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outline of course

1. Legendre families, period map
2. Hodge structure of curves and abelian varieties
3. Hodge decomposition, Kähler manifolds
4. Period domains, mixed Hodge structures
5. Deformation theory intro and variations of the Hodge structure,
p-adic Hodge structures

Before

- Elliptic curves = genus 1 Riemann surfaces, parametrized by $\lambda \in \mathbb{C} \setminus \{0, 1\}$
- $H^1(E_\lambda) = \mathbb{C}[\omega = \frac{dx}{y}] \oplus \mathbb{C}[\overline{\omega}]$ - Hodge structure of weight one
- local period map $P : \mathbb{P}^1 \setminus \{0, 1, \infty\} \to \mathbb{H}$ given by ratio of periods (integrals of ω on the basis of cycles)
- monodromy representation $\rho : \pi_1((\mathbb{P}^1 \setminus \{0, 1, \infty\}), \lambda_0) \to Sl_2(\mathbb{Z})$ (bc we can change the basis)
- global period map $\tilde{P} : \mathbb{P}^1 \setminus \{0, 1, \infty\} \to img\, \rho \backslash \mathbb{H}$
- A real (rational, integer) Hodge structure of weight k is a real vector space $H_\mathbb{R}$ ($H_\mathbb{Q}$, free $\mathbb{Z}$-module $H_\mathbb{Z}$) together with a decomposition:
$$H_\mathbb{C} := H_\mathbb{R} \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$$
for $\mathbb{C}$-subspaces $H^{p,q} \subset H_\mathbb{C}$ st $H^{p,q} = \overline{H}^{q,p}$
- Hodge structure might be defined by **filtration** **(Lecture 2)** or as **real algebraic representation** **(Lecture 3)**

- There is notion of polarization (which is generalization of the intersection form for elliptic curve). it is a quadratic form on $H_{\mathbb{R}}$ which is symmetric (anti-symmetric) if k even (odd) and satisfies $Q(H^{p,q}, H^{p',q}) = 0$ unless $p = q', q = p'$, and $i^{p-q}Q(x, \overline{x}) > 0$ for any $0 \neq x \in H^{p,q}$
- Analogously to the elliptic curve curves of genus g have Hodge structure of weight one given by $H^{1,0}$ – closed holomorphic forms
- The period map $\tilde{P} : \tilde{U} \to \mathfrak{H}_g$ to Siegel upper half-space
- there is a quotient map $P : U \to Sp(g, \mathbb{Z})\backslash\mathfrak{H}_g$ which is holomorphic map of analytic spaces
- there is correspondence between weight one HS and complex tori
- Hodge decomposition for complex torus via translational-invariant forms
- Hodge decomposition in a Kähler case $H^n_{dR}(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}_{\overline{\partial}}(X)$
- Lefschetz decomposition gives $H^n_{dR}(X, \mathbb{C}) = \bigoplus_i L^i H^{m-2i}_{prim}(X)$

Before

- The map $[\omega]^k \cup (-) : H^k_{dR} \to H^{m-2k}_{dR}$ is an isomorphism (Hard Lefschetz theorem)
- Hodge numbers for hypersurfaces in $\mathbb{P}^N$ differ from $\mathbb{P}^N$ only in a middle line (Lefschetz hyperplane theorem)
- The Hodge diamond for K3 has been computed as an example. The middle line is 1, 20, 1.
- Weight two Hodge structures of the type $(1, x, 1)$ are called K3-type HS.
- Kuga-Satake construction assignes complex torus (that means HS of weight one) via Clifford algebras using the view of HS as the real algebraic representation

More on the HS of K3

Last time we get the Hodge diamond for smooth degree 4 surface in $\mathbb{P}^3$, and $h = c_1(\mathcal{O}_X(1))$:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Any projective surface with this Hodge diamond is called **K3 surface**.
Note: $H^2(X, \mathbb{Z})$ has $\mathbb{Z}$-pairing $q(\alpha, \beta) = \int_X \alpha \cup \beta$
This pairing is

