form on $H_{\mathbb{R}}$ which is symmetric (anti-symmetric) if k even (odd) and satisfies $Q(H^{p,q}, H^{p',q}) = 0$ unless $p = q', q = p'$, and $i^{p-q} Q(x, \overline{x}) > 0$ for any $0 \neq x \in H^{p,q}$
- Analogously to the elliptic curve curves of genus g have Hodge structure of weight one given by $H^{1,0}$ – closed holomorphic forms
- The period map $\tilde{P} : \tilde{U} \to \mathfrak{H}_g$ to Siegel upper half-space
- there is a quotient map $P : U \to Sp(g, \mathbb{Z}) \backslash \mathfrak{H}_g$ which is holomorphic map of analytic spaces
- there is correspondence between weight one HS and complex tori
- Hodge decomposition for complex torus via translational-invariant forms
- Hodge decomposition in a Kähler case $H^n_{dR}(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}_{\overline{\partial}}(X)$
- Lefschetz decomposition gives $H^n_{dR}(X, \mathbb{C}) = \bigoplus_i L^i H^{m-2i}_{prim}(X)$

Before

- The map $[\omega]^k \cup (-) : H^k_{dR} \to H^{m-2k}_{dR}$ is an isomorphism (Hard Lefschetz theorem)
- Hodge numbers for hypersurfaces in $\mathbb{P}^N$ differ from $\mathbb{P}^N$ only in a middle line (Lefschetz hyperplane theorem)
- The Hodge diamond for K3 has been computed as an example. The middle line is 1, 20, 1.
- Weight two Hodge structures of the type $(1, x, 1)$ are called K3-type HS.
- Kuga-Satake construction assignes complex torus (that means HS of weight one) via Clifford algebras using the view of HS as the real algebraic representation
- Period domain is homogeneous space
- The Hodge structure might be defined for singular varieties as well via MHS
- Weight filtration via Gysin spectral sequence

Splitting of MHS

Mixed Hodge structure: more examples

Tate twist

Let $H = (H_{\mathbb{Z}}, F, W)$ be a mixed Hodge structure. We define $H(m)_{\mathbb{Z}} := (2\pi i)2m \cdot H_{\mathbb{Z}}, W_k(H(m)) := W_{k+2m}H, F^p(H(m)) := F^{p+m}H$

Then $H(m)$ is also a mixed Hodge structure, the m-th Tate twist of H.

Cohomology of Kähler manifold

Deformation theory: intro

Let $B = \Delta^s$ and $\mathcal{X} \to B$ be a smooth projective map, so L on $\mathcal{X}$ s.t. $L|_{X_s}$ ample, and X_0 is the fiber over $0 \in B$.
We set $H_{\mathbb{Z}} = H^*_{prim}(X_0, \mathbb{Z})$, $Q_{\mathbb{Z}}$ polynomial coming from $c_1(L|_{X_0}), h^{p,q} = dim H^{p,q}_{prim}(X_0)$. Hence we have a map $P : B \to D = \{$ Hodge structures on $H_{\mathbb{Z}}$ plarized by $Q_{\mathbb{Z}}$ with Hodge numbers $h^{p,q}\}$ holomorphic.
Now, we are generalizing this picture: let S be a smooth complex manifold with $\mathcal{X} \xrightarrow{\pi} S$ with the fiber X_0 over s_0.
After considering minimal cover $\tilde{S}$ of S we have analytic continuation, holomorphic map, $\tilde{P} : \tilde{S} \to D$ of $P : S \to \Gamma \backslash D$, where $\Gamma = img(\pi_1(S, s_0) \to G(\mathbb{Z}) = Aut(H_{\mathbb{Z}}, Q_{\mathbb{Z}}))$.

