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outline of course

1. Legendre families, period map
2. Hodge structure of curves and abelian varieties
3. Hodge decomposition, Kähler manifolds, mixed Hodge structures
4. Hodge structures for hypersurfaces, K3, Kuga-Satake, period domains
5. Deformation theory intro and variations of the Hodge structure, p-adic Hodge structures

Hodge decomposition: forms

One of the main applications of Hodge structures is to the study of the cohomology of Kähler manifolds, via the **Hodge decomposition**. This decomposition will be described now.

Basics: Let X be C^∞-manifold. If X is a complex manifold, then the space of smooth $\mathbb{R}$-valued n-forms on X $\mathcal{C}^n_{\mathbb{C}} = \bigoplus_{p+q=n} \mathcal{C}^{p,q}$ ((p,q)-forms which locally have form $\sum f_{I,J} dz_I \wedge d\overline{z}_J$ for holomorphic coordinates z_i).

We have exterior derivative d (determined on functions and 1-forms): $\mathcal{C}^{p,q} \xrightarrow{\partial,\overline{\partial}} \mathcal{C}^{p+1,q} \oplus \mathcal{C}^{p,q+1}$.

$d^2 = \partial^2 = \overline{\partial}^2 = 0$

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Chain complexes

$(\mathcal{C}^\bullet_{\mathbb{R}}, d)$ $((\mathcal{C}^{p,\bullet}_{\mathbb{R}}, \overline{\partial}))$ and $(\Gamma(X, \mathcal{C}^\bullet_{\mathbb{R}}), d)$ $((\Gamma(X, \mathcal{C}^{p,\bullet}_{\mathbb{R}}), \overline{\partial}))$ are chain complexes. The cohomology complexes of the latter two:

$H^\bullet_{dR}(X, \mathbb{R}) := H^\bullet(\Gamma(X, \mathcal{C}^\bullet_{\mathbb{R}}), d)$ **de Rham cohomology** and

$H^{p,\bullet}_{\overline{\partial}}(X) := H^\bullet(\Gamma(X, \mathcal{C}^{p,\bullet}_{\mathbb{R}}), \overline{\partial})$ **Dolbeault cohomology**.

de Rham *vs* Dolbeault

Remark: There is no obvious map between $H^n_{dR}(X, \mathbb{C})$ and $H^{p,q}_{\bar{\partial}}$ in either direction:

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However, both are true in some cases. Later we will study Hodge decomposition in Kähler case, now let us consider complex tori as a an example when we have Hodge structure

Hodge decomposition for tori

Let $T = V/\Lambda$ be a complex torus, where V is n-dimensional vector space and $\Lambda \subset V$ is a lattice isomorphic to $\mathbb{Z}^{2n}$. Then

$$H^n(T, \mathbb{Z}) \text{ carries a Hodge structure of weight } n$$

Hodge decomposition for tori

Decomposition

$$H^n_{dR}(T,\mathbb{C}) \simeq \bigoplus_{p+q=n} H^{p,q}_{\overline{\partial}}(T)$$

Hodge decomposition

Remark: We do not have such decomposition in general, bu we do have for Kähler manifolds.

X be a complex manifold

$T_{\mathbb{R}}X \circlearrowleft I$ where I is endomorphism with $I^2 = -Id$ (like multiplication by i). Then complexification $T_{\mathbb{C}}X$ decomposes as $T^{1,0}X \oplus T^{0,1}X$ with the first subspace having the eigenvalue i and the second eigenvalue $-i$.

A real 2-form ω of type $(1,1)$ is **positive** if $\forall x \neq 0 \in T^{1,0}X, \omega_{\mathbb{C}}(x, \overline{x}) \in i\mathbb{R}_{>0}$.

Example: $X = \mathbb{C}^n, \omega = i \sum_j dz_j \wedge d\overline{z}_j$ is positive. Indeed, for $x = \sum a_j \frac{\partial}{\partial z_j}$ we have $\omega(x, \overline{x}) = i \sum |a_i|^2$.