- unimodular bc of Poincare duality
- even (ie $2|q(\alpha, \alpha)$). For $\alpha = c_1(L)$, where L is the line bundle it follows from Riemann-Roch formula $\chi(L) = \frac{1}{2}c_1(L)^2 + 2$. In general, it follows from Wu's formula.
- signature (3, 19). It follows from the Index Theorem that $\tau = b_2^+ - b_2^- = \frac{1}{8}p_1(X) = \frac{1}{3}(c_1^2 - 2c_2) = -16$. In detail, $H^2(X, \mathbb{Q}) = H^2_{prim}(X, \mathbb{Q}) \oplus \mathbb{Q}h$, where $\int h^2 > 0$. Also $(H^{2,0} \oplus H^{0,2})_{\mathbb{R}}$ is 2-dimensional and $H^{1,1}_{\mathbb{R}}$ is 19-dimensional.
- lattice Λ_{K3} is determined uniquely by these data: $H^2(X, \mathbb{Z}), q) \simeq U^3 \oplus E_8(-1)^2$, where U is hyperbolic matrix and the second factor is the minus Cartan matrix of E_8 root diagram.

Period domain for K3 surface

- $Aut(\Lambda_{K3})$ act transitively on primitive $h \in \Lambda_{K3}$ with $h^2 = 2d$
- If X has ample class h with $h^2 = 2d$:
  $H^2_{prim}(X, \mathbb{Z}) \simeq \Lambda^{2d}_{K3} := (-2d) \oplus U^2 \oplus E_8(-1)^2$

Period domain

$D = \{$ Hodge structures on Λ^{2d}_{K3} polarized by q with Hodge numbers $(h^{2,0}, h^{1,1}, h^{0,2}) = (1, 19, 1)\} = \{\mathbb{C}\sigma \subset \Lambda^{2d}_{K3} \otimes \mathbb{C} | q(\sigma, \sigma) = 0, q(\sigma, \overline{\sigma}) > 0\} \hat{D} = \{\mathbb{C}\sigma \subset \Lambda^{2d}_{K3} \otimes \mathbb{C} | q(\sigma, \sigma) = 0\} \subset \mathbb{P}(\Lambda^{2d}_{K3} \otimes \mathbb{C})$

Remark: $\hat{D}$ is smooth quadric. Hence, period domain is analtically open set in a projective variety $\hat{D}$. In particular it is a complex manifold.

Period map

$P : \{$ marked K3 surfaces/ isomorphism $\} \to D$, where marking is isometry between $H^2(X, \mathbb{Z})$ and lattice Λ_{K3}.

Torelli theorems

Let A and A' be free $\mathbb{Z}$-modules of finite rank, endowed with Hodge structures and bilinear forms. A Hodge isometry $A \to A'$ is an isomorphism that respects both the Hodge structures and the bilinear forms.

Torelli for K3

Two complex K3 surfaces X and X' are isomorphic if and only if there is a Hodge isometry $H^2(X, \mathbb{Z}) \simeq H^2(X', \mathbb{Z})$.

Remark: Theorem was proved by Shapiro-Shafarevich (algebraic case) and Burns-Rapoport (analytic case). It is named for its analogy to the original Torelli theorem for curves: two complex curves X and X' are isomorphic if and only if their Jacobians $Jac(X)$ and $Jac(X')$ are isomorphic as polarized abelian varieties.
Remark: Next time we will discuss the deformation theory and formulate local Torelli theorem.

Deformation theory: Intro

A **deformation** of X is a proper smooth family $X \to S$ with a distinguished point $0 \in S$ and a given isomorphism $X_0 \simeq X$.

Remark: In fact we are only interested in the germ of the family around 0. A deformation $X \to S$ is **universal** if any other deformation $X' \to S'$ is (on germs) the pullback of $X \to S$ along a unique map $S' \to S$.

Usually people ask about the existence of universal deformation, and if deformation theory is unobstructed

Example: K3 surfaces **(Kodaira-Spencer)**. Let X be a K3 surface. Then it has a universal deformation $X \to Def(X)$. It is a universal deformation for each of its fibers. The deformation space $Def(X)$ is a smooth complex manifold of dimension 20.

Indeed,the dimension 20 originates from isomorphisms $T_0 Def(X) = H^1(X, TX) \simeq H^1(X, \Omega^1 X)$.

The deformation space $Def(X)$ is simply connected, so given a marking of X we have a period map $Def(X) \to D$

(local Torelli) Let (X, φ) be a marked K3 surface. The period map $Def(X) \to D$ is a local isomorphism on $Def(X)$.