Intro example

Let $F, G \in \mathbb{C}[x_0, ..., x_N]$ be a degree d homogeneous polynomials cutting out smooth hypersurfaces.
Conside $\mathcal{X} = V(tF + sG) \in \mathbb{P}^n \times \mathbb{P}^1_{t,s}$ with map $\pi : \mathcal{X} \to \mathbb{P}^1$.
Note: The map π might have singular fibers along the divisor $\Delta \subset \mathbb{P}^1$ so $\mathcal{X}_U = \pi^{-1}(u) \xrightarrow{\pi} U = \mathbb{P}^1 \setminus \Delta$.
As we have seen any intersecting Hodge structure is in weight $N - 1$.

Another example

Degree d homogeneous polynomials in $\mathbb{C}[x_0, ..., x_N]$

Consider all degree d polynomials in $\mathbb{C}[x_0, ..., x_N]$. Then $\mathbb{X} \xrightarrow{\pi} \mathbb{P} := \mathbb{P}H^0(\mathbb{P}^N, \mathcal{O}(d))^\vee$, where $\mathcal{X} = V(\sum_{|I|=d} a_I x^I) \subset P^N \times \mathbb{P}$. The dimension of ¶ is $C^d_{N+d} - 1$. Then, for the divisor $\Delta \subset \mathbb{P}$ where fiber is singular we have $\mathcal{X}_U \to U = \mathbb{P} \setminus \Delta$.

Example: Degree 4 in $\mathbb{P}^3$ (K3 surface).

Then $dimU = C^4_7 - 1 = 34$. It gives map $P : U \to \Gamma \backslash D$, where D – polarized Hodge structures on Λ^4_{K3} with Hodge numbers (1, 19, 1) (it is open set in $\hat{D}$, which is quadric in $\mathbb{P}(\Lambda^4_{K3} \otimes \mathbb{C})$. In particular, $dimD = 19$.

Note: $PGL_4(\mathbb{C}) \circlearrowleft U$, i.e. g acts in a way that X and gX have isomorphic polarized Hodge structures. Hence, P factors via $\hat{P} : PGL_4(\mathbb{C}) \backslash U \to \Gamma \backslash D$. The $dimPGL_4(\mathbb{C}) \backslash U = 34 - 15 = 19$. And $\hat{P}$ is close to $\simeq$.

Kodaira-Spencer

liftability of the field to family

Let $\mathcal{X} \xrightarrow{\pi} \Delta$ be a smooth projective family and ξ tangent field on *Delta*, then the lift $\tilde{\xi}$ to the family $\mathcal{X}$ if

1. locally π is projective free product
2. path with partition of unity

There is a flow φ_t trivializing $\mathcal{X}$ in direction of ξ.
If ξ is holomorphic, we can take $\tilde{\xi}$ of type (1,0) if we can take $\tilde{\xi}$ holomorphic, then we will have holomorphic trivialization φ_t, which is not the case in general since (2) fails. But (1) still holds so local holomorphic lifts $(U_i, \tilde{\xi}_i)$ exist. Therefore, obstruction to holomorphically trivializing π in the direction of ξ is measured by $(U_i \cap U_j, \tilde{\xi}_i - \tilde{\xi}_j) \in H^1(\mathcal{X}, T_{\mathcal{X}/\Delta} = Ker(T_{\mathcal{X}} \to \pi^* T_\Delta))$

Kodaira-Spencer map

For family $\mathcal{X} \xrightarrow{\pi} S$ at $s_0 \in S$ the KS map is
$\kappa : T_{S,s_0} \to H^1(X_0, T_{X_0})$.

Remark: It measures whether family is trivializable to 1st order.

Deformations

Definition

A deformation of a smooth projective X_0 is a map $\pi : \mathcal{X} \to S$ with π smooth projective and X_0 as the fiber over s_0.