Kähler manifolds

Definition

A Kähler manifold is a complex manifold which possesses a global closed positive real (1,1)-form. Such form is a Kähler form

Examples:

- $i \sum d_j \wedge d\bar{z}_j$ descends to $X = V/\Lambda$ so any complex torus is Kähler
- $X = \mathbb{P}^1$. A form $\omega_{FS} := \frac{idz \wedge d\bar{z}}{(1+|z|^2)^2}$ is called the Fubini-Study form.
- $X = \mathbb{P}^n$ has a Fubini-Study form ω_{FS} making it Kähler
- $X \subset \mathbb{P}^n$ any projective variety is Kähler by retracting ω_{FS}.

Note: A hermitian metric h on $T^{1,0}X$ is a $(\mathbb{C}, \overline{\mathbb{C}}$-bilinear form such that $h(x, y) = \overline{h(y, x)}, h(x, x) > 0 \forall x \in T_p^{1,0}X$ (any p)
Using the action of I on $T_{\mathbb{R}}X$ one can write $h = Reh - i(-Imh)$.
We denote Reh by g and $(-Imh)$ by ω.
Remark: g is Riemannian metric and ω is a non-degenerate 2-form, h is determined by (and *vice versa*) ω since $g(x, y) = \omega(Ix, y)$.

Laplacian

Let X be a compact complex n-dimensional manifold with hermitian metric h. Then we have metrics on $\Lambda^{p,q}T^*X$ and $\Lambda^n T^*X$, so in fact:

Scalar product

$\langle \alpha, \beta \rangle = \int g(\alpha, \beta) vol_g$, where $vol_g = \omega^n / n!$ (the canonical volume form) and α, β are n-forms
$\langle \alpha, \beta \rangle = \int h(\alpha, \beta) vol_g$, where $vol_g = \omega^n / n!$ (the canonical volume form) and α, β are (p, q)-forms

$d, \partial, \overline{\partial}$

We can define the adjoint operators $d^*, \partial^*, \overline{\partial}^*$ in a way that
$\langle d\alpha, \beta\alpha = \langle \alpha, d^*\beta \rangle$

Definition

Define Laplace operators as follows
$\Delta_d = dd^* + d^*d, \Delta_\partial = \partial\partial^* + \partial^*\partial, \Delta_{\overline{\partial}} = \overline{\partial\partial}^* + \overline{\partial}^*\overline{\partial}$

Laplace operators

Proposition

A n-form α is Δ_d-harmonic if equivalently:

(1) $\Delta_d \alpha = 0$

(2) $d\alpha = d^*\alpha = 0$

(3) $d\alpha = 0$ and $||\alpha||^2 = \langle \alpha, \alpha \rangle$ is minimal in $[\alpha] \in H^n_d R(X, \mathbb{R})$

Proof.

(1) $\leftrightarrow$ (2): one direction is obvious, the opposite can be seen as follows: $0 = \langle \alpha, \Delta_d \alpha \rangle = ||d\alpha||^2 + ||d^*\alpha||^2$
(1) $\leftrightarrow$ (3): $||\alpha + td\beta||^2 = ||\alpha||^2 + 2t\langle \alpha, d\beta \rangle + O(t^2)$. Hence it is minimal iff $\langle \alpha, d\beta \rangle = \langle d^*\alpha, \beta \rangle = 0$ (for any β). That is equivalent to $d^*\alpha = 0$. □

Remark: Same true for Δ_∂- and $\Delta_{\overline{\partial}}$-harmonic.

Harmonic decomposition

Theorem

Any class in H^n_{dR} and $H^{p,q}_{\overline{\partial}}$ is uniquely represented by a Δ_d-, $\Delta_{\overline{\partial}}$-harmonic class.
Denote $\mathcal{H}^{p,q}_{\overline{\partial}}(X)$ – the space of $\overline{\partial}$-harmonic (p,q)-forms.
Indeed, $\mathcal{C}^{p,q}(X) = \mathcal{H}^{p,q}_{\overline{\partial}}(X) \oplus Im(\Delta_{\overline{\partial}} : \mathcal{C}^{p,q} \to \mathcal{C}^{p,q}) = \mathcal{H}^{p,q}_{\overline{\partial}}(X) \oplus \overline{\partial}(\mathcal{C}^{p,q-1}) \oplus \overline{\partial}^*(\mathcal{C}^{p,q+1})$.
Moreover,

$$ker(\overline{\partial} : \mathcal{C}^{p,q} \to \mathcal{C}^{p,q+1}) = \mathcal{H}^{p,q} \oplus \overline{\partial}(\mathcal{C}^{p,q-1})$$