Quintic 3-folds

What if we have quintic 3-fold in $\mathbb{P}^4$ instead of K3 surface? When the middle line of Hodge diamond is $(1, 101, 101, 1)$.

And $(H^3(X,\mathbb{Z}), q) = \left(\mathbb{Z}^{204}, \begin{bmatrix} 0 & Id \\ -Id & 0 \end{bmatrix}\right) =: (\Lambda_{CY3}, q)$

Question: What is period domain in that case?

Answer: It appears that in this (and a lot of other cases) period domains are better phrased in terms of Hodge filtration.

So, recap of filtrations...

Filtrations recap

Suppose H has a weight k Hodge structure. Then

$$F^i H_{\mathbb{C}} = \bigoplus_{p+q=k, p \geqslant i} H^{p,q}$$

gives descending filtration (**Hodge filtration**)

$$H_{\mathbb{C}} = F^0 H_{\mathbb{C}} \supset F^1 H_{\mathbb{C}} \supset \ldots \supset F^k H_{\mathbb{C}} \supset F^{k+1} H_{\mathbb{C}}$$

Hodge structure could be recovered from the filtration as follows: $H_{\mathbb{C}} = F^p H_{\mathbb{C}} \oplus \overline{F^{k-p+1}} H_C, H^{p,q} = F^p H_{\mathbb{C}} \cap \overline{F^q H_{\mathbb{C}}}$.

Motivation (Griffiths): The Hodge filtration exists naturally on the cohomology of a smooth compact complex algebraic variety X and it is defined by a natural filtration on the algebraic de Rham complex.

Also, $Gr_F^p H_{\mathbb{C}} := F^p H_{\mathbb{C}} / F^{p+1} H_{\mathbb{C}} \simeq H^{p,q}$ is the object of quotient complex with the induced filtration.

Period domain for quintic 3-fold

Let $X \subset \mathbb{P}^4$ is quintic 3-fold

Recall $(H^3(X,\mathbb{Z}), q) = \left(\mathbb{Z}^{204}, \begin{bmatrix} 0 & Id \\ -Id & 0 \end{bmatrix}\right) =: (\Lambda_{CY3}, q)$

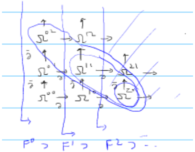
Period domain

$D = \{$ flags $\Lambda_{CY3} \otimes \mathbb{C} = F^0 \supset F^1 \supset F^2 \supset F^3 \supset 0, dimF^0/F^1 = dimF^3 = 1, dimF^1/F^2 = dimF^2/F^3 = 101 | F^\bullet$ is isotropic, h is postive-definite $\} \subset \hat{D} = \{$ flags $F^\bullet | F^\bullet$ is isotropic $\}$.

Remark: The latter one is the projective variety!

Hodge filtration for cohomology

WARNING: We are over the field of characteristic zero. Namely, if X is nonsingular over a field k,$\Omega^\bullet X_{/k}$ (note that this is a complex of coherent sheaves although the differential is of course not $\mathcal{O}_X$-linear). The cohomology of this complex, $H^m_{dR}(X)$ is a vector space over k and in *char p* differentiating can get very tricky.



Deligne theorem

Let X be a smooth compact complex algebraic variety, then the filtration F by subcomplexes (**truncation**) of the de Rham complex: $F^p\Omega^* X := \Omega^{*\geqslant p} X = 0 \to 0 \to ... \to 0 \to \Omega^p X \to \Omega^{p+1} X \to ... \to \Omega^n X \to 0$ induces a Hodge filtration of a Hodge structure on the cohomology of X

This is the *"stupid" filtration.*

Filtration on cohomology

The filtration on de Rham complex induces the one on the cohomology.

Filtration on de Rham cohomology

The Hodge filtration F is defined on de Rham cohomology as follows: $F^pH^i(X, \mathbb{C}) = F^pH^i(X, \Omega^*X) := Im(H^i(X, F^p\Omega^*X) \to H^i(X, \Omega^*X))$.