Definition

The deformation is called

- complete if for any other deformation $\mathcal{X}' \xrightarrow{\pi'} T$ there is a polydisc $t_o \in B \subset T$ and a map $B \xrightarrow{f} S, t_0 \mapsto s_0$ such that $\mathcal{X}'|_B \simeq f^*\mathcal{X}$.
- universal if it is complete and f is unique
- versal if it is complete and f is unique to first order

Fact

- If $\kappa : T_{S,s_0} \twoheadrightarrow H^1(X_0, T_{X_0})$ then π is complete
- If κ is $\simeq$ then π is versal
- If $H^2(X_0, T_{X_0}) = 0$ then a versal deformation exists with KS map an isomorphism
- If further $H^0(X_0, T_{X_0}) = 0$ then that deformation is universal

Deformations

Notations:

$Def(X_0) = H^1(X_0, T_{X_0})$ measures first order deformations of X_0,
$Obs(X_0) = H^2(X_0, T_{X_0})$ measures obstruction to constructing with a given first order behavior
$H^0(X_0, T_{X_0})$ – holomorphic vector fields, ie infinitesimal automorphisms of X_0

Example:

If X is Calabi-Yau (for instance, K3), then $\omega_X = \Omega_X^{dimX} \simeq \mathcal{O}_X$.
Then $T_X \simeq \Omega_X^{dimX-1}$. So,
$H^i(X, T_X) \simeq H^i(X, \Omega_X^{dimX-1}) \simeq H^{dimX-1,i}(X)$

A few more words on K3 surfaces

Deformations

For any K3 surface there is an unobstructed 20-dimensional universal deformation

local Torelli

The map $S \to D$ is a local isomorphism.

Period map in terms of deformation theory

Consider a smooth projective family $\mathcal{X}$ over a disk B:
Shrinking B we make π trivial as a C^∞-family. Hence, there is a smooth family of diffeomorphisms $\varphi_t : X_0 \xrightarrow{\simeq} X_t$. It induces $\varphi_t^* : H^k(X_t, \mathbb{Z}) \xrightarrow{\simeq} H^k(X_0, \mathbb{Z})$.
Note: This preserves polarizing form q_t (induced by $h_t = c_1(L|_{X_t})$, where L is given by restricting $\mathcal{O}(1)$).

Period map

$P : B \to D$ with $t \mapsto [\varphi_t^* H^*_{prim}(X_t, \mathbb{Q})]$ (in terms of filtrations: $P^\bullet(t) := \varphi_t^* F^\bullet H^k_{prim}(X_t, \mathbb{C})$)

Proposition

- The map P is holomorphic: $\frac{\partial}{\partial \bar{z}} P^\bullet(z)|_{z=0} = 0$
- (Griffiths transversality) $\frac{\partial}{\partial z} P^i(z)|_{z=0} \subset P^{i-1}(0) = F_0^{i-1}$

Deformation theory

Nontriviality of a family *vs* nontriviality of one of its period maps

Setup: D – Hodge structures on $H_{\mathbb{Z}}$ polarized by $Q_{\mathbb{Z}}$ with given Hodge numbers, there is an action of $G(\mathbb{R})$ on D (last lecture), and D is open in $\hat{D}$ (where $G(\mathbb{C})$ acts). Denote the Lie algebra of $G(\mathbb{R})$ as $\mathfrak{g}_{\mathbb{R}} \subset End(H_{\mathbb{R}})$ ($\mathfrak{g}_{\mathbb{C}}$ respectively for $G(\mathbb{C})$)

weight 0 HS on $\mathfrak{g}_{\mathbb{R}}$

For $x \in D$ one have
$\mathfrak{g}_x^{p,-p} = \{\xi \in \mathfrak{g}_{\mathbb{R}} | \xi(H_x^{r,s}) \subset H_x^{r+p,s-p}\}, \mathfrak{g}_x^{\geqslant 0} = \bigoplus_{p \geqslant 0} \mathfrak{g}_x^{p,-p}$.

Gauss-Manin connection

From the last diagram we have a natural map:

$$T^{hor}_X D \xrightarrow{\oplus_p \sigma^p} \bigoplus_p Hom(F^p_x/F^{p+1}_x, F^{p-1}_x/F^p_x)$$

Definition

$\mathcal{H}_{\mathbb{C}} := R^k_{prim}\pi_* \mathbb{C}$ – sheaf whose sections over $U \subset B$ are families $\alpha_z \in H^k_{prim}(X_z, \mathbb{C})$ with $\varphi^*_z \alpha_z$ constant.
$A^0(\mathcal{H}_{\mathbb{C}})$:= sheaf whose sections are all smooth families $\alpha_z \in H^k_{prim}(X_z, \mathbb{C})$.