So, in conclusion the Dolbeaux cohomology group is isomorphic to $\mathcal{H}^{p,q}_{\overline{\partial}}(X)$

Kähler case

In the case of Kähler manifold one have $\Delta_d = 2\Delta_{\overline{\partial}}$
In particular, it follows that $\Delta_{\overline{\partial}}$-harmonic (p,q)-form is Δ_d-harmonic, and (p,q)-part of $\Delta_{\overline{\partial}}$. Moreover, the conjugate of a $\Delta_{\overline{\partial}}$-harmonic form.

decomposition

$$H^n_{dR}(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}_{\overline{\partial}}(X), \quad H^{p,q}_{\overline{\partial}} = \overline{H^{q,p}_{\overline{\partial}}}$$

So that gives a Hodge structure of weight n on $H^n(X,\mathbb{Z})$.

Note: $H^{p,q}_{\overline{\partial}}(X) = H^q(X,\Omega^p_X)$ via the natural map.

Corollary

Every global holomorphic p-form is d-closed and de Rham cohomologous holomorphic p-forms are equal.

- If n is odd, then $H^n(X,\mathbb{Z})$ has even rank.

non-Kähler manifolds

Sure, there are non-Kähler manifolds. Among them are *Hopf surface*:

$$M = \frac{\mathbb{C}^2 \setminus \{(0,0)\}}{(z_1, z_2) \sim (2z_1, 2z_2)}$$

One can check that M is diffeomorphic to $S^3 \times S^1$ and Künneth formula shows that $b_3(M) = 1$ so M is not Kähler.
and
Kodaira surface, which is (nilmanifold) elliptic fibration over an elliptic curve E. Namely, for a line bundle L with $c_1(L) > 0$ we can consider the complement L^* to the zero loci, there is natural action of $\mathbb{C}^*$ there. The quotient $S := L^*/\mathbb{Z}$ is the Kodaira surface with fibration $S \xrightarrow{E_L = \mathbb{C}^*/\mathbb{Z}} E$. It has $b_1 = 3$ and it is also non-Kähler.

Harmonic forms on complex torus

Example: Complex torus $X = V/\Lambda$ where $V \simeq \mathbb{C}^g$.
Forms $dz_{I,J} := \frac{dz_{i_1}\wedge\ldots\wedge dz_{i_k}\wedge d\overline{z_{j_1}}\ldots\wedge d\overline{z_{j_l}}}{2^{(|I|+|J|)/2}}$ are o/n basis.
In this case we can write explicitly
$\partial^* g dz_{I,J} = \sum_{i\in I} \frac{\partial g}{\partial \overline{z_i}}(-1)^{|I|<i}\sqrt{2}dz_{I-i,J}$ and
$\overline{\partial}^* g dz_{I,J} = \sum_{i\in I} \frac{\partial g}{\partial z_i}(-1)^{|I|+|J<j|}\sqrt{2}dz_{I,J-j}$.
Note: This is the direct computation.
In the case of torus it is also not that hard to compute the laplace operator $\Delta_{\overline{\partial}}$:
$\overline{\partial}\overline{\partial}^* g dz_{I,J} = \overline{\partial}\left(\sum_{j\in J} \frac{\partial g}{\partial z_j}(-1)^{|I|+|J<j|}dz_{I,J-j}\right) =$
$\sum_{j\in J, k\notin J-j} \frac{\partial^2 g}{\partial z_j \partial \overline{z_k}}(-1)^{|I|+|J<j|}(-1)^{|I|+|(J-j)<k}dz_{I,J-j+1}$
After the same calculation for $\overline{\partial}^*\overline{\partial} g dz_{I,J}$ we have that the term with $j \neq k$ appear with opposite sign, so
$\Delta_{\overline{\partial}} g dz_{I,J} = \left(\sum \frac{\partial^2 g}{\partial z_j \partial \overline{z_j}}\right) dz_{I,J}$
Harmonic function on a compact manifold are constant, so harmonic functions are exactly the translation-invariant ones