Remark: The first isomorphism is defined by holomorphic Poincare lemma on the resolution of the constant sheaf C by the analytic de Rham complex Ω^*.

isomorphism

If X is a nonsingular over $\mathbb{C}$, then $H^m(X; \mathbb{C}) = H^m(X, \Omega^\bullet X_{/\mathbb{C}})$

Remark: The beauty of the theorem is that the group on the LHS is the singular cohomology of X, which is all about the (classical) topology of X, where as you can compute the group on the RHS purely from the Zariski topology of X.

Spectral sequence as the Hodge filtration

Hodge-to-de Rham spectral sequence

The proof of Deligne theorem is based on the degeneration at rank one of the spectral sequence with respect to F which is defined as follows:
${}_F E^1_{p,q} := H^{p+q}(X, Gr^p_F \Omega^* X) \simeq H^q(X, \Omega^p X) \Rightarrow$
$Gr^p_F H^{p+q}(X, \Omega^* X) = H^{p+q}_{dR}(X)$

When X is complex projective (or compact Kähler) you can interpret Hodge theory as saying that for all $r \geqslant 1$ $E_1 = E_r$ – that is, the spectral sequence degenerates on the first page. The reason for this degeneration is deep and it is the most crucial way in which topological spaces underlying algebraic varieties are special.

Weil operator

Descending filtration $F^\bullet$ on $H_\mathbb{C}$ is isotropic with respect with quadratic form q if the following condition applies $q(F^p, F^{q+1}) = 0, \forall p+q=k$.

Condition

$F^\bullet H_\mathbb{C}$ is isotropic iff $q(H^{p,q}, H^{p',q'}) = 0$ unless $(p,q) = (q',p')$.

Weil operator

The Weil operator $C \in End(H_\mathbb{R})$ is defined by
$C|_{H^{p,q}} = i^{p-q},\ C = \bigoplus C|_{H^{p,q}}$

Hodge metric

The Hodge metric is the hermitian form: $h(x,y) := q(x, C\overline{y})$

Remark: C is defined over reals. Indeed, for $x \in H^{p,q}$ we have
$C\overline{x} = i^{q-p}\overline{x} = \overline{Cx}$

Remark: $H_\mathbb{Q}$ is polarized by q iff h is positive-definite.

Period domains via filtrations

Let $H_{\mathbb{Z}}$ be free $\mathbb{Z}$-module of a finite rank, $q_{\mathbb{Z}}$ a $(-1)^k$ alternating form, and $h^{p,q}$ is the set of Hodge numbers with the condition $h^{p,q} = h^{q,p}$ such that $\sum h^{p,q} = rkH_{\mathbb{Z}}$.
This gives integral Hodge structure polarized by $q_{\mathbb{Z}}$.

Period domain

$D = \{F^{\bullet}$ descending length k filtrations on $H_{\mathbb{C}}$ with $dimGr_F^p H_{\mathbb{C}} = h^{p,q} | F^{\bullet}$ is isotropic, $q(x, C\overline{y})$ is positive-definite $\} \subset \hat{D} = \{F^{\bullet} | F^{\bullet}$ is isotropic $\}$
In particular, for a point $x_0 \in D$ gives $H_{\mathbb{C}} = \bigoplus H_0^{p,q}, F_0^{\bullet} H_{\mathbb{C}}, C_0, h_0$.

Group action on D

Let $G(\mathbb{R}) = Aut(H_{\mathbb{R}}, q_{\mathbb{R}})$ be an automorphism group. Then it acts transitively on D, so $D \simeq G(\mathbb{R})/Stab_{G(\mathbb{R})}(x_0)$. And for Weil operator and metric we have $C_{gx_0} = gC_0g^{-1}$ and $h_{gx_0}(x, y) = h_0(g^{-1}x, g^{-1}y)$.
What is the action?
Consider $g \in G(\mathbb{R}), x \in D$, then $H_{gx}^{p,q} = gH_x^{p,q}$ (the same for filtration)

Group action on D

Let $G(\mathbb{R}) = Aut(H_{\mathbb{R}}, q_{\mathbb{R}})$ be an automorphism group. Then it acts transitively on D, so $D \simeq G(\mathbb{R})/Stab_{G(\mathbb{R})}(x_0)$. And for Weil operator and metric we have $C_{gx_0} = gC_0g^{-1}$ and $h_{gx_0}(x, y) = h_0(g^{-1}x, g^{-1}y)$.