Definition (Gauss-Manin connection)

For a given α_z lift it to $\alpha_z = [\omega|_{X_z}]$ for ω on $\mathcal{X}$ closed on fibers, lift ξ to $\tilde{\xi}$ on $\mathcal{X}$, $\nabla_\xi \alpha = [(L_{\tilde{\xi}}\omega)|_{X_z}]$ is a section of $A^0(\mathcal{H}_z)$.

Lemma

Gauss-Manin connection satisfies the Leibnitz rule:
$\nabla_\xi(f\alpha_z) = df(\xi)\alpha_z + f\nabla_\xi \alpha_z$.

Proof of Leibnitz rule

Griffiths transversality

Definition

Let $\mathcal{F}^p$ is a sheaf whose sections are $\alpha_z \in F^p H^k_{prim}(X_z, \mathbb{C})$ s.t. $\varphi_z^* \alpha_z$ is holomorphic. It could be described as the sheaf of sections of holomorphic vector bundle because such sections are pulled-back from $\hat{D}$ (P is holomorphic).

Griffiths transversality

If ξ is a holomorphic vector bundle on B, then we can take $\tilde{\xi}$ of type (1,0) and we have Griffiths transversality: $\nabla_{\xi} \mathcal{F}^p \subset \mathcal{F}^{p-1}$.

Hence, we obtain $\mathcal{O}$-linear map:

$$T_B \xrightarrow{\nabla^p} Hom(Gr^p_{\mathcal{F}}, Gr^{p-1}_{\mathcal{F}})$$

Indeed,

Property of ∇^p

map i_κ^p

There is a natural map:
$i_\kappa^p : H^1(X_0, T_{x_0}) \to Hom(H^q(X_0, \Omega_{X_0}^p), H^{q+1}(X_0, \Omega_{X_0}^{p+1}))$
which is defined by taking cup-product with $\kappa(\xi)$ and contracting.

Proposition

$i_\kappa^p = \nabla^p$

Period map: revision

Definition

The map $P : B \to D$ is **horizontal** if dP factors through $T^{hor}D$.

Examples

Consider curve C of genus $g \geqslant 2$. Then $H^2(T_C) = 0$. Therefore, there is versal $\mathcal{C} \to Def(C)$ with $\kappa : T_0 Def(C) \xrightarrow{\simeq} H^1(T_C)$.
Let us consider the period map:
$P : Def(C) \to D = \mathfrak{H}_g \subset Gr(g, H^1(C, \mathbb{C}))$ with
$dP = i_\kappa : T_0 Def(C) \simeq H^1(T_C) \simeq H^0(\omega_C^{\otimes 2})^\vee \to$
$Hom(H^0(\omega_C), H^1(\omega_t)) \simeq H^0(\omega_C)^\vee \otimes H^0(\omega_C)^\vee$

Proposition

The map i_κ is dual to multiplication:
$m : H^0(\omega_C) \otimes H^0(\omega_C) \to H^0(\omega_C^{\otimes 2})$

Bunch of classical theorems

Theorems

- (Noether) If C is non-hyperelliptic, m is surjective
- (Petri) If C is non-hyperelliptic, non-trigonal, not a plane quintic, it is determined by $Ker(m)$.
- (infinitesimal Torelli) If C is non-hyperelliptic, then $P : Def(C) \to \mathfrak{H}_g$ is immersive at 0.
- (generic Torelli) If C is non-hyperelliptic, non-trigonal, not a plane quintic, it is determined by dP_0.

Deformations of CYs

Variations of HS, local systems

Let X be a manifold, R is $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$, denote by R_X sheaf of locally constant functions into R, and analogously Λ_X is the sheaf of functions into module.

Definition

An R-local system$/X$ is a sheaf $\mathcal{L}$ of R_X-modules locally isomorphic to Λ_X.