Lefschetz operators on Kähler manifolds

Let X be a manifold of dimension m with Kähler class ω.
Consider $L : \Gamma(X, \mathcal{C}^{p,q} \to \Gamma(X, \mathcal{C}^{p+1,q+1})$ operator acting $\alpha \mapsto \omega \wedge \alpha$ and L^* with respect to $\langle , \rangle$.
Example: Let $X = V/\Lambda$ be torus:
$Ldz_{I,J} = i \sum_{i \notin I \cup J} dz_{I+i,J+i} (-1)^{|I|+|J<i|}(-1)^{|I<i|}$, $L^* dz_{I,J} = -i \sum_{i \in I \cap J} dz_{I-i,J-i} (-1)^{|I|+|J<i|}(-1)^{|I<i|}$ and L, L^* preserve harmonicity, $[L^*, L] = m - degree$
$\langle LL^* dz_{I,J}, dz_{I',J'} \rangle = \langle L^* dz_{I,J}, L^* dz_{I',J'} \rangle = \sum_{i \in I \cap J, j \in I' \cap J', I'-j=I-i, J'-j=J-j} (-1)^{|I|+|I<i|+|J<i|}(-1)^{|I'|+|I'<j|+|J'<j|}$,
$\langle L^* L dz_{I,J}, dz_{I',J'} \rangle = \sum_{i \notin I \cup J, j \notin I' \cup J', I'+i=I+i, J'+i=J+i} (-1)^{|I|+|J<j|+|I<j|}(-1)^{|I'|+|I'<j|+|J'<j|}$.
After cancelling terms with $i \neq j$ (when $i = j$ we have $I' = I, J' = J$):
$\langle [L^*, L] dz_{I,J}, dz_{I,J} \rangle = \sum_{i \notin I \cup J} 1 + \sum_{i \in I \cap J} (-1) = m - |I| - |J|$

Primitive forms

A k-form α is primitive if $L^{m-k+1}\alpha = 0$. We write Prim$\mathcal{C}^k$ for primitive k-forms.

Remark: Since L is real, the form α is primitive if and only if $\overline{\alpha}$ is primitive. Since L has bidegree (1, 1), the form α is primitive if and only if all of its (p, q)-components are primitive.

Lefschetz for forms

$\mathcal{C}^k = \bigoplus_{i \geqslant 0} L^i \text{Prim}(\mathcal{C}^{k-2i})$ for $0 \leqslant k \leqslant m$.

The proof follows by the induction.

Remark: A k-form α is primitive iff $L^*\alpha = 0$.

Definition

$H^{m-k}_{prim}(X, \mathbb{C})$ = lowest weights in $H^{m-k}_{dR}(X, \mathbb{C}) = ker(L^* : H^{m-k}_{dR} \to H^{m-k-2}_{dR}) = ker(L^{k+1} : H^{m-k}_{dR} \to H^{m+k+2}_{dR})$ - the set of primitive cohomology classes

Remark: The primitive part of $H^k(M)$ is the piece not accounted for by lower degree cohomology:

$\dim H^k_{prim}(M) = b_k(M) - b_{k-2}(M)$

Lefschetz decomposition

L preserves harmonicity

We have $[\Delta, L] = 0$

Indeed, $\Delta = \Delta_{\overline{\partial}}$ then $[\Delta_{\overline{\partial}}, L] = \overline{\partial}[\overline{\partial}^*, L] + [\overline{\partial}^*, L]\overline{\partial}$. Using $\overline{\partial}L = L\overline{\partial}$ we have $[\Delta_{\overline{\partial}}, L] = \overline{\partial}i\partial + i\partial\overline{\partial} = 0$.

Hard Lefschetz

For $k \leqslant m$, the map $[\omega]^k \cup (-) : H^k_{dR} \to H^{m-2k}_{dR}$ is an isomorphism.

Remark: The fact that the spaces are isomorphic follows from Poincare duality as well, but Hard Lefschetz is a much more useful statement. For example, it implies for $k \leqslant m-2$ that $b_k(M) \leqslant b_{k+2}(M)$.

Then we have the **Lefschetz decomposition for cohomology** $H^k_{dR} = \bigoplus_{i \geqslant 0} L^i H^{k-2i}_{prim}$ (for $k \leqslant m$) which is a decomposition into polarized sub-Hodge structures (since L preserves harmonic forms).