This gives D the structure of a manifold. A useful equivalent set-theoretic identification is $D = \{$ set of conjugacy classes $g^{-1}H_{x_0}g$ of H_{x_0} in $G(\mathbb{R})\}$

Period domains as homogeneous spaces

The action on Weil operator and metric follow from the direct computation.

Remark: If $g \in G(\mathbb{C})$ and it is not real, then gx_0 may no longer be a Hodge structure, but we still have the analogous result.

Group action on $\hat{D}$

Group $G(\mathbb{C})$ acts transitively on $\hat{D}$, so $\hat{D} \simeq G(\mathbb{C})/Stab_{G(\mathbb{C})}x_0$ with the action defined on filtration as follows: $F^{\bullet} \mapsto gF^{\bullet}$

Examples:

- For weight 1 structures with Hodge decomposition $H = H^{1,0} \oplus H^{0,1}$ the period domain is $D = Sp(g, \mathbb{R})/U(g)$, where $g = dimH^{1,0}$. This is Siegel's upper half-space $\mathfrak{H}_g$
- For weight 2 structures with Hodge decomposition $H^2 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$ we have $D = SO(2a, b)/U(a) \times SO(b)$, where $a = dimH^{2,0}$ and $b = dimH^{1,1}$.

Period domains

As homogeneous spaces

The resulting parameter space of Hodge structures D can be represented as a complex homogeneous space G/V, where G is a Lie group and V is a compact subgroup.

Remark: However, V is rarely a maximal compact subgroup, and so D is rarely hermitian symmetric. In some special cases D nonetheless hermitian symmetric. In particular, in the cases of K3 surfaces and of the cyclic cubic threefolds associated to cubic surfaces.

More on the period domains

Fact

The D is a homogeneous complex manifold.

Indeed, consider the compact dual $\hat{D}$. We have seen that $\hat{D}$ is acted on transitively by $G(\mathbb{C})$ with stability group of $F \in \hat{D}$ be a parabolic subgroup P. Hence, $\hat{D} = G(\mathbb{C})/P$ is a compact, complex manifold.

Remark: It is a rational, projective variety defined over $\mathbb{Q}$, since as may be seen from the $G(\mathbb{C})$-equivariant embeddings $\hat{D} \subset \prod_{p=n}^{[\eta/2]} Grass(f^p, V_{\mathbb{C}}) \subset \prod_{p=n}^{[\eta/2]} \mathbb{P}(\Lambda^{f_p} V_{\mathbb{C}})$, where $f_p = h^{n,0} + ... + h^{p,n-p}$, and where the second inclusion is the Plücker embedding.

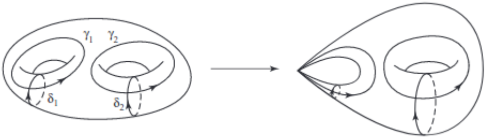
Then we have $D \subset \hat{D}$ as an open $G(\mathbb{R})$-orbit of a fixed point $x_0 \in D$, and as such has an induced complex structure.

Remark: It is very important and of the independent interest to consider *Mumford-Tate domains*.

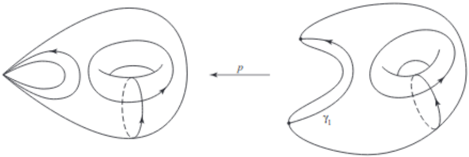
Singular case?

Let us consider $y^2 = (x - a_1)...(x - a_5)(x - t)$ and assume $a_1 = 0$. We are studying the behaviour of the Hodge structure of a surface when t is approaching 0.

Degeneration



Normalization



Question: Whether it makes sense to take a limit of the Hodge structure $H^1(S_t)$, and whether it is possible to define a Hodge structure forthe singular variety S_0.

Note: Meromorphic differential $\omega_2(0) = \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}$ makes sense as meromorphic differentials on the elliptic curve E defined by $y^2 = (x-a)(x-b)(x-c)(x-d)$. The Riemann surface $\tilde{E}$ is the normalization of the algebraic curve $E = S_0$ defined by $y^2 = x^2(x-a)(x-b)(x-c)(x-d)$.