Note:

For $R = \mathbb{C}$: it is equivalent to $\mathbb{C}$-vector bundles whose transition functions can be chosen to be constant.
It is also equivalent to representation $\pi_1(X, x) \to GL(L_x)$.
Hence, we can define $A^0(\mathcal{L}) = \mathcal{L} \otimes_{R_X} \mathcal{C}_R^\infty$ – sheaf of sections $\sum f_i e_i$ with e_i sections of $\mathcal{L}$ and f_i R-valued functions. And $\Omega^0(\mathcal{L}) = \mathcal{L} \otimes_{R_X} \mathcal{O}$ – the same with f_i holomorphic.
Remark: $A^0(\mathcal{L})$ comes with an unique connection $\nabla : A^0(\mathcal{L}) \to A^1(\mathcal{L}) = \mathcal{L} \otimes A^1_X$ such that $\nabla e = 0$ for $\forall e \in \mathcal{L}$.
Moreover, there is connection on $\Omega^0(\mathcal{L})$
Example: For $\mathcal{X} \xrightarrow{\pi} S$ smooth, ∇ associated to $R^k \pi_* \mathbb{C}$ is Gauss-Manin connection.

Variations of HS

Definition

An R-variation of Hodge structure/S is $(\mathcal{H}_R, \mathcal{F}^\bullet)$, where

- $\mathcal{H}_R$ is an R-local system
- $\mathcal{F}^\bullet$ is a descending filtration of $\Omega^0(\mathcal{L})$ in the category of $\mathcal{O}$-modules such that $((\mathcal{H}_R)_s, \mathcal{F}^\bullet_s)$ is an R-Hodge structure for each $S \in S$ and $\nabla(\mathcal{F}^\bullet) \subset \Omega^1(\mathcal{F}^{\bullet -1}) = \mathcal{F}^\bullet \otimes \Omega^1_S$

Examples:

- $(R^*\pi_*\mathbb{Z}, F^\bullet R^*\pi_*\mathbb{C})$ is a $\mathbb{Z}$-variation of Hodge structure.
- Given two variations $\mathcal{V}, \mathcal{V}'$ of hodge structure over S of weights k and k', there is a natural structure if variation of Hodge structure on the local systems of $\mathcal{V} \otimes \mathcal{V}'$ and $Hom(\mathcal{V}, \mathcal{V}')$ of weights $k + k'$ and $k - k'$ respectively
- Let $\mathcal{V}$ be a Hodge structure of weight k and $s_0 \in S$. Suppose we have a representation $\rho : \pi_1(S, s_0) \to Aut(V)$. Then the local system $\mathcal{V}(\rho)$ associated to ρ gives a locally constant variation of HS.

VHS

Theorem

Let $\mathcal{V}$ be a variation of HS of weight k on a complex manifold S which is Zariski open in a compact complex manifold. Then $H^0(S, \mathcal{V})$ admits a HS of weight k. The evaluation map at a point $s \in S$ gives an isomorphism of $H^0(S, \mathcal{V})$ with the subspace of $\mathcal{V}_s$ left fixed by the action of $\pi_1(S, s)$. The inclusion of this subset into $\mathcal{V}_s$ is a morphism of HS. In other words, the variation of HS on $\mathcal{V}$ restricts to a constant variation of HS on its maximal constant local system.

Corollary

If $a \in H^0(S, \mathcal{V})$ has Hodge type (p, q) at some point $s \in S$, it has Hodge type (p, q) everywhere.

Fact

A variation of HS $\mathcal{V}$ over S with $img(\pi_1(S, s) \to GL(\mathcal{V}_s)) \subset \Gamma$ is equivalent to a locally liftable horizontal holomorphic map $X \to \Gamma \backslash D$, where D is associated period domain.

Hodge numbers

Theorem

The Hodge numbers $h^{p,q}(X_t) = dim_{\mathbb{C}} H^{p,q}(X_t)$ are constant.

Gauss-Manin connection for curves

Short review for elliptic curves

A few words on *p*-adic Hodge Theory

Thanks!