Hodge-Riemann bilinear relations

Since ω is a real $(1,1)$-form, the Lefschetz decomposition is compatible with the real structure and the decomposition into (p,q)-subspaces.

Polarization: $Q(\alpha,\beta) = (-1)^{\frac{k(k-1)}{2}} \int \alpha \wedge \beta \omega^{m-k}$ on H^k_{dR} for $k \leqslant m$.

Q polarizes H^k_{prim} for $k \leqslant m$ and $Q(n)^i$ polarizes $L^i H^k_{prim}$

Remark: Q does not give a polarization directly on H^k, because for ω we have $Q(\omega,\omega) = -\int \omega^m < 0$). That's why we have $(-1)^i Q$ on $L^i H^k_{prim}$.

Hodge-Riemann bilinear relations

For $k \leqslant m$ and (p,q) with $p+q=k$ the form $i^{p-q}Q$ is positive definite on $H^{p,q}_{prim}(M)$.

Remark: If $\omega \in H^2_{dR}(X,\mathbb{Z})$, then Q is defined over $\mathbb{Q}$ and $\mathfrak{sl}_2$-spliting given by L is defined over $\mathbb{Q}$ so

$$H^k_{dR} = \bigoplus L^i H^{k-2i}_{prim} \simeq \bigoplus H^{k-2i}_{prim}(X,\mathbb{Q})(-i)$$

decomposition of polarized $\mathbb{Q}$-Hodge structures.

$m = 2$

First of all,
$H^0(X,\mathbb{Q}) = \mathbb{Q}, H^1(X,\mathbb{Q}) = H^1_{prim}, H^2(X,\mathbb{Q}) = \mathbb{Q}h \oplus H^2_{prim}$, where
$h = c_1(L) = [H]$ th class of ample line bundle L on smooth projective X.
We need to check: $Q(H^{p,q}, H^{p',q'}) = 0$ unless
$(p+p', q+q') = (k, k')$ and $i^{p-q}Q(x,\overline{x}) > 0$ for $x \in H^{p,q}$
Indeed, the first is automatic,and the second is the straightforward computation.
Note: Q does not polarize $H^2(X,\mathbb{Q})$ since $0 < \int h^2$.

Computation

$H^1_{prim}(X,\mathbb{Q})$: $Q(\alpha,\beta)=\int\alpha\wedge\beta\wedge h$, locally
$h=idz\wedge d\overline{z}+id\omega\wedge\overline{\omega},\alpha=dz\in H^{1,0}(X)$ so
$0<i\int dz\wedge d\overline{z}\wedge h=\int i(dz\wedge d\overline{z})\wedge i(d\omega\wedge d\overline{\omega})$
$H^2_{prim}(X,\mathbb{Q})$: $q(\alpha,\beta)=-\int\alpha\wedge\beta$, $\alpha\in H^{2,0}_{prim}(X)=H^{2,0}(X)$,
locally $\alpha=dz\wedge d\omega$, then
$0<i^2\int(dz\wedge d\omega)\wedge(d\overline{z}\wedge d\overline{\omega}=\int(idz\wedge d\overline{z})\wedge(d\overline{z}\wedge d\overline{\omega})$
If $\alpha\in H^{1,1}_{prim}(X)=ker(h\cup -)$, locally
$h=idz\wedge d\overline{z}+id\omega\wedge\overline{\omega},\alpha=adz\wedge d\overline{z}+bdz\wedge d\overline{\omega}+cd\omega\wedge d\overline{z}+dd\omega\wedge d\overline{\omega}$.
Because α lies in kernel we have $a=-d$.
So $0<-i^0\int\alpha\wedge\overline{\alpha}=(2|a|^2+|b|^2+|c|^2)(dz\wedge d\overline{z})(d\omega\wedge d\overline{\omega})$.