- Since the cohomology of S_0 has rank 3, it cannot carry a Hodge structure of weight 1
- To understand the topology,consider the normalization map $p : \tilde{S}_0 \to S_0$ and its induced map on cohomology, $p^* : H^1(S_0) \to H^1(\tilde{S}_0)$
- p^* is surjective. Indeed, the corresponding map p^* on homology is injective
- there is an exact sequence $0 \to K \to H^1(S_0) \to H^1(\tilde{S}_0) \to 0$, where the kernel K is the $\mathbb{Z}$-module generated by γ_1.

Note: The quotient by K is isomorphic to $H^1(\tilde{S}_0)$, which carries a Hodge structure.

MHS makes its first appearance

Let F^1 be the span of the meromorphic differential $\omega_2(0)$, viewed as a subspace of either $H^1(S_0)$ or $H^1(\tilde{S}_0)$.

Hence, we now have the following data:

(a) a subspace $F^1 \subset H^1(S_0, \mathbb{C})$ defined by the complex structure of the central fiber,

(b) a subspace $K \subset H^1(S_0, \mathbb{Z})$ defined by the topology of the normalization map.

with the properties

(c) the subspace which F^1 defines on $H^1(S_0)/K \simeq H^1(\tilde{S}_0)$ is the natural subspace $F^1 H^1(\tilde{S}_0)$,

(d) $F^1 \cap K = 0$.

Property (c) asserts that the data (K, F^1) define a Hodge structure of weight 1 on $H^1(\tilde{S}_0)$. It is the natural one with $H^{1,0} = F^1, H^{0,1} = \overline{F^1}$.

Then property (d) says that if the pullback of a holomorphic 1-form is 0 as a cohomology class on $\tilde{S}_0$, then it is 0 as a cohomology class on S_0.

MHS, motivation

Filtrations on $H^1(\tilde{S}_0)$

The filtration on $H^1(S_0)$ induces filtration on the quotient $H^1(S_0)/K \simeq H^1(\tilde{S}_0)$, which is isomorhic to the natural one on $H^1(\tilde{S}_0)$.

Filtration on the kernel

On K, the filtration is $F^1 \cap K = 0, F^0 \cap K = K$. Hence, K carries a Hodge structure of weight 0, where $K_{\mathbb{C}} = K^{0,0}$. Thus the data (K, F^1) define two Hodge structures, one of weight 0, the other of weight 1.

Mixed Hodge structure

Definition (Deligne, '71)

A mixed Hodge structure consists of a triple $(H_{\mathbb{Z}}, F^{\bullet}, W^{\bullet})$, where

(i) $H_{\mathbb{Z}}$ is a $\mathbb{Z}$-module of finite rank,

(ii) $F^{\bullet}$ is a finite decreasing filtration on $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} Z_{\mathbb{C}}$, **the Hodge filtration**,

(iii) $W^{\bullet}$ is a finite increasing filtration on $H_{\mathbb{Q}} = H_Z \otimes_{\mathbb{Z}} \mathbb{Q}$, **the weight filtration**, satisfying in addition the requirement that the graded quotients for the weight filtration, $Gr_k^W H = W^k / W^{k-1}$, together with the filtration induced by $F^{\bullet}$, form a (pure) Hodge structure of weight k.

Example: The Hodge filtration on singular Riemannian surface is already defined and the weight filtration is given by $W^1 = H^1(S_0, \mathbb{Q}), W^0 = K$. By construction, $F^{\bullet}$ defines a Hodge structure of weight 1 on $W^1/W^0 \simeq H^1(\tilde{S}_0)$ and a Hodge structure of weight 0 on W_0.

Mixed Hodge Structures

Remark: The induced filtration on $W^m/W^{m-1} = Gr_m^W$ is $F^p Gr_m^W H_{\mathbb{C}} = F^p \cap W^m + W^m/W^{m-1}$.
Remark: In general, one may assume that $H_{\mathbb{Z}}$ is free. Indeed, we can replace $H^*(X.\mathbb{Z})$ by its image in $H^(X, \mathbb{R})$.
Remark: For mixed Hodge structures, the notion of Hodge number still makes sense: $h^{p,q}$ is the dimension of the (p, q)-component of the pure Hodge structure on the graded quotient Gr_{p+q}^W. In particular, in the case of a Riemann surface of genus 2 that has acquired a node, $h^{1,0} = h^{0,1} = 1$ and $h^{0,0} = 1$.