Hypersurfaces in $\mathbb{P}^{m+1}$ of degree d

The **Hodge diamond** of a smooth Kähler manifold X (with all symmetries) looks like

$$
\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & {\color{red}h^{1,0}} & & h^{0,1} & & & \\
 & & h^{2,0} & & {\color{red}h^{1,1}} & & h^{0,2} & & \\
 & .. & & .. & & {\color{red}..} & & .. & \\
h^{m,0} & & & & & & & & h^{0,m} \\
 & .. & & .. & & .. & & .. & \\
 & & {\color{blue}h^{m,m-2}} & & {\color{blue}h^{m-1,m-1}} & & {\color{blue}h^{m-2,m}} & & \\
 & & & h^{m,m-1} & & h^{m-1,m} & & & \\
 & & & & h^{m,m} & & & &
\end{array}
$$

where red alternative sum is $\chi(\Omega^1_X)$, and sum is $\dim H^{2m-2}_{\mathbb{Q}}(X, \mathbb{Q})$.

Note: $h^{p,q} = h^{q,p} = h^{m-p,m-q}$

Example: $X = V/\Lambda$ abelian surface has the Hodge diamond

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 2 & & 2 & \\
1 & & 4 & & 1 \\
 & 2 & & 2 & \\
 & & 1 & &
\end{array}
$$

Lefschetz hyperplane

For $\mathbb{P}^{m+1}$:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & .. & & & .. & & & .. & \\
0 & & & 0 & 1 & 0 & & & 0 \\
 & .. & & & .. & & & .. & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

Lefschetz hyperplane

Let $X \subset \mathbb{P}^{m+1}$ smooth hypersurface. Then $i^* : H^k(\mathbb{P}^{m+1}, \mathbb{Q}) \xrightarrow{\simeq} H^k(X, \mathbb{Q}), \quad k < m$ is am isomorphism of Hodge structures.

As a corollary of that we have the following look of the Hodge diamond of $X \subset \mathbb{P}^{m+1}$:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & .. & & & .. & & & .. & \\
* & & & .. & * & .. & & & * \\
 & .. & & & .. & & & .. & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

middle Hodge numbers of hypersurfaces

Idea: Enough to compute $\chi(X, \Omega_X^p)$. To do this we have powerful instrument – Grothendieck-Riemann-Roch theorem:

$$\chi(X, E) = \int_X ch(E) \cdot td(T_X)$$

Recall: for $0 \to E' \to E \to E'' \to 0$. Then $ch(E) = ch(E') + ch(E'')$. Also $ch(E \otimes E') = ch(E) \cup ch(E'), ch(f^*E) = f^*ch(E), ch(L) = e^{c_1(E)}$ for a line bundle L.
for $0 \to E' \to E \to E'' \to 0$
Then $td(E) = td(E') \cup td(E'')$. Also $td(f^*E) = f^*td(E), td(L) = \frac{c_1(L)}{1-e^{-c_1(L)}}$ for a line bundle L.

Example: We have $0 \to \Omega^1_{\mathbb{P}^{\mathbb{N}}}(1) \to \mathcal{O}_{\mathbb{P}^N}^{N+1} \to \mathcal{O}_{\mathbb{P}^N}(1) \to 0$
Hence, $td(T_{\mathbb{P}^N}) = (td\mathcal{O}_{\mathbb{P}^N}(1)^{N+1}), ch(\Omega^1_{\mathbb{P}^N}) = (N+1)e^{-h} - 1$, where h is the first Chern class of $H \to \mathbb{P}^N$, the hyperplane line bundle. Its sections can be identified with linear maps $\mathbb{C}^{N+1} \to \mathbb{C}$. As is known we have the equality $\langle h^N, [\mathbb{P}^N] \rangle = \int_{\mathbb{P}^N} h^N = 1$ and an isomorphisms of rings $H^\bullet(\mathbb{P}^N, \mathbb{Z}) \simeq \mathbb{Z}[h]/(h^{N+1})$.

middle Hodge numbers of hypersurfaces

In particular, for hypersurface X of degree d:
$0 \to \mathcal{O}_X(-d) \to \Omega^1_{\mathbb{P}^N}|_X \to \Omega^1_X \to 0$. We have the natural epimorphism $\Omega^1_{\mathbb{P}^N}|_X \to \Omega^1_X$ given by the restriction of 1-forms. Let f be a defining polynomial of X, and let $M = \Gamma_*(\Omega^1_{\mathbb{P}^N})$ and $\overline{M} = \Gamma_*(\Omega^1_X)$. We embed M as a submodule of $\Omega_S(-1)$ (where $S = k[x_1, ..., x_n]$. In particular, the symbols dx_i have degree 1.Then $ker[M/fM \to M]$ is a free $S/(f)$-module generated by $df = \sum_i \partial f/\partial x_i dx_i$.Thus it is isomorphic to $S/(f)(-d)$.
$td(T_X) = \frac{i^* td(T_{\mathbb{P}^N}}{i^* td(\mathcal{O}(d))} = i^* \frac{td\mathcal{O}(1)^{N+1}}{td\mathcal{O}(d)}, ch(\Omega^1_X) = i^* ch(\Omega^1_{\mathbb{P}^N}) - i^* ch(\mathcal{O}(-d)) = (N+1)e^{-h} - 1 - e^{-dh}$.
where $i : X \to \mathbb{P}^N$ is inclusion.
Example: Quartic X in $\mathbb{P}^3$
We have $td(T_X) = 1 + \frac{1}{2}(i^* h^2), ch(\Omega^1_X) = 2 - 6(i^* h^2)$.
Hence, $\chi(\mathcal{O}_X) = \int ch(\mathcal{O}_X) td(T_X) = \int td(T_X) = 2$
and $\chi(\Omega^1_X) = \int (1 - 6i^* h^2)(1 + 1/2 i^* h^2) = -5 \int i^* h^2 = -20$
Then, $h^{2,0} = h^{0,2} = 1, h^{1,1} = 20$
Such quartic is called K3 surface.

Kuga-Satake construction

Last time we have noticed the correspondence between integral Hodge structures of weight one and complex tori. There is some way to construct tori from the Hodge structure of weight 2 of specific type.

K3-type Hodge structure

We call V a Hodge structure of K3-type if V is a (rational or integral) Hodge structure of weight two with $\dim_{\mathbb{C}}(V^{2,0}) = 1$ and $V^{p,q} = 0$ for $|p - q| > 2$.
The motivation for this definition is, of course, that $H^2(X, \mathbb{Q})$ and $H^2(X, \mathbb{Z})$ of a complex K3 surfaces (or a two-dimensional complex torus) X are rational resp. integral Hodge structures of K3-type.

There is the Kuga-Satake torus associated with the each K3-type

Hodge structures as algebraic representations

Any rational Hodge structure of weight n gives a real representation of $\mathbb{C}^*$, namely the group homomorphism

$$\rho : \mathbb{C}^* \to GL(V_{\mathbb{R}}), z \mapsto \rho(z) : v \mapsto (z^p \overline{z}^q) \cdot v$$

for $v \in V^{p,q}$. This representation is real. Indeed, take $v \in V_{\mathbb{R}}$ and consider its decomposition $v = \sum v^{p,q}$ with $\overline{v^{p,q}} = v^{q,p}$.
Then $\rho(z)(v) = (z^p \overline{z}^q) v^{p,q}$ is still real, as $\overline{(z^p \overline{z}^q) v^{p,q}} = (z^p \overline{z}^q) v^{p,q}$.
Note that the induced representation $\rho|_{\mathbb{R}}^*$ is given by $\rho(t)(v) = t^n \cdot v$.
There is a natural bijection between rational Hodge structures of weight n on a rational vector space V and algebraic $GL(V_{\mathbb{R}})$ with R^* acting by representations $\rho : \mathbb{C}^* \rho(t)(v) = t^n \cdot v$.
Let us denote the $\mathbb{C}$-linear extension of ρ by $\rho_{\mathbb{C}} : \mathbb{C}^* \to GL(V_{\mathbb{C}})$ and let

$$V^{p,q} := \{v \in V_{\mathbb{C}} | \rho_{\mathbb{C}}(z)(v) = (z^p \overline{z}^q) \cdot v, \forall z \in \mathbb{C}^*\}$$

Then $\rho_{\mathbb{C}}$ splits into a sum of one-dimensional representations $\lambda_i(z) = z^p \overline{z}^q$ for some $p + q = n$

Algebraic representations: converse

To construct $\lambda_i(z)$ we use algebraicity of ρ:
As an $\mathbb{R}$-linear algebraic representation
$\mathbb{C}^* = \{z = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}\} \subset GL_2(\mathbb{R})$
Hence, $\lambda_i(z)$ must be polynomial in $z, \overline{z}, (z\overline{z})$. Therefore, it is of the form $z^p\overline{z}^q$ for some p, q with $p + q = n$.
Remark: It is easier to say in terms of the Deligne torus.