Weight filtration and spectral sequences

Now we are going to discuss particular example of the weight filtration following Deligne.

Let X be a smooth complex algebraic variety of dimension n and $j : X \hookrightarrow \overline{X}$ be a smooth compactification of X such that $D = \overline{X} \setminus X$ is a normal crossings divisor (locally isomorphic to the union of hyperplanes). We may write $D = D_1 \cup \ldots \cup D_N$ as the union of irreducible smooth divisors meeting transversally.

Let $D(0) = X$ and for $0 < p \leqslant N$ let $D(p)$ be the disjoint union of all p-fold intersections $D_{i_1} \cap \ldots \cap D_{i_p}$ with $\{i_1, \ldots, i_p\} \subset \{1, \ldots, N\}$. Since D is a ncd, each $D(p)$ is a smooth projective variety of dimension $n - p$.

Definition

The weight spectral sequence is given by
$E_1^{-p,q}(X) = H^{q-2p}(D(p); \mathbb{Q}) \Rightarrow H^{q-p}(X; \mathbb{Q})$.

The differential $d_1 : E_1^{-p,q}(X) \to E_1^{-p+1,q}(X)$ is defined by the sum of Gysin morphisms
$i_*(j) : H^q(D_{i_1} \cap \ldots \cap D_{i_p}) \to H^{q+2}(D_{i_1} \cap \ldots \cap \hat{D}_{i_j} \cap \ldots \cap D_{i_p}$.

Weight filtration and spectral sequences

This spectral sequence degenerates at the second page, and it induces a filtration on the cohomology of X. The **weight filtration** W is defined by a shift of this filtration: $Gr_p^W H^{p+q}(X,\mathbb{Q}) \simeq E_2^{p,q}(X)$. For all $n \geqslant 0$ the weight filtration satisfies $0 = W_{n-1}H^n(X,\mathbb{Q}) \subset W_n H^n(X,\mathbb{Q}) \subset \ldots \subset W_{2n}H^n(X,\mathbb{Q}) = H^n(X,\mathbb{Q})$

$$
E_1(X) = \left.\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
H^1(D^{(1)}) & \xrightarrow{d_1} & H^3(D^{(0)}) & \xrightarrow{d_1} & H^5(\overline{X}) \\
H^0(D^{(1)}) & \xrightarrow{d_1} & H^2(D^{(0)}) & \xrightarrow{d_1} & H^4(\overline{X}) \\
0 & & H^1(D^{(0)}) & \longrightarrow & H^3(\overline{X}) \\
0 & & H^0(D^{(0)}) & \longrightarrow & H^2(\overline{X}) \\
0 & & 0 & & H^1(\overline{X}) \\
0 & & 0 & & H^0(\overline{X}) \\
\hline
\end{array}\right| \Longrightarrow H(X) = \left.\begin{array}{ccc}
\vdots & \vdots & \\
Gr_5^W H^3 & Gr_5^W H^4 & Gr_5^W H^5 \\
Gr_4^W H^2 & Gr_4^W H^3 & Gr_4^W H^4 \\
Gr_3^W H^1 & Gr_3^W H^2 & Gr_3^W H^3 \\
0 & Gr_2^W H^1 & Gr_2^W H^2 \\
0 & 0 & Gr_1^W H^1 \\
0 & 0 & Gr_0^W H^0 \\
\hline
\end{array}\right|
$$

Weight filtration, example

Let $X = \mathbb{C}^* \hookrightarrow \overline{X} = \mathbb{P}^1_{\mathbb{C}}$.
Then $D = \{**\}$.
So

$$E_1^{*,*}(X) = \begin{array}{cc} 2 & 1 \\ 0 & 0 \\ 0 & 1 \end{array} \Rightarrow H^1(X) = \begin{array}{cc} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}$$

Because $E_2^{*,*}(X)$ is the cohomology of $X \simeq S^1$, the only non-trivial Gysin map of $E_1^{*,*}(X)$ must be onto. We find $Gr_2^W H^1(X, \mathbb{Q}) = \mathbb{Q}$.

Next lecture

- Deformation theory
- Variations of Hodge structures
- p-adic Hodge structures

Thanks!