Algebraic representations: examples

Examples:

- If $V_{\mathbb{C}} = \bigoplus V^{p,q}$ is a Hodge structure given by $\rho : \mathbb{C}^* \to GL(V_{\mathbb{R}})$ then the dual Hodge structure V defined earlier corresponds to the dual representation $\rho : \mathbb{C}^* \to GL(V_{\mathbb{R}}^*)$ which is explicitly given by $\rho^*(z)(f) : v \mapsto f(\rho(z)^{-1}v)$.
- A polarization is in this language described by a bilinear map $\Psi : V \otimes V \to \mathbb{Q}$ with $\Psi(\rho(z)v, \rho(z)w) = (z\overline{z})^n \Psi(v, w)$ and such that $\Psi(v, \rho(i)w)$ defines a positive definite symmetric form on $V_{\mathbb{R}}$.
- iii) The Tate Hodge structure $\mathbb{Q}(1)$ corresponds to $\mathbb{C}^* \to \mathbb{R}^*, z \mapsto (z\overline{z})^{-1}$. Thus if a Hodge structure on V corresponds to a representation ρ_V, then the Tate twist $V(1)$ corresponds to $\rho_{V(1)} : z \mapsto (z\overline{z})^{-1}\rho V(z)$

Clifford algebra

- *Tensor algebra*: $T(V) = \bigoplus_{i \geqslant 0} V^{\otimes i}$ with $V^{\otimes 0} = K$ is a graded non-commutative K-algebra.
- *Clifford algebra*: $Cl(V, q) = T(V)/I(q)$, where $I(q)$ is the two-sided ideal generated by the even elements $v \otimes v - q(v)$.

Remark: Clifford algebra no longer has $\mathbb{Z}$-grading. However, since $I(q)$ is generated by even elements, it still has a natural $\mathbb{Z}/2\mathbb{Z}$-grading and we write $Cl(V, q) = Cl^{+}(V) \oplus Cl^{-}(V)$.

From weight 2 to weight 1

The Hodge structure V of K3-type induces a decomposition of the real vector space: $V_{\mathbb{R}} = (V^{1,1} \cap V_{\mathbb{R}}) \oplus \left((V^{2,0} \oplus V^{0,2}) \cap V_{\mathbb{R}}\right)$
If we pick a generator $\sigma = e_1 + ie_2 \in V^{2,0}$ with $q(e_1) = 1$, then $q(\sigma) = 0$ gives $q(e_1, e_2) = 0, q(e_2) = 1$ so that e_1, e_2 is o/n basis of $(V^{2,0} \oplus V^{0,2}) \cap V_{\mathbb{R}}$. Hence, $e_1 \cdot e_2 = -e_2 \cdot e_1$ in $Cl(V_{\mathbb{R}})$.
Therefore, multiplication by $J := e_1 e_2$ induces a complex structure on the real vector space $Cl(V_{\mathbb{R}})$, that means $J^2 = -Id$

Kuga-Satake torus

Remark: J is independent on the choice of the basis e_1, e_2. And it preserves odd and even parts of Clifford algebra.

KS Hodge structure

The Kuga-Satake Hodge structure is the Hodge structure of weight one on $Cl^+(V)$ given by

$$\rho : \mathbb{C}^* \to Gl(Cl^+(V)_\mathbb{R}), x + yi \mapsto x + Jy$$

It gives the Hodge structure of weight one on the full $Cl(V)$.

KS variety

The Kuga-Satake variety associated with the integral Hodge structure V of K3-type is the complex torus $KS(V) := Cl^+(V_\mathbb{R})/Cl^+(V)$

Note: $\dim KS(V) = 2^{n-2}$ for $\dim_\mathbb{C}(V_\mathbb{C}) = n$.

Remark: There is the generalization of Kuga-Satake to higher dimensional analogs of K3 surfaces (hyperkähler manifolds) described by Soldatenkov, Verbitsky and myself.

Next lecture

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Thanks!